Physics at High Baryon Density (PHD2026), Dalian

Generative Modeling of Critical Phenomena

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and **Gert Aarts**, **Diaa E. Habibi**, **Biagio Lucini** [arXiv:2607.08505],
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November 13, 2026

DM-QFT

dm-qft.github.io/homepage

Outline

Critical phenomena and universality

Part I: diffusion models and equilibrium sampling

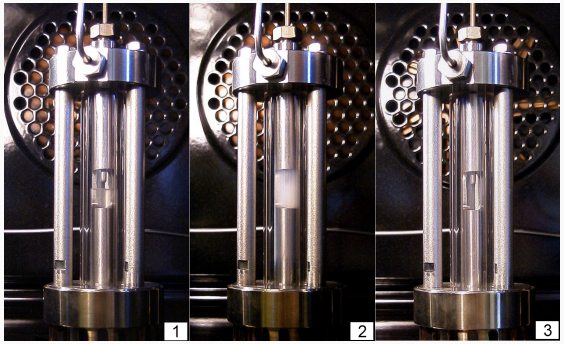
- Diffusion model on the lattice
- Critical scaling and volume extrapolation
- Towards QCD sampling

Part II: real-time critical dynamics

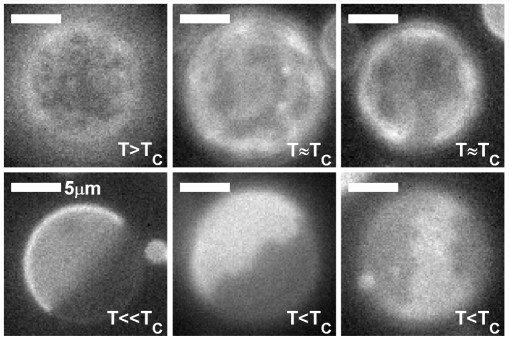
- Conserved dynamics on the lattice
- Dynamic critical scaling
- Phase separation
- Colloid experiment
- Outlook

Critical phenomena and universality

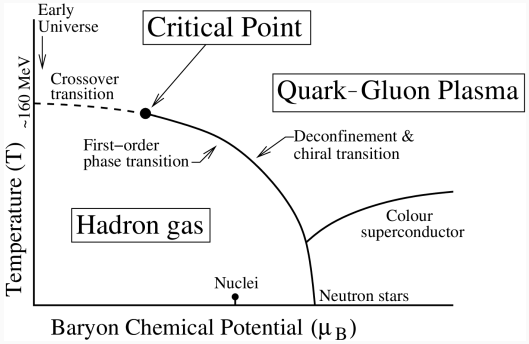
Critical phenomena across physics



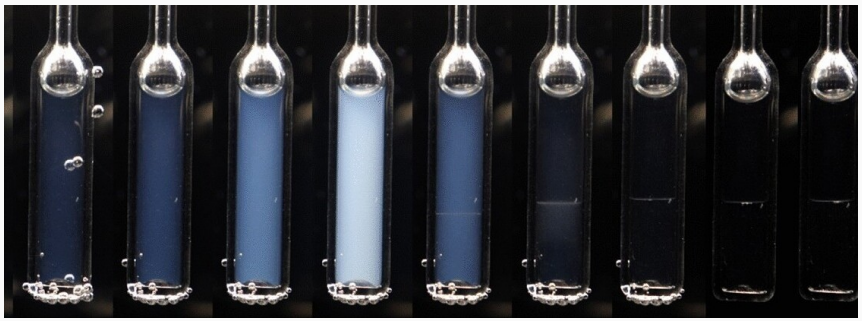
liquid–gas: ethane at T_c



cell membranes, $d = 2$



QCD: critical end point



binary mixture 3-picoline–D_2O at T_c

Universal features

no characteristic length, $\xi \to \infty$

fluctuations on all scales

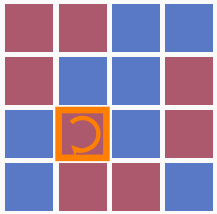
power-law singularities at T_c

$\chi \sim |T - T_c|^{-\gamma}$, $\xi \sim |T - T_c|^{-\nu}$

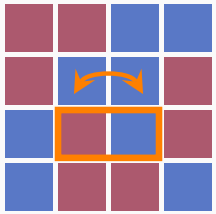
Ethane: Horstmann, Wikimedia Commons. **Membranes:** Gray, Karslake, Machta, Veatch, Biophys. J. 105, 2751 (2013). **Mixture:** Kalabiński, Drozd-Rzoska, Rzoska, EPJ E 45, 73 (2022). **QCD:** Bhalerao, doi:10.5170/CERN-2014-001.219; Stephanov, Rajagopal, Shuryak, PRL 81, 4816 (1998).

Dynamic universality

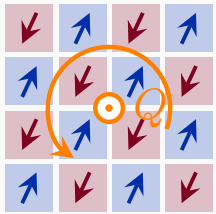
	conservation laws	examples	z
Model A	nonconserved ϕ	magnets, order–disorder alloys	$\simeq 2.0$
Model B	conserved ϕ	alloys, diffusive mixtures	$4-\eta$
Model G	nonconserved ϕ + conserved charges	antiferromagnets, chiral transition	$d/2$
Model H	conserved ϕ + conserved momentum	fluids, binary liquids, QCD end point	$\simeq 3.0$



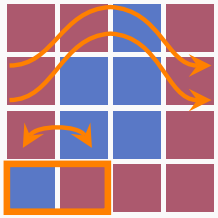
A: flip a site



B: swap a pair



G: precession with conserved Q



H: diffusion + advection

Same statics, different dynamics

the equilibrium weight e^{-S} is identical

relaxation time $\tau \sim \xi^z$, $\omega(k) \sim k^z$

conservation laws and mode couplings determine z

Model B: conserved ϕ, $z = 4 - \eta$

a local Monte Carlo update is Model A

critical slowing down, $\tau \propto \xi^{z \simeq 2}$

Classification: Hohenberg, Halperin, Rev. Mod. Phys. 49, 435 (1977). **Model B exponent:** Halperin, Hohenberg, Ma, PRB 10, 139 (1974). **Model G and chiral QCD:** Rajagopal, Wilczek, Nucl. Phys. B 399, 395 (1993). **Model H and QCD:** Son, Stephanov, PRD 70, 056001 (2004).

Part I: diffusion models and equilibrium sampling

Diffusion models generate images, video and text

Images and video DALL-E 3, Stable Diffusion, Sora

add noise step by step, the images dissolve



learn to undo each step, new images emerge from noise

Text Gemini Diffusion, Mercury

learn to fill them in

			Physics				
			Physics			Baryon	
		Workshop	Physics			Baryon	Density
		Workshop	Physics		High	Baryon	Density
		Workshop	on Physics		High	Baryon	Density
	International	Workshop	on Physics		High	Baryon	Density
	International	Workshop	on Physics	at	High	Baryon	Density

mask word by word

Lattice field theory

a configuration is an image

one pixel per site, value ϕ_x

known target $p(\phi) \propto e^{-S[\phi]}$

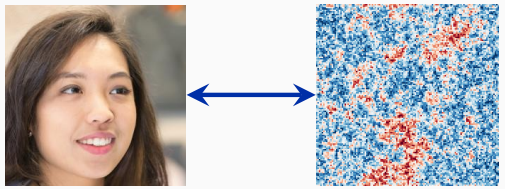


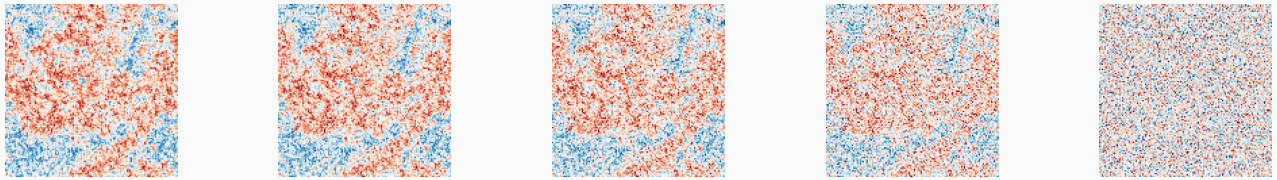
image configuration

Diffusion models: Sohl-Dickstein et al., ICML (2015). **Faces:** Song et al., ICLR (2021).
Stable Diffusion: Rombach et al., CVPR (2022). **Text:** Lou, Meng, Ermon, ICML (2024).

