Anomalies:Fujikawa's Method and Atiyah-Singer Index Theorem

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Chiral Symmetry in Classical Theory

Background

Massless QED Model

Consider the QED action for a massless fermion:

$$S = \int \mathrm{d}^4 x \bar{\psi} \mathrm{i} \gamma^\mu D_\mu \psi$$

where $D_\mu = \partial_\mu + \mathrm{i}eA_\mu$ is the electromagnetic covariant derivative.

- Global Chiral Transformation: Since the fermion mass vanishes ($m = 0$), the action remains invariant under

$$\psi \to \mathrm{e}^{i\alpha\gamma^5} \psi, \qquad \bar{\psi} \to \bar{\psi} \mathrm{e}^{i\alpha\gamma^5}.$$

- Noether's Theorem: This classical symmetry gives rise to a conserved axial-vector current:

$$J_5^\mu = \bar{\psi}\gamma^\mu\gamma^5\psi \implies \partial_\mu J_5^\mu = 0.$$

- Physical Consequence: Classical theory predicts that the chiral charge $Q_5 = \int \mathrm{d}^3 x J_5^0$ is exactly conserved.

Quantization and Fujikawa's Key Insight

Path Integral Quantization

In quantum field theory, the complete physical information of a system is encoded in the path integral:

$$Z = \int \mathscr{D}\psi \mathscr{D}\bar{\psi}\, \mathrm{e}^{\mathrm{i}S}.$$

- Limitation of the Classical Perspective: Traditional derivations only examine whether the classical action S remains invariant under a given transformation.
- Fujikawa's Crucial Observation: A quantum theory is fundamentally composed of two ingredients:

$$Z = \underbrace{\mathscr{D}\psi \mathscr{D}\bar{\psi}}_{\text{Measure}} \times \underbrace{\mathrm{e}^{\mathrm{i}S}}_{\text{Action}}.$$

- New Criterion for Quantum Symmetry: A symmetry survives quantization only if
  1. the action is invariant (classical symmetry),
  2. the path-integral measure is invariant (quantum dynamical symmetry).

Eigenfunction Expansion of Fermionic Fields

To analyze the effect of chiral transformations on the measure, we introduce the eigenvalue equation of the Dirac operator

$$\mathrm{i}\not{D} = \mathrm{i}\gamma^\mu D_\mu :$$

$$\mathrm{i}\not{D}\phi_n = \lambda_n \phi_n.$$

- Complete Orthonormal Basis: The eigenfunctions $\{\phi_n\}$ form a complete orthonormal basis of the Hilbert space.
- Field Expansion: Expand the fermion fields and their conjugates in this basis (where a_n, b_n are Grassmann variables):

$$\psi(x) = \sum_n a_n \phi_n(x), \qquad \bar{\psi}(x) = \sum_n b_n \phi_n^\dagger(x).$$

- Rewriting the Functional Measure: The path-integral measure can then be expressed as

$$\mathscr{D}\psi\mathscr{D}\bar{\psi} = \prod_n da_n\, db_n.$$

Effect of Chiral Transformations on the Measure and the Jacobian

Consider an infinitesimal local chiral transformation:

$$\psi \to \psi' = (1 + \mathrm{i}\alpha(x)\gamma^5)\psi.$$

Substituting the mode expansion induces a mixing of basis coefficients [B]:

$$a'_n = \sum_m M_{nm} a_m, \qquad M_{nm} = \delta_{nm} + \mathrm{i}\int \mathrm{d}^4x\,\alpha(x)\phi_n^\dagger(x)\gamma^5\phi_m(x).$$

- Measure Transformation and Jacobian: The fermionic measure transforms according to the inverse square of the Jacobian:

$$\mathscr{D}\psi'\mathscr{D}\bar{\psi}' = (\det M)^{-2}\mathscr{D}\psi\mathscr{D}\bar{\psi} \implies J = (\det M)^{-2}.$$

- Using the Trace Formula: Since

$$\det M = \mathrm{e}^{\mathrm{Tr}\ln M},$$

to first order one obtains [C]

$$J = \exp\left(-2\mathrm{i}\int \mathrm{d}^4x\,\alpha(x)\sum_n \phi_n^\dagger(x)\gamma^5\phi_n(x)\right).$$

- Divergence Problem: The coincident-point sum

$$\sum_n \phi_n^\dagger \gamma^5 \phi_n$$

formally takes the indeterminate form $\infty - \infty$, and therefore requires regularization.

