The Optical Theorem, Cutkosky Rules and Unstable Particles

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Overview and Motivation

Motivation:the problem of unstable particles

In free field theory, particles are defined as strict eigenstates of the Hamiltonian with definite energy and infinite lifetime. However, in interacting field theory, particles decay through quantum interactions, so they are no longer eigenstates of the Hamiltonian, and their energy is no longer a definite real number.

We must abandon the definition of "free single-particle states" and instead use analytic singularities (poles) of the scattering amplitude to define particles:

- Stable particles: correspond to poles on the first Riemann sheet of the complex energy plane on the real axis
- Unstable particles: correspond to complex poles on the second Riemann sheet, whose imaginary part is directly related to the decay width and lifetime

Overview and Motivation

Properties of S-matrix

- Causality: Cause must precede effect. Mathematically, this requires scattering amplitudes to be analytic functions in the complex plane.
- Unitarity: The sum of probabilities for all possible physical processes must be 1.

Outline

- The Optical Theorem
- Cutkosky Rules
- Decay Width and Lifetime
- Poles on the Second Riemann Sheet
- Preview of Study

The Optical Theorem

Definition

The Optical Theorem is a direct consequence of S-matrix unitarity. Mathematically, it equates the imaginary part of the forward scattering amplitude of a particle beam to the particle's total cross-section (the total probability of any interaction occurring).

$$\mathrm{Im}\,\mathcal{M}_{\mathrm{forward}} = 2E_{\mathrm{CM}}|\vec{p}_i|\sigma_{\mathrm{tot}} \quad (1)$$

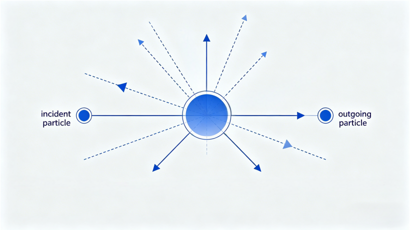


Figure 1: The optical theorem

The Optical Theorem

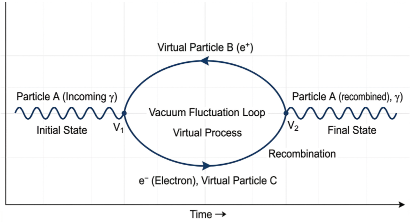
Examples

- Classical analogy: When light passes through an absorbing medium, the decrease in forward light intensity is the result of destructive interference between the secondary waves scattered by atoms in the medium and the original incident wave. The interference intensity is exactly equal to the total light intensity absorbed and scattered.
- Quantum analogy: When the wave function of an incident particle propagates forward, its imaginary part describes the probability that the particle "disappears". These disappeared particles are either scattered to other directions or transformed into other particles, and the total probability of all processes is the total cross section.

Cutkosky Rules

Cutting rules

- 1.Cut through the diagram in any way that can put all of the cut propagators on-shell without violating momentum conservation.
- 2.For each cut, replace $\frac{1}{p^2-m^2+i\varepsilon} \to -2i\pi\delta(p^2-m^2)\theta(p^0)$.
- 3.Sum over all cuts.
- 4.The result is the discontinuity of the diagram, where $\text{Disc}(i\mathcal{M}) = -2\text{Im}\mathcal{M}$.



Cutkosky Rules

Example:1-Loop Self-Energy Diagram

Imagine the simplest scenario: a particle A flies through space, splits into two particles B and C midway, and then B and C immediately recombine back into particle A. Drawn as a Feynman diagram, this is a line with a single circle (loop) in the middle.

Conventional calculation (very difficult): You have to perform an extremely complicated integral over the momenta of B and C inside the loop, because as virtual particles, they can have any possible momentum.

Cutkosky Rules

Using the Cutting Rules (simpler and physically meaningful):

- We "make a cut" across the middle loop, severing both the B and C lines.
- After cutting, the left side becomes the process: A decays into B + C. The right side becomes the process: B + C combine into A.
- At this point, if the mass of A (M_A) is greater than the sum of the masses of B and C ($m_B + m_C$), then B and C can exist as real physical particles (energy conservation is satisfied, satisfying Rule 1).
- According to Rule 4, the imaginary part of this complicated "loop diagram" satisfy $\operatorname{Im}\Sigma(M_A^2) = -M_A\Gamma$,then the decay probability per unit time is obtained.

Decay Width and Lifetime

Untable particles

Unstable particles are subatomic particles that cannot exist independently indefinitely and spontaneously decay into other lighter particles through strong, electromagnetic, or weak interactions. In contrast, stable particles (such as electrons, photons, and protons) have an infinite lifespan in their natural state.

Decay width

The decay width represents the probability of a particle decaying per unit time.

$$\Gamma = \frac{\hbar}{\tau} \tag{2}$$

τ is the average lifespan of particles.

Unstable Particles and Poles on the Second Riemann Sheet

Square of the total energy in the center-of-mass frame (complex variable) : $s = (p_1 + p_2)^2$,Scattering amplitude (multivalued function) : $A(s)$.

The stable particle $\Gamma = 0$, with mass below the threshold and the pole located within the analytic region of the first Riemann sheet, is a physical asymptotic state.

The unstable particle $\Gamma \neq 0$ has a mass above the threshold, and its pole is blocked by the branch cut outside the first Riemann sheet, only to be found within the analytic region of the second Riemann sheet. Therefore, its rigorous mathematical definition is a pole on the second Riemann sheet.

Preview of Study

Next, we will:

- Start from the unitarity and causality of the S-matrix to complete the full mathematical derivation of the optical theorem.
- Learn the specific operational method of the Cutkosky rules, analyze the cutting procedure through examples, and apply this approach to general loop diagram scenarios.
- Use the above theoretical tools to address the issue of unstable particles, derive the quantitative relationship between the imaginary part of the self-energy, decay width , and particle lifetime , and reveal how the finite lifetime of unstable particles originates from quantum loop effects.
- Analyze the analytic structure of scattering amplitudes in the complex energy plane, leading to the definition of unstable particles as complex poles on the second Riemann sheet.

Reference

[1] M. D. Schwartz, *Quantum Field Theory and the Standard Model*, Cambridge University Press, Section 24.1 (The Optical Theorem) and Section 24.2 (Unstable Particles).

[2] S. Willenbrock, *Unstable Particles in Quantum Field Theory*, arXiv:2511.16941 (Pedagogical introduction to poles and branch cuts).

[3] R. E. Cutkosky, *Singularities and discontinuities of Feynman amplitudes*, J. Math. Phys. **1**, 429 (1960).

Thanks
for listening!