Forward and reverse diffusion

Generative model learn $p(\phi)$ from its samples, then draw new ones

forward: add Gaussian noise until nothing of ϕ is left



reverse: a Langevin equation with a learned force turns noise back into a sample

Forward $\phi_t = \phi_0 + \Sigma(t)\,\eta$, a heat equation for p_t, nothing to learn

$$\mathrm{d}\phi = \sigma^t\,\mathrm{d}w, \qquad \Sigma^2(t) = \int_0^t \sigma^{2s}\,\mathrm{d}s = \frac{\sigma^{2t}-1}{2\ln\sigma}, \qquad \Sigma(1) \gtrsim \text{data spread}$$

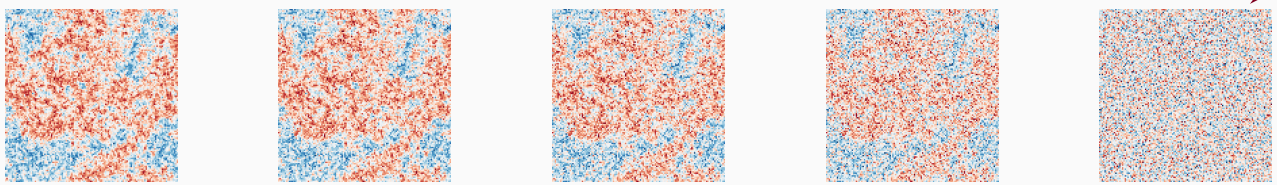
so $p_{t=1}$ is a broad Gaussian, the same whatever ϕ_0 was

Reverse-time diffusion: Anderson, Stoch. Proc. Appl. 12, 313 (1982). **Denoising diffusion:** Ho, Jain, Abbeel, NeurIPS (2020). **Score SDE:** Song et al., ICLR (2021). **DSM:** Vincent, Neural Comput. 23, 1661 (2011).

Forward and reverse diffusion

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Training approximate the drift by a network

$$s_\theta(\phi_t, t) \approx \nabla_\phi \log p_t(\phi_t)$$

it points from the noised ϕ_t back to the data ϕ_0, $-\eta/\Sigma(t)$

$$\mathcal{L}(\theta) = \left\langle \left\| \Sigma(t)\, s_\theta(\phi_t,\, t) + \eta \right\|^2 \right\rangle_{t,\phi_0,\eta}$$

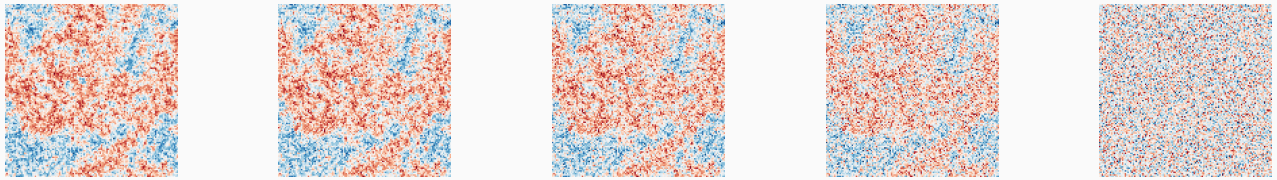
the network learns by predicting the noise it was given

Reverse-time diffusion: Anderson, Stoch. Proc. Appl. 12, 313 (1982). **Denoising diffusion:** Ho, Jain, Abbeel, NeurIPS (2020). **Score SDE:** Song et al., ICLR (2021). **DSM:** Vincent, Neural Comput. 23, 1661 (2011).

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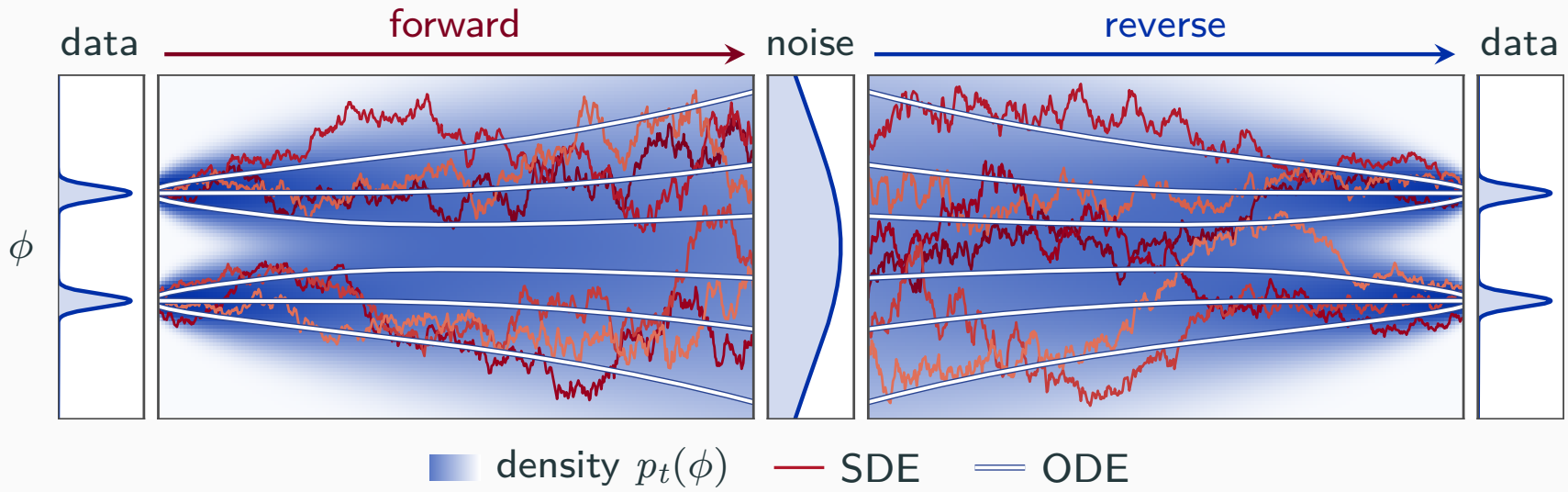
Reverse the same diffusion run backwards, with the learned drift

$$\mathrm{d}\phi = -g^2(t)\, s_\theta(\phi, t)\, \mathrm{d}t + g(t)\, \mathrm{d}w, \qquad \frac{\mathrm{d}\phi}{\mathrm{d}t} = -\tfrac{1}{2} g^2(t)\, s_\theta(\phi, t)$$

SDE and ODE share the marginals p_t, $g(t) = \sigma^t$

Reverse-time diffusion: Anderson, Stoch. Proc. Appl. 12, 313 (1982). **Denoising diffusion:** Ho, Jain, Abbeel, NeurIPS (2020). **Score SDE:** Song et al., ICLR (2021). **DSM:** Vincent, Neural Comput. 23, 1661 (2011).

Forward and reverse diffusion



Irreversible and reversible

irreversible: each noisy trajectory (SDE)

reversible: probability distribution (ODE)

Small-time limit the drift of stochastic quantization: is it reproduced?

$$s_\theta(\phi, t \to 0) \;\to\; \nabla_\phi \log p_{\text{data}} = -\nabla_\phi S[\phi]$$

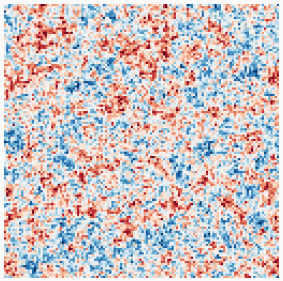
Figure after: Song et al., ICLR (2021), Fig. 2. **Stochastic quantization:** Parisi, Wu, Sci. Sin. 24, 483 (1981). **DM as stochastic quantization:** Wang, Aarts, Zhou, JHEP 05, 060 (2024); Fukushima, Kamata, JPSJ 94, 031010 (2025); Hirono, Tanaka, Fukushima, arXiv:2403.11262.