Heat Kernel Regularization and the ABJ Anomaly Formula

Heat Kernel Regularization

Fujikawa introduces an ultraviolet cutoff scale Λ and exponentially suppresses high-energy modes:

$$\sum_n \phi_n^\dagger \gamma^5 \phi_n \longrightarrow \lim_{\Lambda\to\infty} \sum_n \phi_n^\dagger \gamma^5 e^{-\lambda_n^2/\Lambda^2} \phi_n = \lim_{\Lambda\to\infty} \sum_n \phi_n^\dagger \gamma^5 e^{-(\mathrm{i}\not{D})^2/\Lambda^2} \phi_n .$$

- Dirac Matrix Trace: In the large-Λ asymptotic expansion, only the $1/\Lambda^4$ contribution survives after cancellation with the operator part. Using

$$\mathrm{tr}\left(\gamma^5\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\right) = 4\mathrm{i}\epsilon^{\mu\nu\rho\sigma},$$

one obtains the final result.

- Birth of the ABJ Anomaly: The non-invariance of the measure contributes to the divergence of the axial current and destroys the classical conservation law D:

$$\partial_\mu J_5^\mu = \frac{e^2}{32\pi^2}\epsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma} = \frac{e^2}{16\pi^2} F_{\mu\nu}\tilde{F}^{\mu\nu}.$$

Here $\tilde{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}$ is the dual electromagnetic field-strength tensor.

Atiyah-Singer Index Theorem

Atiyah–Singer Index Theorem

$$\operatorname{ind} D_E = \int_M \widehat{A}(TM)\,\mathrm{ch}(E)$$

where

$$\widehat{A}(TM) = 1 - \frac{p_1(TM)}{24} + \cdots, \mathrm{ch}(E) = \mathrm{Tr}\exp\left(\frac{\mathrm{i}F}{2\pi}\right)$$

In ABJ-anomaly, E is a $\mathrm{U}(1)$-line bundle, hence $\mathrm{rk}(L) = 1$, thus

$$\mathrm{ch}(E) = 1 + \frac{\mathrm{i}F}{2\pi} + \frac{1}{2}\left(\frac{\mathrm{i}F}{2\pi}\right)^2 + \cdots$$

Atiyah-Singer Index Theorem

In flat spacetime, $\widehat{A}(TM) = 1$, thus

$$\begin{aligned} \mathrm{ind}(D) &= \int_M \mathrm{ch}(E)_{(4)} \\ &= -\frac{1}{8\pi^2}\int_M F \wedge F = -\frac{1}{32\pi^2}\int_M \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma} \mathrm{d}^4 x \end{aligned}$$

Two different conventions:

- Mathematical: $\nabla = \mathrm{d} + A$.
- Physical: $D_\mu = \partial_\mu + \mathrm{i}eA_\mu$.

Hence

$$\mathrm{ind}(D) = \frac{e^2}{32\pi^2}\int_M \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma} \mathrm{d}^4 x$$

