Entanglement in Quantum Field Theory

From Reeh-Schlieder Theorem to Modular Theory & Relative Entropy

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Overview: The Logic of Vacuum Entanglement

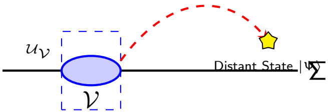
1.1 The Core Statement: Local Density of States

Reeh-Schlieder Theorem (1961)

Let $\mathcal{V} \subset \Sigma$ be an **arbitrarily small** open set, and $\mathcal{U}_\mathcal{V}$ its spacetime neighborhood. The subspace of states

$$\mathcal{D}(\mathcal{U}_\mathcal{V}) = \{a|\Omega\rangle : a \in \mathcal{A}(\mathcal{U}_\mathcal{V})\}$$

is **dense** in the vacuum sector $\mathcal{H}_0$.



- **Physical Astonishment:** Acting on the vacuum strictly within a microscopic local lab can approximate any arbitrary global state (e.g., creating the "Moon" afar).
- **Prerequisite:** Relativistic spectral condition (P^μ lies in the forward lightcone, $H \geq 0$).

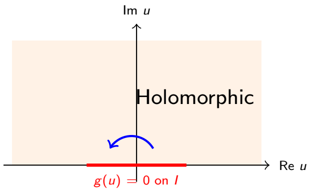
1.2 Sketch of Proof: The Power of Analyticity

Proof Strategy: Analytic Continuation via Time Translation

Suppose $\exists|\chi\rangle \neq 0$ orthogonal to $\mathcal{D}(\mathcal{U}_\mathcal{V})$. Then for all $x_i \in \mathcal{U}_\mathcal{V}$:

$$g(u) = \langle\chi|\phi(x_1)\cdots\exp(iHu)\phi(x_n)|\Omega\rangle = 0 \quad \text{(for small } u\text{)}$$

- **Holomorphy:** Since $H \geq 0$ and $H|\Omega\rangle = 0$, $g(u)$ is holomorphic in the upper half complex u-plane.
- **Identity Theorem:** By the reflection principle/Cauchy integral formula, vanishing on a real segment implies $g(u) \equiv 0$ identically.
- **Spacetime Extension:** Repeating zig-zag timelike shifts extends the vanishing matrix elements to all $x_i \in M_D$. Thus $|\chi\rangle \perp \mathcal{H}_0 \implies |\chi\rangle = 0$.



1.3 Core Corollary: Cyclic and Separating Vectors

- **Cyclic Property:** $\mathcal{A}_{\mathcal{U}}|\Omega\rangle$ is dense in $\mathcal{H}_0$ (Direct from RS Theorem).
- **Separating Property:** Let $\mathcal{U}$ and $\mathcal{U}'$ be spacelike separated. Microcausality implies $[\mathcal{A}_{\mathcal{U}}, \mathcal{A}_{\mathcal{U}'}] = 0$.

Proof of Separating Property

Suppose $a \in \mathcal{A}_{\mathcal{U}}$ annihilates the vacuum: $a|\Omega\rangle = 0$. For any operator $b \in \mathcal{A}_{\mathcal{U}'}$, microcausality yields:

$$a(b|\Omega\rangle) = b(a|\Omega\rangle) = 0$$

Since $b|\Omega\rangle$ generates a dense set in $\mathcal{H}_0$ by the RS theorem applied to $\mathcal{U}'$, a must vanish identically: $a = 0$.

Consequences

1. Local energy density operators $T_{00}(x)$ are **not positive semi-definite**.
2. Resolution of the "Moon Paradox": Local physical operations are **unitary** perturbations of H. For any distant measurement M, $\langle a\Omega|M|a\Omega\rangle = \langle\Omega|a^\dagger Ma|\Omega\rangle = \langle\Omega|M|\Omega\rangle$. No superluminal signaling occurs.

2.1 The Tomita Operator S_Ψ

- **The Bridge:** The "Cyclic and Separating" property guarantees that the antilinear mapping $a|\Psi\rangle \mapsto a^\dagger|\Psi\rangle$ is uniquely defined.
- **Definition:** The Tomita operator S_Ψ is densely defined on $\mathcal{A}_\mathcal{U}|\Psi\rangle$ by:

$$S_\Psi a|\Psi\rangle = a^\dagger|\Psi\rangle, \quad \forall a \in \mathcal{A}_\mathcal{U} \tag{1}$$

- **Inherent Unboundedness:** In QFT, S_Ψ is inherently **unbounded** due to short-wavelength UV modes near the boundary (Type III von Neumann algebra). We extend it to a closed operator via graph limits.
- **Polar Decomposition:** Unique decomposition of closed operators:

$$S_\Psi = J_\Psi \Delta_\Psi^{1/2} \tag{2}$$

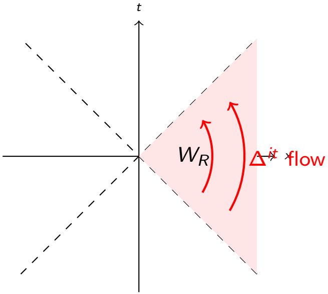
where $\Delta_\Psi \equiv S_\Psi^\dagger S_\Psi$ is the **Modular Operator** (> 0, self-adjoint), and J_Ψ is the **Modular Conjugation** (antiunitary, $J_\Psi^2 = 1$).