Lattice ϕ^4 theory

$$S[\phi] = \sum_x \Big[-2\kappa \sum_{\mu=1}^{d} \phi_x \phi_{x+\hat{\mu}} + \phi_x^2 + \lambda\big(\phi_x^2 - 1\big)^2 \Big]$$

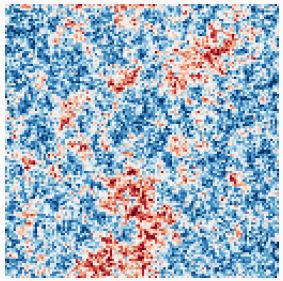
periodic L^d lattice, symmetric under $\phi \to -\phi$

$\lambda \to \infty$ leaves Ising spins $\phi_x = \pm 1$, $\lambda = 0$ a Gaussian field



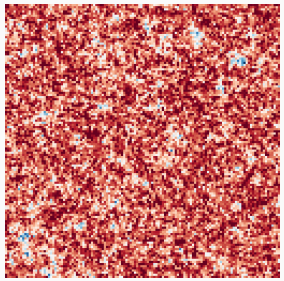
symmetric

$\kappa = 0.26 < \kappa_c$



critical

$\kappa = 0.2705 \simeq \kappa_c$

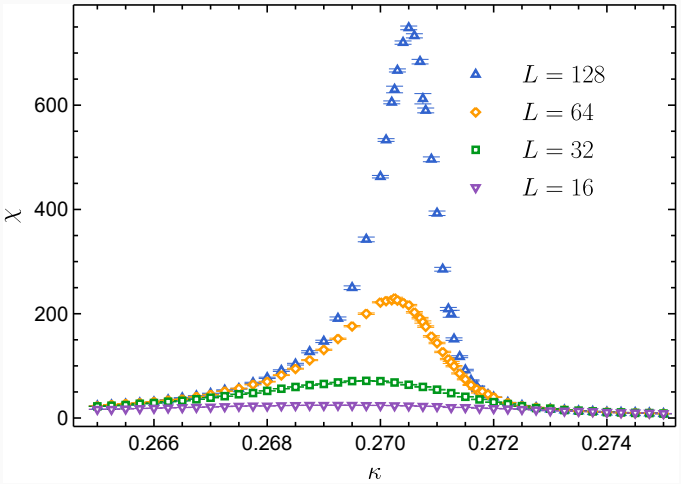


broken

$\kappa = 0.28 > \kappa_c$

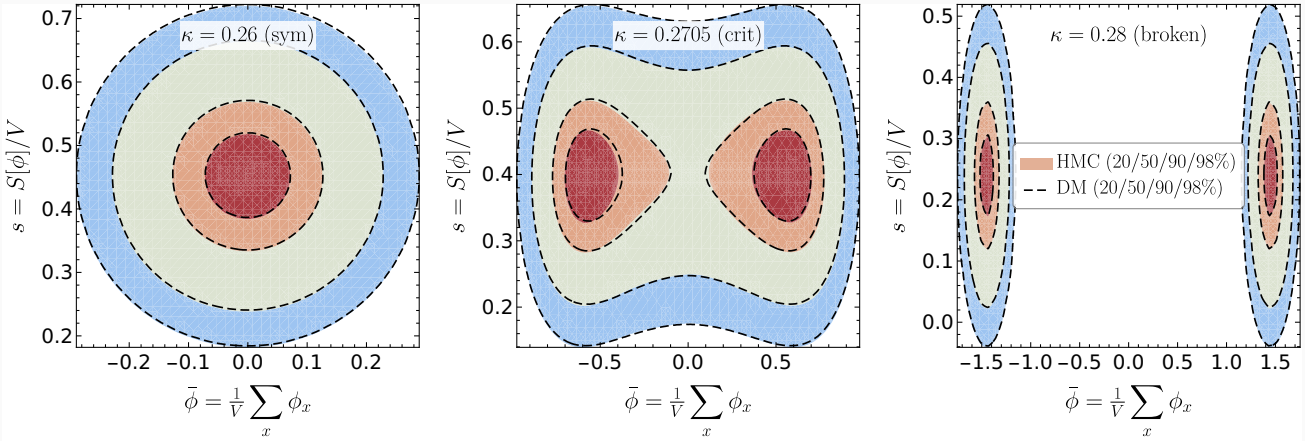
Couplings

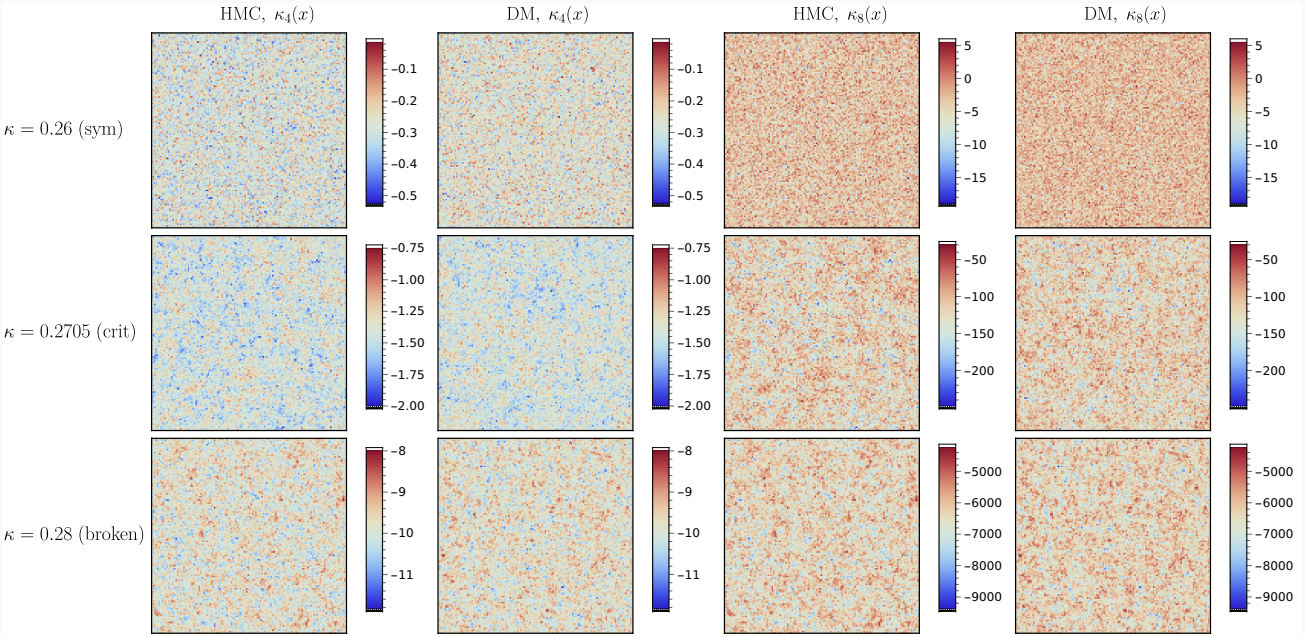
	λ	κ_c
2D	0.022	0.27088
3D	0.9	0.19225



2D, $\lambda = 0.022$

Comparison with HMC





2D, $L = 128$

HMC filled

DM dashed

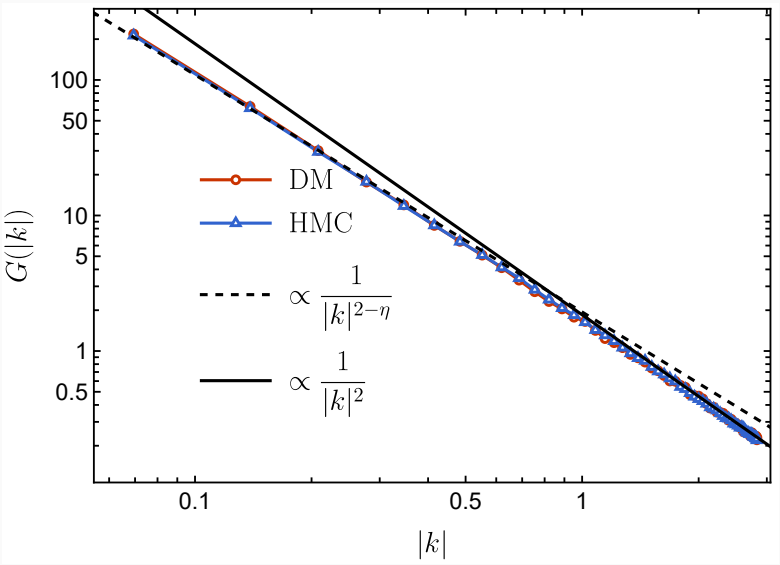
Cumulants

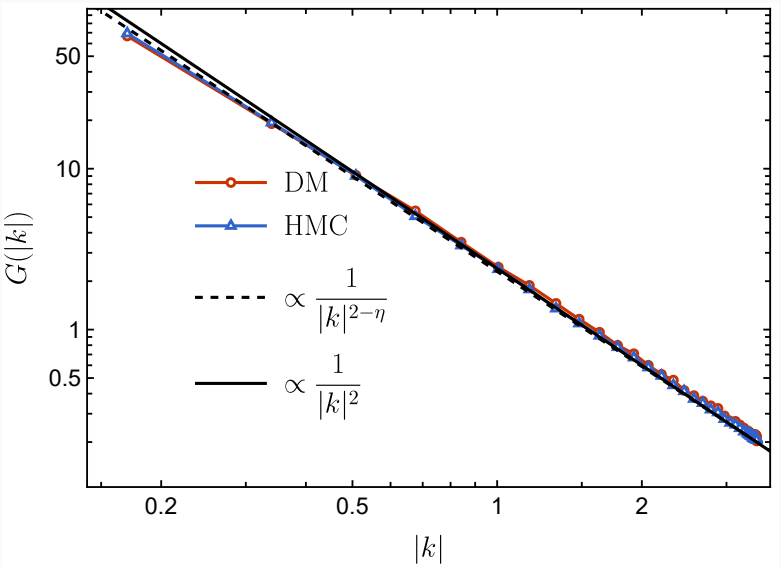
$\kappa_n(x) = \langle \phi_x^n \rangle_c$