Compared with the result of Fujikawa's, we obtain exactly

$$\boxed{\text{ABJ anomaly} = \text{local index density}}$$

Generalization

We have just obtain $\operatorname{ind} D$ in 4D flat spacetime, by the method of Atiyah-Singer index theorem, it can be easily generalized to curved spacetime and higher dimensional case.
In curved spacetime, 4-form

$$[\widehat{A}(TM)\,\mathrm{ch}(L)]_{(4)} = \frac{e^2}{8\pi^2} F \wedge F - \frac{1}{24} p_1(TM)$$
$$\Rightarrow \partial_\mu J_5^\mu = \frac{e^2}{16\pi^2} F_{\mu\nu} \widetilde{F}^{\mu\nu} + \frac{1}{384\pi^2} R_{\mu\nu\rho\sigma} \widetilde{R}^{\mu\nu\rho\sigma}$$

where $\widetilde{R}^{\mu\nu\rho\sigma} = \frac{1}{2} \epsilon^{\mu\nu\alpha\beta} R_{\alpha\beta}{}^{\rho\sigma}$.
In higher dimension, for example, $\dim M = 8$, then

$$(\widehat{A}(TM)\,\mathrm{ch}(E))_{(8)} = \frac{1}{5760}(7p_1^2 - 4p_2) - \frac{1}{48} p_1 (c_1^2 - 2c_2)$$
$$+ \frac{1}{24}(c_1^4 - 4c_1^2 c_2 + 2c_2^2 + 4c_1 c_3 - 4c_4)$$

Physical Significance of the Chiral Anomaly

Classical Picture: Exact Conservation

$$Q_5 = N_R - N_L.$$

The difference between the numbers of right-handed and left-handed fermions remains unchanged during time evolution.

Quantum Picture: Vacuum Pumping Effect

$$\frac{dQ_5}{dt} = \int d^3x \frac{e^2}{16\pi^2} F_{\mu\nu}\tilde{F}^{\mu\nu}.$$

- Role of Parallel Electric and Magnetic Fields: Since

  $$F_{\mu\nu}\tilde{F}^{\mu\nu} = -4\vec{E}\cdot\vec{B},$$

  the above equation becomes

  $$\frac{dQ_5}{dt} = -\frac{e^2}{4\pi^2}\int d^3x\, \vec{E}\cdot\vec{B}.$$

- Physical Interpretation: In the presence of parallel electric and magnetic fields, the external field continuously pumps states from left-handed modes into right-handed modes through quantum fluctuations of the Dirac sea.
- Conclusion: This quantum-topological effect implies that the chiral charge difference $N_R - N_L$ is no longer conserved.

Summary

- Classical massless QED has exact global chiral symmetry: $\partial_\mu J_5^\mu = 0, \quad Q_5 = N_R - N_L$.
- Quantum path integral introduces a subtlety: measure is not invariant under local chiral transformations.
- Fujikawa method:

$$\mathscr{D}\psi\mathscr{D}\bar{\psi} \to \mathscr{D}\psi'\mathscr{D}\bar{\psi}' = (\det M)^{-2}\mathscr{D}\psi\mathscr{D}\bar{\psi},$$

$$\det M = e^{\mathrm{Tr}\ln M}, \quad J = \exp\left(-2i\int d^4x \sum_n \phi_n^\dagger \gamma^5 \phi_n \alpha(x)\right)$$

- Heat kernel regularization resolves $\infty - \infty$ divergence:

$$\partial_\mu J_5^\mu = \frac{e^2}{32\pi^2}\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma} = \frac{e^2}{16\pi^2}F_{\mu\nu}\tilde{F}^{\mu\nu}.$$

- Geometric view via Atiyah-Singer index theorem: $\mathrm{ind}(D_E) = \int_M \widehat{A}(TM)\,\mathrm{ch}(E)$.
- Physical consequence: ABJ anomaly allows $N_R - N_L$ to change in parallel E and B fields.
- Takeaway: Path integral measure + topology $\to$ quantum anomalies; classical symmetries may be broken by quantum effects.

Appendix: Background

Back

In the action

$$S = \int \mathrm{d}^4 x \bar{\psi} \mathrm{i} \gamma^\mu D_\mu \psi$$

where ψ is the Dirac spinor field, and $\bar{\psi}$, defined by $\bar{\psi} = \psi^\dagger \gamma^0$, is the Dirac adjoint. γ^μ are Dirac gamma matrices which admit a structure of Clifford algebra: $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu} I$. And $\gamma^5 := \mathrm{i}\gamma^0\gamma^1\gamma^2\gamma^3$.