Tomita-Takesaki Main Theorems

1. **Modular Automorphism:** $\Delta_\Psi^{it} \mathcal{A}_\mathcal{U} \Delta_\Psi^{-it} = \mathcal{A}_\mathcal{U}$.
2. **Commutant Mapping:** $J_\Psi \mathcal{A}_\mathcal{U} J_\Psi = \mathcal{A}'_\mathcal{U}$.

Physical Realization: Bisognano-Wichmann

- For the **Rindler Wedge** ($x > |t|$), the modular flow Δ^{it} corresponds strictly to **Lorentz Boosts**.
- **Modular Hamiltonian:** $H_{mod} = -\log \Delta = 2\pi K$ (K is the boost generator).
- **Unruh Effect:** Vacuum restricted to the wedge is a thermal KMS state at temperature $T = 1/2\pi$.



2.3 Relative Modular Operator & Araki Relative Entropy

- **The UV Obstruction:** Standard entropy $S = -\mathrm{Tr}(\rho \log \rho)$ is universally UV divergent across the boundary in QFT. Density matrices do not strictly exist.
- **Relative Tomita Operator:** For two states $|\Psi\rangle$ (cyclic/separating) and $|\Phi\rangle$:

$$S_{\Psi|\Phi} a|\Psi\rangle = a^\dagger|\Phi\rangle \implies \Delta_{\Psi|\Phi} \equiv S^\dagger_{\Psi|\Phi} S_{\Psi|\Phi}$$

Araki's Relative Entropy (1976)

The relative entropy for measurements in region U is rigorously defined as:

$$S_{\Psi|\Phi}(U) \equiv -\langle\Psi| \log \Delta_{\Psi|\Phi} |\Psi\rangle$$

- **Strict Positivity:** Using the scalar inequality $\log \lambda \leq \lambda - 1$, we get $-\log \Delta_{\Psi|\Phi} \geq 1 - \Delta_{\Psi|\Phi}$. Taking expectation values yields $S_{\Psi|\Phi}(U) \geq \langle\Psi|\Psi\rangle - \langle\Phi|\Phi\rangle = 0$.
- **Distinguishability:** $S_{\Psi|\Phi}(U) = 0 \iff |\Phi\rangle = a_0'|\Psi\rangle$ (a_0' unitary in $\mathcal{A}_U'$).

2.4 Monotonicity of Relative Entropy

Monotonicity Theorem (Araki)

For nested subregions $\widetilde{U} \subset U$, information decreases as the region shrinks:

$$S_{\Psi|\Phi}(U) \geq S_{\Psi|\Phi}(\widetilde{U})$$

- **Operator Monotonicity:** Since $\log P = \int_0^\infty ds[1/(s+1) - 1/(s+P)]$, the function $\log(x)$ preserves operator orderings. Proving monotonicity strictly reduces to proving:

$$\Delta_{\widetilde{U}} \geq \Delta_U \iff \log \Delta_{\widetilde{U}} \geq \log \Delta_U \tag{4}$$

- **Physical Analogy:** S_U is an **extension** of $S_{\widetilde{U}}$ because its domain $\mathcal{A}_U|\Psi\rangle$ is larger ($\mathcal{A}_U \supset \mathcal{A}_{\widetilde{U}}$). Imposing subregion restrictions acts like moving from Neumann (Δ_N) to Dirichlet (Δ_D) boundary conditions for a Laplacian. Extra boundary constraints naturally lift the ground state energy threshold: $\Delta_D \geq \Delta_N$.

2.5 Ultimate Rigorous Proof via Graph Projectors

- **Graph Subspaces:** Let $\Gamma_U = \{(\psi, S_U \psi)\} \subset \mathcal{H} \oplus \mathcal{H}$ be the closed graph subspace of S_U. Since $S_U \supset S_{\widetilde{U}}$, the graph subspaces satisfy $\Gamma_U \supset \Gamma_{\widetilde{U}}$.
- **Projector Ordering:** The exact orthogonal projector $\Pi_U : \mathcal{H} \oplus \mathcal{H} \to \Gamma_U$ is explicitly given by the bounded 2×2 block operator matrix:

$$\Pi_U = \begin{pmatrix} (1 + \Delta_U)^{-1} & (1 + \Delta_U)^{-1} S_U^\dagger \\ S_U (1 + \Delta_U)^{-1} & S_U (1 + \Delta_U)^{-1} S_U^\dagger \end{pmatrix} \tag{5}$$

- **The Proof Lineage:** Subspace inclusion strictly forces the projector inequality $\Pi_U \geq \Pi_{\widetilde{U}}$. Evaluating the upper-left block element on a test state $(\psi, 0)^T$ yields:

$$\langle \psi | (1 + \Delta_U)^{-1} | \psi \rangle \geq \langle \psi | (1 + \Delta_{\widetilde{U}})^{-1} | \psi \rangle$$

- Rescaling $S \to S/\sqrt{s}$ directly proves $(s + \Delta_U)^{-1} \geq (\mathrm{s} + \Delta_{\widetilde{U}})^{-1}, yielding \Delta_{\widetilde{U}} \geq \Delta_U$.
- **Synthesis:** Combined with Chapter 2, the intense local entanglement guarantees cyclic/separating states, opening the door to modular operators, which rigorously quantify local distinguishability via monotonic Relative Entropy.