at each site x

same colour scale

Correlation function at the critical point





2D, $L = 128$, $\kappa = 0.2705$, $\eta = 1/4$

3D, $L = 64$, $\kappa = 0.1923$, $\eta \simeq 0.036$

Critical scaling

$G(k) \propto k^{-(2-\eta)}$

$\eta = 0$ for a free field

2D Ising

$\eta = 1/4$ resolved

off the free $1/k^2$

3D Ising

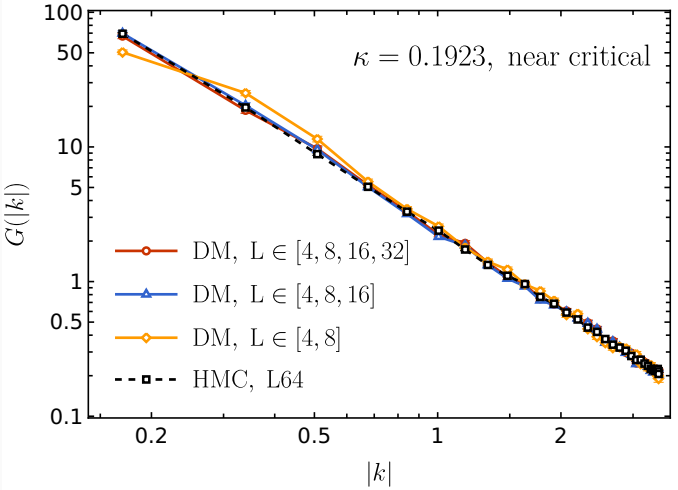
$\eta \simeq 0.036$, lines close

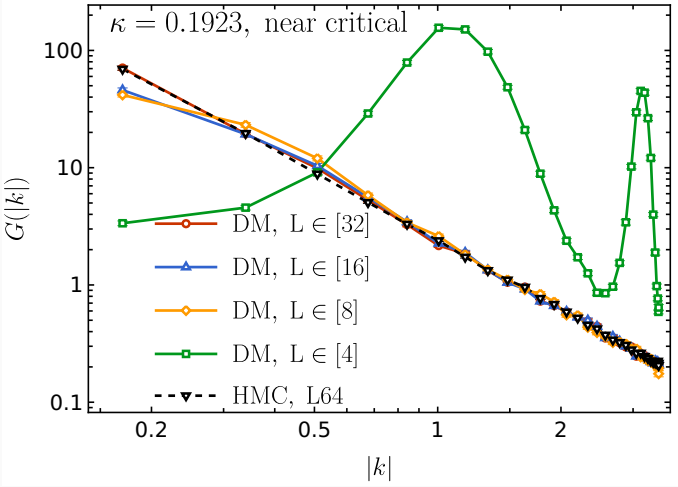
DM and HMC coincide

the interacting fixed point leaves its fingerprint in η, and the samples carry it

Exponents: Pelissetto, Vicari, Phys. Rept. 368, 549 (2002); Hasenbusch, J. Phys. A 32, 4851 (1999). Converged models, trained and sampled at the same L.

Train on small lattices, sample a lattice never seen





multi-L training $\rightarrow L = 64$

3D, $\kappa = 0.1923$: $L_{\text{train}} \leq 32 \implies L = 64$

$\{4, 8, 16, 32\}$ tracks HMC down to $k_{\min}$

$\{4, 8\}$ misses the infrared rise

single L: 32 closest, 4 fails

several volumes teach the infrared

single-L baselines $\rightarrow L = 64$

training L	acceptance
$\{4, 8\}$	$0.578(8)$
$\{4, 8, 16\}$	$0.830(6)$
$\{4, 8, 16, 32\}$	$0.877(6)$
$\{64\}$, in distribution	$0.911(9)$

Multi-lattice generative models: Máté, Fleuret, MLST 5, 045053 (2024); Bauer, Kapust, Pawlowski, Temmen, SciPost Phys. 19, 077 (2025); Singha et al., ICLR (2025); Zhu et al., JHEP 03, 111 (2026). **Cost of large-volume training:** Abbott et al., EPJA 59, 257 (2023). No $L = 64$ information enters training or sampling.

Towards QCD sampling

SU(N) gauge theory

$$U \to e^{\mathrm{i}\,(g^2 s_\theta^a \Delta t + g\sqrt{\Delta t}\,\eta^a)\,T^a}\,U$$

diffuse on the SU(N) group

Fermions Yukawa, Gross–Neveu

$$p(\phi) \propto \det M[\phi]\; e^{-S[\phi]}$$

nonlocal, generation without M^{-1}

Schwinger model gauge links with dynamical fermions

$$p(U) \propto \det M[U]\; e^{-S_g[U]}, \quad \text{U(1) in 2D}$$

QCD-like theories nonlinear sigma model, CP^{N-1}

asymptotic freedom, mass gap, topology

Full QCD gauge links with dynamical quarks

the same weight, SU(3) in 4D

Diffusion on manifolds: De Bortoli et al., NeurIPS (2022). **Diffusion for gauge fields:** Aarts et al., arXiv:2601.19552. **Yukawa model:** Albergo et al., PRD 104, 114507 (2021).

Towards QCD sampling

SU(N) gauge theory

$U \to e^{\mathrm{i}\,(g^2 s_\theta^a \Delta t + g\sqrt{\Delta t}\,\eta^a)\,T^a}\,U$

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nonlocal, generation without M^{-1}

The sign problem

$\mu_B \neq 0, \quad \det M[U]$ is complex

Monte Carlo sampling fails

Physics-informed kernels

Jan Pawlowski, Friederike Ihssen and Renzo Kapust

arXiv:2510.26678, arXiv:2603.03159

Full QCD gauge links with dynamical quarks

the same weight, SU(3) in 4D

Part II: real-time critical dynamics

From statics to dynamics

Statics

Boltzmann weight $e^{-S[\phi]}$

however it was reached

no time

Dynamics

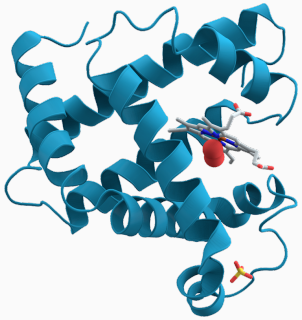
equation of motion $\partial_t \phi = \mathcal{F}[\phi, \xi]$

stochastic trajectories $\phi(t)$

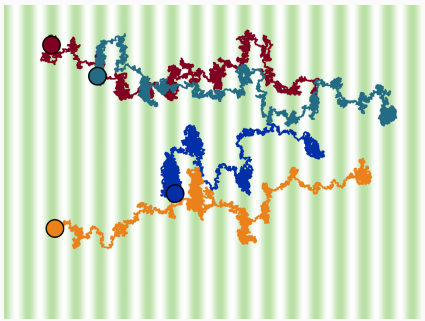
$\mathcal{F}$ does not depend on t



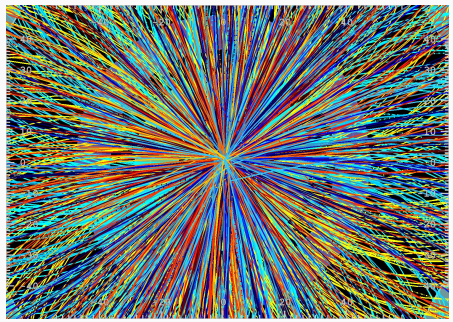
weather



protein dynamics



driven colloids



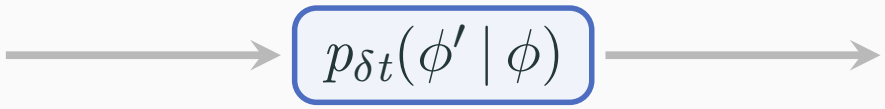
heavy-ion collision

Hurricane Isabel: NASA. **Myoglobin:** Wikimedia Commons. **ALICE event display:** CERN. **Colloids:** Dieball et al., Soft Matter 21, 2170 (2025). **Learned kernels:** weather, Price et al., Nature 637, 84 (2025). Proteins, Schreiner, Winther, Olsson, NeurIPS (2023).