Noether theorem:

$$\begin{aligned} \delta\psi = \mathrm{i}\alpha(x)\gamma^5\psi, \delta\bar{\psi} = \mathrm{i}\alpha(x)\bar{\psi}\gamma^5, \\ \delta\mathscr{L} = -(\partial_\mu \alpha)\bar{\psi}\gamma^\mu\gamma^5\psi, \\ J_5^\mu = \bar{\psi}\gamma^\mu\gamma^5\psi, \partial_\mu J_5^\mu = 0 \end{aligned}$$

The matrix γ^5 distinguishes left- and right-handed components of a Dirac spinor. Consequently, the conserved axial charge$Q_5 = \int d^3x J_5^0$ measures the difference$Q_5 = N_R - N_L$. Therefore, classical chiral symmetry implies that the net conversion between left-handed and right-handed fermions is forbidden.

B

◂ Back

$$\psi'(x) = (1 + \mathrm{i}\alpha(x)\gamma^5)\psi(x)$$
$$= (1 + \mathrm{i}\alpha(x)\gamma^5)\sum_m a_m\phi_m(x)$$
$$= \sum_m a_m\phi_m(x) + \mathrm{i}\sum_m a_m\alpha(x)\gamma^5\phi_m(x)$$

$$\gamma^5\phi_m(x) = \langle x|\gamma^5|\phi_m\rangle$$
$$= \sum_n \langle x|\phi_n\rangle\langle\phi_n|\gamma^5|\phi_m\rangle$$
$$= \sum_n \phi_n(x)\int \mathrm{d}^4y\,\phi_n^\dagger(y)\gamma^5\phi_m(y)$$

$$\psi'(x) = \sum_n \phi_n(x)[a_n + \mathrm{i}\sum_m a_m \int \mathrm{d}^4y\,\alpha(y)\phi^\dagger(y)\gamma^5\phi_m(y)]$$

$$a_n' = a_n + \mathrm{i}\sum_m a_m \int \mathrm{d}^4y\,\alpha(y)\phi^\dagger(y)\gamma^5\phi_m(y)$$
$$= \sum_m M_{nm}a_m$$

$$M_{nm} = \delta_{nm} + \mathrm{i}\int \mathrm{d}^4x\,\alpha(x)\phi^\dagger(x)\gamma^5\phi_m(x)$$

Back

$$
\begin{aligned}
J &= (\det M)^{-2} \Rightarrow \ln J = -2 \ln \det M \\
\det M &= \mathrm{e}^{\mathrm{Tr} \ln M} \Rightarrow \ln J = -2 \, \mathrm{Tr}(\ln M) \\
M &= 1 + \epsilon, \text{where} \\
\epsilon_{mn} &= \mathrm{i} \int \mathrm{d}^4 x \alpha(x) \phi_n^\dagger(x) \gamma^5 \phi_m(x) \\
\ln(M) &= \ln(1+\epsilon) \approx \epsilon \\
J &= \exp\left(-2 \, \mathrm{Tr} \, \epsilon\right) \\
&= \exp\left(-2\mathrm{i} \int \mathrm{d}^4 x \alpha(x) \sum_n \phi_n^\dagger(x) \gamma^5 \phi_n(x)\right)
\end{aligned}
$$

D

Back

$$\mathscr{A}(x) = \lim_{\Lambda\to\infty} \mathrm{tr}\left\langle x \middle| \gamma^5 \mathrm{e}^{-(\mathrm{i}\not{D})^2/\Lambda^2} \middle| x \right\rangle$$
$$= \frac{\mathrm{e}^2}{32\pi^2}\epsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma},$$

$$\partial_\mu J_5^\mu = \frac{\mathrm{e}^2}{32\pi^2}\epsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma} = \frac{\mathrm{e}^2}{16\pi^2} F_{\mu\nu} \tilde{F}^{\mu\nu}.$$