From statics to dynamics

Transition kernel the probability to go from ϕ to ϕ' in a time δt

equation of motion
$\delta t \to 0$



statics $e^{-S[\phi]}$
$\delta t \to \infty$

From statics to dynamics

Transition kernel the probability to go from ϕ to ϕ' in a time δt

equation of motion $\delta t \to 0$ $\longrightarrow$ $p_{\delta t}(\phi' \mid \phi)$ $\longrightarrow$ statics $e^{-S[\phi]}$ $\delta t \to \infty$

$$\phi(t) \xrightarrow{p_{\delta t}} \phi(t+\delta t) \xrightarrow{p_{\delta t}} \phi(t+2\delta t) \xrightarrow{p_{\delta t}} \cdots$$

Chapman–Kolmogorov: $p_{2\delta t}(\phi'|\phi) = \int \mathrm{d}\phi''\, p_{\delta t}(\phi'|\phi'')\, p_{\delta t}(\phi''|\phi)$. Colored noise: enlarge the state, $X = (\phi, \eta)$, and the joint kernel composes again.

From statics to dynamics

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$$\phi(t) \xrightarrow{p_{\delta t}} \phi(t+\delta t) \xrightarrow{p_{\delta t}} \phi(t+2\delta t) \xrightarrow{p_{\delta t}} \cdots$$

Learned transition kernel

pairs $(\phi_t, \phi_{t+\delta t})$ suffice, from a simulation or an experiment

each kernel step spans δt, far beyond the step of a direct integrator

Chapman–Kolmogorov: $p_{2\delta t}(\phi'|\phi) = \int \mathrm{d}\phi''\, p_{\delta t}(\phi'|\phi'')\, p_{\delta t}(\phi''|\phi)$. Colored noise: enlarge the state, $X = (\phi, \eta)$, and the joint kernel composes again.

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An effective theory in real time

$$e^{-S_{\delta t}[\phi_<]} = \int \mathcal{D}\phi_> \; e^{-S_{\rm dyn}[\phi_< + \phi_>]}, \qquad \phi_> = \phi\big|_{\omega > 1/\delta t}$$

$$\omega_k \propto k^z > 1/\delta t \;\Leftrightarrow\; k > k_{\delta t} \propto \delta t^{-1/z}$$

Chapman–Kolmogorov: $p_{2\delta t}(\phi'|\phi) = \int \mathrm{d}\phi''\, p_{\delta t}(\phi'|\phi'')\, p_{\delta t}(\phi''|\phi)$. Colored noise: enlarge the state, $X = (\phi, \eta)$, and the joint kernel composes again.

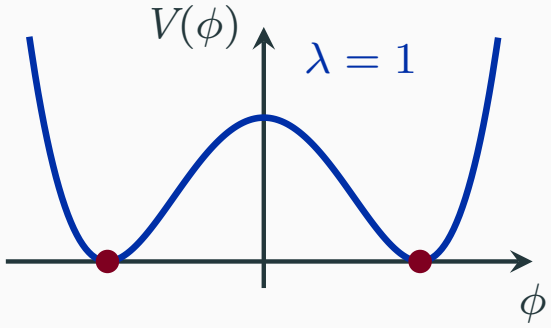
Model B: conserved dynamics

2D conserved ϕ^4 field

$$\partial_t \phi(\boldsymbol{x}, t) = \Gamma \nabla^2 \frac{\delta S[\phi]}{\delta \phi(\boldsymbol{x}, t)} + \zeta(\boldsymbol{x}, t),$$

$$\langle \zeta(\boldsymbol{x}, t) \zeta(\boldsymbol{x}', t') \rangle = -2\Gamma \nabla^2 \delta^{(d)}(\boldsymbol{x} - \boldsymbol{x}') \delta(t - t'),$$

$S[\phi]$ as in Part I, now at $\lambda = 1$.



Model B: Hohenberg, Halperin, Rev. Mod. Phys. 49, 435 (1977). Reference trajectories from a direct stochastic integrator (SOSRA) of these equations, on the lattice action of Part I. **Kernel:** $-\Gamma\nabla^2$ in front of $\delta S/\delta\phi$ leaves the equilibrium e^{-S} unchanged and sets the dynamics. **Model B for the QCD critical point:** Nahrgang et al., PRD 99, 116015 (2019); Schweitzer, Schlichting, von Smekal, NPB 984, 115944 (2022); Chattopadhyay et al., PRD 108, 074004 (2023).

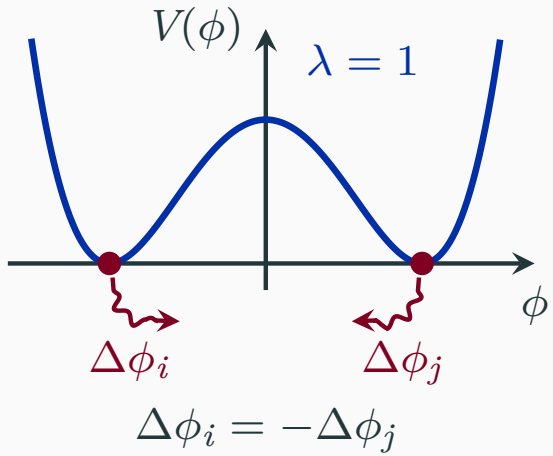
Model B: conserved dynamics

2D conserved ϕ^4 field

$$\partial_t \phi(\boldsymbol{x}, t) = \Gamma \nabla^2 \frac{\delta S[\phi]}{\delta \phi(\boldsymbol{x}, t)} + \zeta(\boldsymbol{x}, t),$$

$$\langle \zeta(\boldsymbol{x}, t) \zeta(\boldsymbol{x}', t') \rangle = -2\Gamma \nabla^2 \delta^{(d)}(\boldsymbol{x} - \boldsymbol{x}') \delta(t - t'),$$

$S[\phi]$ as in Part I, now at $\lambda = 1$.



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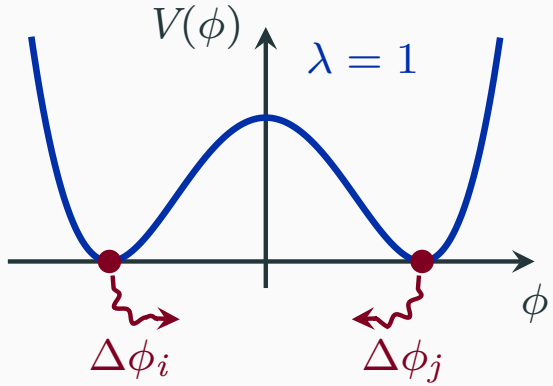
Model B: conserved dynamics

2D conserved ϕ^4 field

$$\partial_t \phi(\boldsymbol{x}, t) = \Gamma \nabla^2 \frac{\delta S[\phi]}{\delta \phi(\boldsymbol{x}, t)} + \zeta(\boldsymbol{x}, t),$$

$$\langle \zeta(\boldsymbol{x}, t) \zeta(\boldsymbol{x}', t') \rangle = -2\Gamma \nabla^2 \delta^{(d)}(\boldsymbol{x} - \boldsymbol{x}') \delta(t - t'),$$

$S[\phi]$ as in Part I, now at $\lambda = 1$.



$$\Delta\phi_i = -\Delta\phi_j$$

Conserved mean

$$M(t) \equiv V^{-1} \int d^d x \, \phi(\boldsymbol{x}, t)$$

$$M(t) = M(0)$$

Critical dynamics

$$\tau \propto \xi^z \propto k^{-z}$$

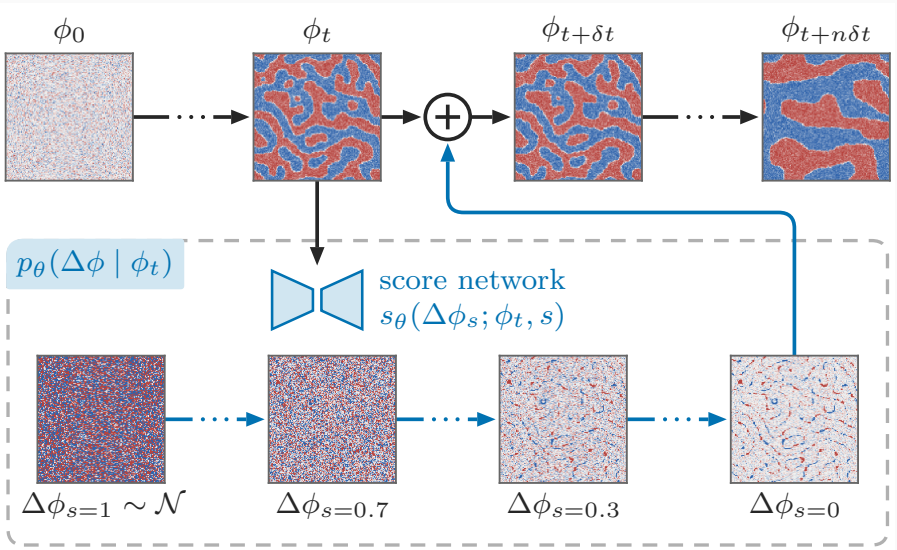
$$z = 4 - \eta$$

Equilibrium

$$P_{\mathrm{eq}}[\phi] \propto e^{-S[\phi]}$$

Model B: Hohenberg, Halperin, Rev. Mod. Phys. 49, 435 (1977). Reference trajectories from a direct stochastic integrator (SOSRA) of these equations, on the lattice action of Part I. **Kernel:** $-\Gamma\nabla^2$ in front of $\delta S/\delta\phi$ leaves the equilibrium e^{-S} unchanged and sets the dynamics. **Model B for the QCD critical point:** Nahrgang et al., PRD 99, 116015 (2019); Schweitzer, Schlichting, von Smekal, NPB 984, 115944 (2022); Chattopadhyay et al., PRD 108, 074004 (2023).

Conditional diffusion as the transition kernel



$$\Delta\phi \sim p_\theta(\,\cdot \mid \phi_t)$$
$$\phi_{t+\delta t} = \phi_t + \Delta\phi$$

Condition

the present field ϕ_t

Target

the increment $\Delta\phi$, diffused and denoised as in Part I

Physics learned

$p(\Delta\phi \mid \phi_t)$ from pairs alone

no drift, noise or free energy

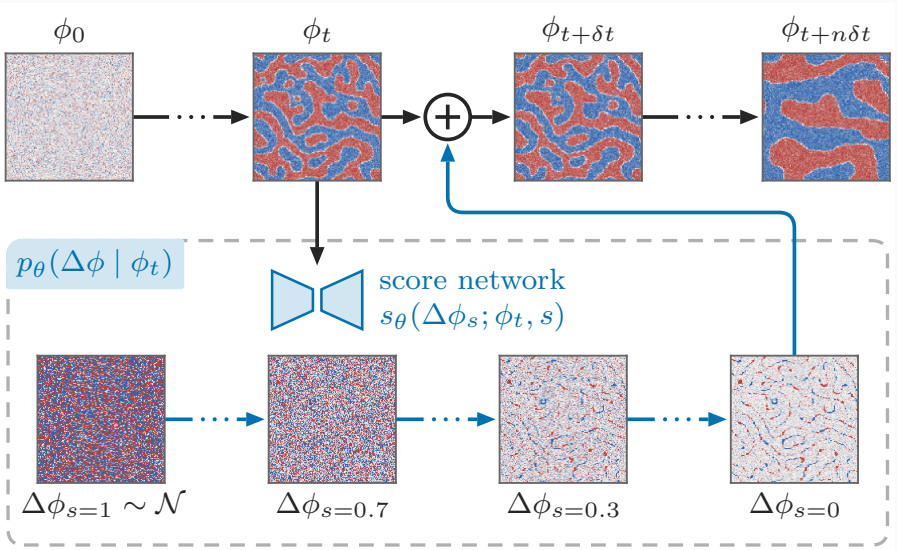
Physics built in

$\sum_x \Delta\phi_x = 0$ by projection

M conserved exactly

Same U-Net as in Part I with ϕ_t as a second input channel.

Conditional diffusion as the transition kernel



$$\Delta\phi \sim p_\theta(\,\cdot \mid \phi_t)$$
$$\phi_{t+\delta t} = \phi_t + \Delta\phi$$

Condition

the present field ϕ_t

Target

the increment $\Delta\phi$, diffused and denoised as in Part I

Physics learned

$p(\Delta\phi \,|\, \phi_t)$ from pairs alone

no drift, noise or free energy

Physics built in

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M conserved exactly

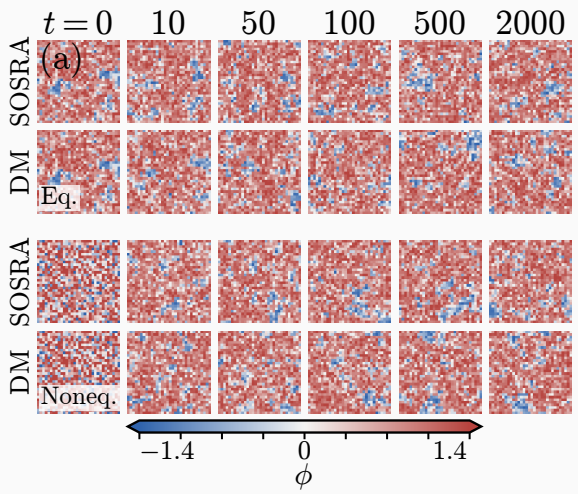
Projection, three times $P\,v = v - \frac{1}{V}\sum_x v_x$, no retraining

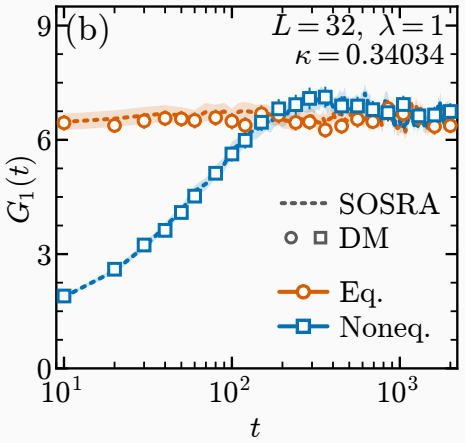
$$\mathrm{d}\Delta\phi_s = -g^2(s)\, P\, s_\theta(\Delta\phi_s; \phi_t, s)\,\mathrm{d}s + g(s)\, P\,\mathrm{d}\bar{w}, \qquad \Delta\phi_{s=1} = P\,\Sigma(1)\,\eta$$

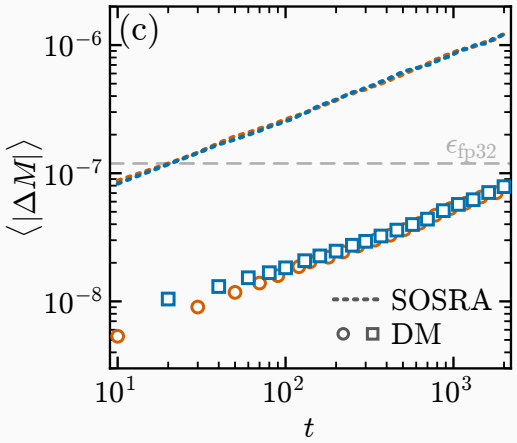
initial noise, score, noise increments: $\sum_x \Delta\phi_x = 0$ exactly

Same U-Net as in Part I with ϕ_t as a second input channel.

Equilibrium and nonequilibrium evolution





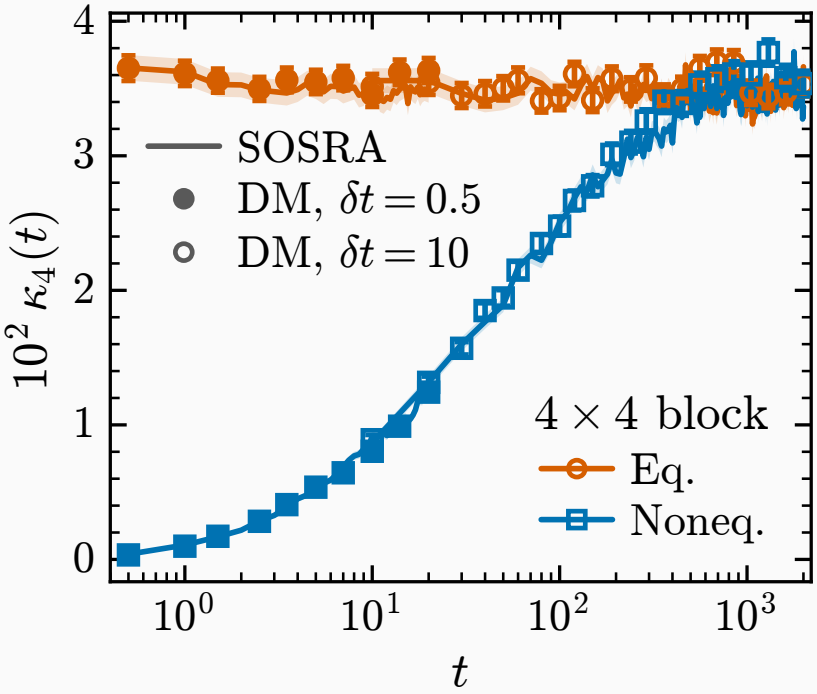


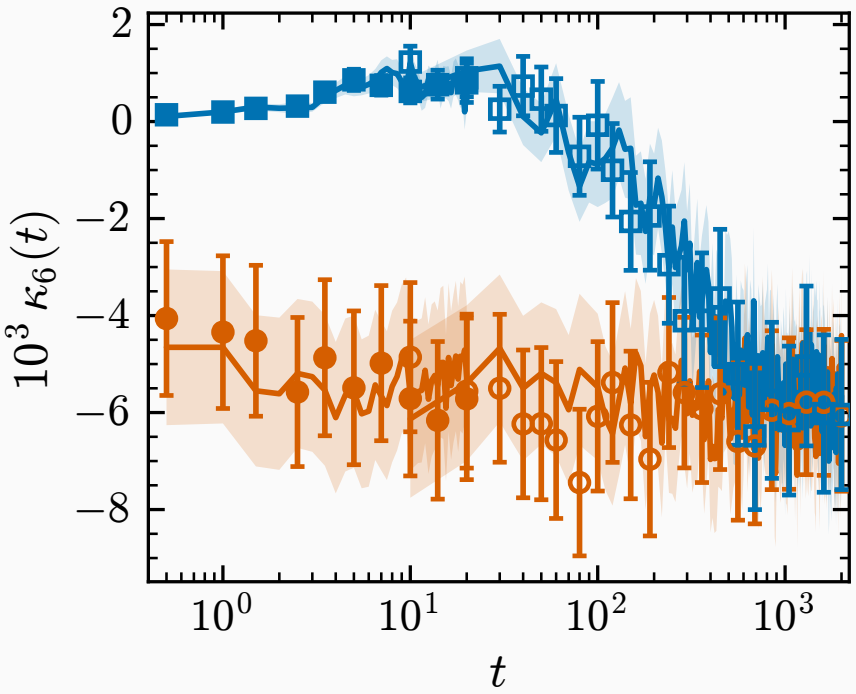
$$G_n(t) = V^{-1}\left\langle|\phi_n(t)|^2\right\rangle, \quad \phi_n(t) = \sum_x e^{-\mathrm{i}k_n\cdot x}\,\phi(x,t), \quad k_n = \frac{2\pi n}{L}(1,1)$$

G_1 for $n=1$, the softest mode, $\quad \Delta M(t) = M(t) - M(0)$

Critical point	**Initial ensembles**	**Composition**	**Against SOSRA**
2D, $L=32$	HMC equilibrium	**200** kernel steps	$G_1(t)$ agrees
strongly coupled	Gaussian quench	$t=2000$	$M(t)=M(0)$

High order fluctuations





$$\kappa_n = \langle \phi_B^n \rangle_c, \qquad \phi_B = \frac{1}{\ell^2} \sum_{x \in B} \left(\phi_x - M \right), \quad \ell = 4$$

Subvolume

4×4 block

excess over mean M

Gaussian quench

$\kappa_4 = \kappa_6 = 0$ at $t = 0$

κ_4 builds up until $t \simeq 500$

κ_6 changes sign

Time correlations in equilibrium

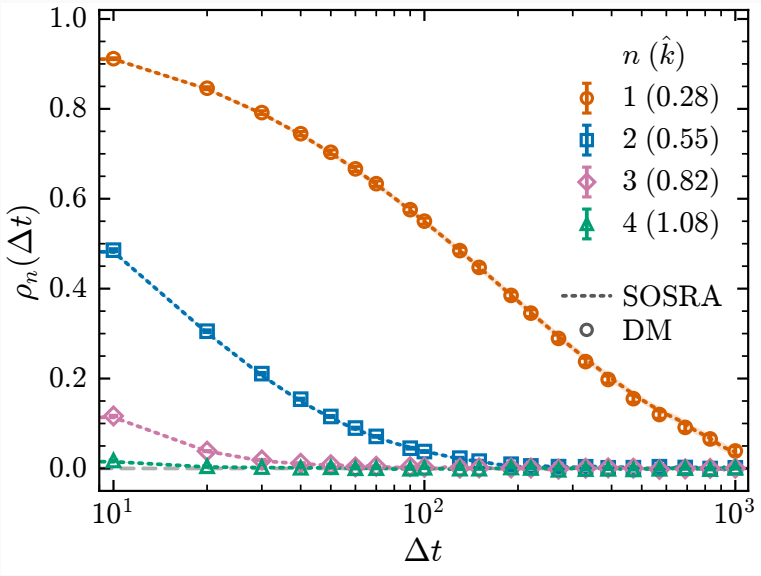
Two-time correlation

$$\rho_n(\Delta t) = \frac{V^{-1}\left\langle \mathrm{Re}[\phi_n(t+\Delta t)\,\phi_n^*(t)]\right\rangle_t}{\sqrt{\left\langle G_n(t+\Delta t)\right\rangle_t \left\langle G_n(t)\right\rangle_t}}$$

$\langle\cdot\rangle_t$ over trajectories and time origins t

Conserved dynamics is slow

$\tau_n \propto k_n^{-z}, \quad z \simeq 4$



2D, $L = 32$, critical point

The critical exponent emerges

Relaxation time

$$\rho_n(\tau_n) = e^{-1}$$

Dynamic scaling

$$\rho_n(\Delta t) = f(\Delta t/\tau_n), \qquad \tau_n \propto k_n^{-z}$$

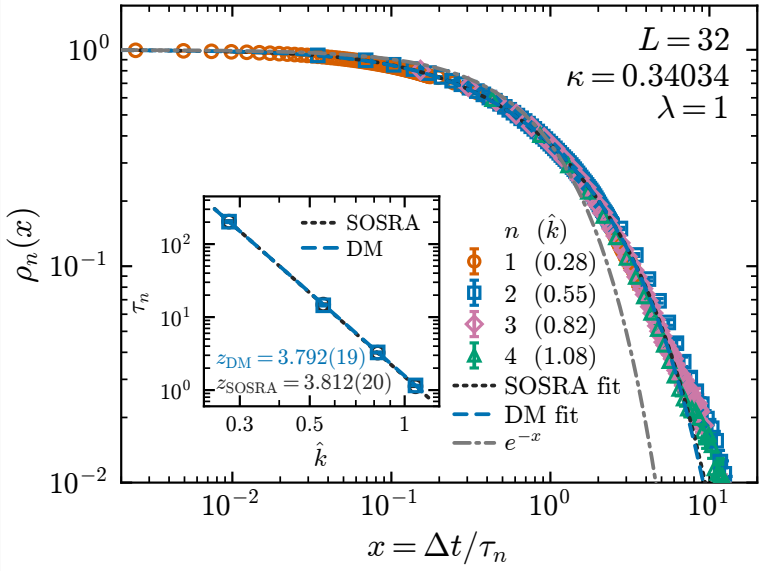
all modes on a common curve

Fits at $L = 32$

$z_{\mathrm{DM}} = \mathbf{3.792(19)}$

$z_{\mathrm{SOSRA}} = \mathbf{3.812(20)}$

exact: $z = 4 - \eta = 3.75$



$z = 4 - \eta = 15/4 = 3.75$. **Exact.** Halperin et al., PRB 10, 139 (1974).

$z = 3.83(10),\ 3.716(17)$. **MC.** Schweitzer et al., NPB 984, 115944 (2022).

$z = 3.95(10)$. **Kawasaki MC.** Zheng, PLA 277, 257 (2000), erratum 282, 132 (2001).

$z = 3.79 \pm 0.05$. **MC RG.** Moseley et al., J. Stat. Phys. 57, 421 (1989).

$z = 3.80$. **MC RG.** Yalabik and Gunton, PRB 25, 534 (1982).

The critical exponent emerges

The scaling function

$$f(x) = e^{-\lambda_0 x}(1+ax)^{-b}$$

$x = \Delta t/\tau_n$, four modes, one fit

a tail slower than e^{-x}

Sum rule

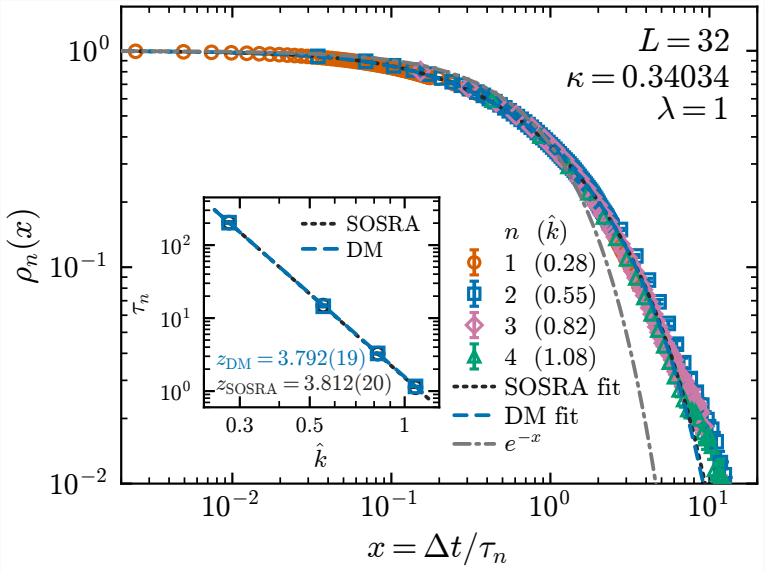
$$-f'(0^+) = \lambda_0 + ab = r$$

$r = \tau_n \hat{k}_n^2 / G_n$, the initial decay rate of ρ_n, set by the statics

two free parameters λ_0, a

$b = (r - \lambda_0)/a$ follows

free fits hit r within a few %



$z = 4 - \eta = 15/4 = 3.75$. **Exact.** Halperin et al., PRB 10, 139 (1974).

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$z = 3.80$. **MC RG.** Yalabik and Gunton, PRB 25, 534 (1982).

Prediction on unseen $L = 64$

Size extrapolation

$$L_{\text{train}} = 8, 16, 32 \implies \boldsymbol{L = 64}$$

no $L = 64$ trajectory ever seen

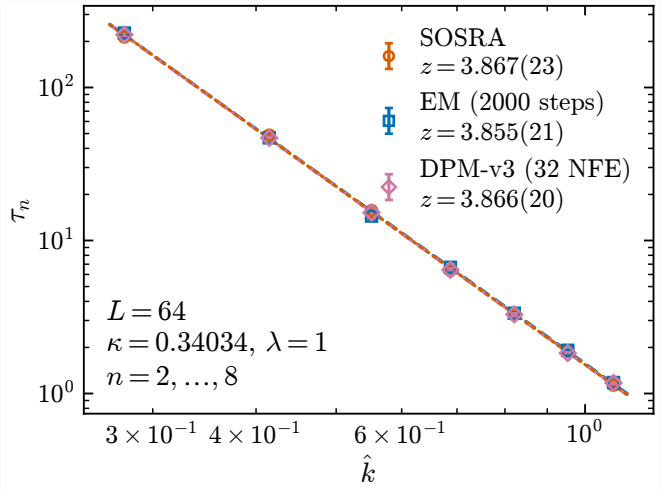
Fits at $L = 64$

$$\boldsymbol{z_{\text{DM}} = 3.855(21)}$$

$$\boldsymbol{z_{\text{Model B}} = 3.867(23)}$$

As fast as the best direct solver

512 trajectories, $t = 100$, GH200, matched accuracy



Method	Wall time (s) equil.	noneq.
DPM-Solver-v3 (This work)	3.24	3.26
SOSRA (This work)	1.30	3.28
SOSRA (StochasticDiffEq.jl)	12.36	30.67
ShARK (Diffrax)	5.56	17.76
EM (Diffrax)	19.45	38.75
EM (other GPU/CPU solvers)	> 100	> 200

Phase separation on unseen $L = 128$

Sign-field correlation

$$\sigma(\boldsymbol{x}, t) = \operatorname{sgn} \phi(\boldsymbol{x}, t)$$

$$C(r, t) = \langle \sigma(\boldsymbol{x}, t)\, \sigma(\boldsymbol{x} + \boldsymbol{r}, t) \rangle$$

$L(t)$: first zero of $C(r, t)$

Phase-ordering scaling

$$C(r, t) = g\big(r / L(t)\big)$$

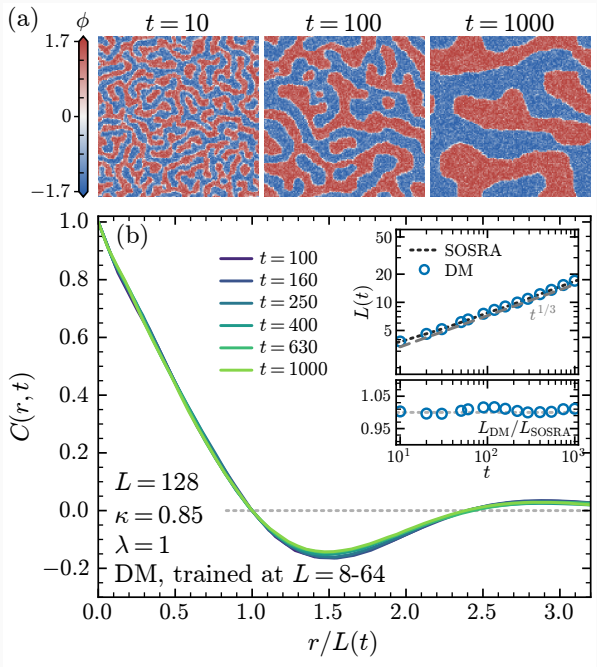
$$L(t) \propto t^{1/3}$$

Size extrapolation

$L_{\text{train}} \leq 64 \implies \boldsymbol{L = 128}$

$\boldsymbol{L(t) \propto t^{1/3}}$

$L_{\text{DM}} / L_{\text{SOSRA}} = 1$ within 2%



$t^{1/3}$. **LSW theory.** Lifshitz and Slyozov, JPCS 19, 35 (1961).

Asymptotic $t^{1/3}$, corrections. **Conserved Ising.** Huse, PRB 34, 7845 (1986).

$t^{1/3}$, scaling. **2D critical Model B.** Rogers et al., PRB 37, 9638 (1988).

$t^{1/3}$ scaling. **Review.** Bray, Adv. Phys. 43, 357 (1994).

$t^{1/3}$. **Polymer experiment.** Krausch et al., PRL 71, 3669 (1993).

Critical $\alpha = 1/3$. **2D CH.** König et al., PCCP 23, 24823 (2021).

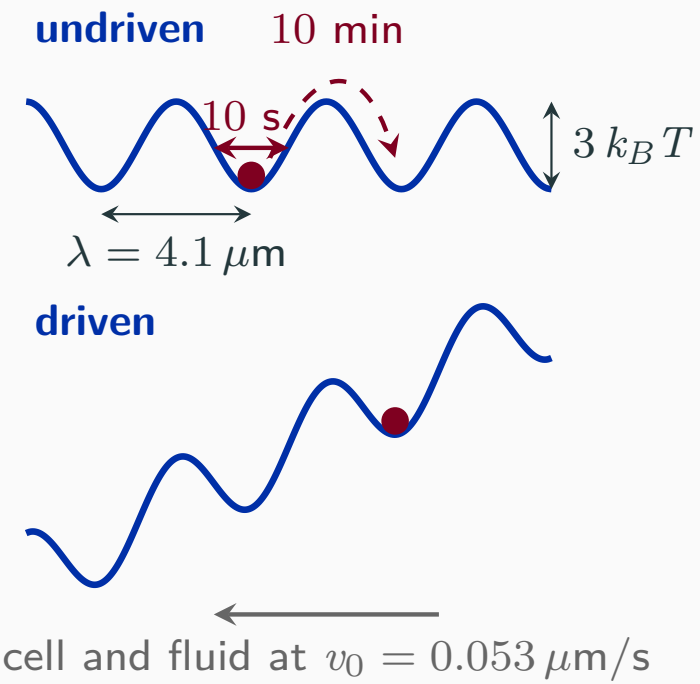
$\alpha_L = 0.34, 0.37$ from perimeter. **Membrane.** Zhiliakov et al., BBA 1863, 183446 (2021).

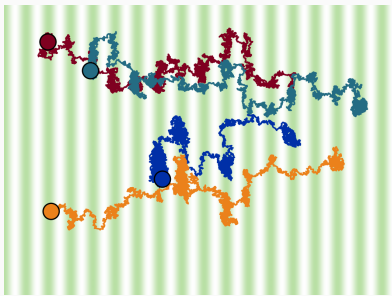
$t^{1/3}$. **Passive membrane, CH.** Arnold et al., PRL 131, 128402 (2023).

$t^{1/3}$. **HP1 Model B.** Tortora et al., PNAS 120, e2211855120 (2023).

Long-time $t^{1/3}$. **Diffusive polymer Model B.** Garcia et al., JCP 159, 214904 (2023).

The colloid experiment





The experiment

polystyrene spheres in a quasi-2D cell
a periodic optical potential along x
1200 tracks of 30 to 60 min

Two time scales

in a well ~ 10 s, escape ~ 10 min
the drive tilts the washboard

Pairs, as before

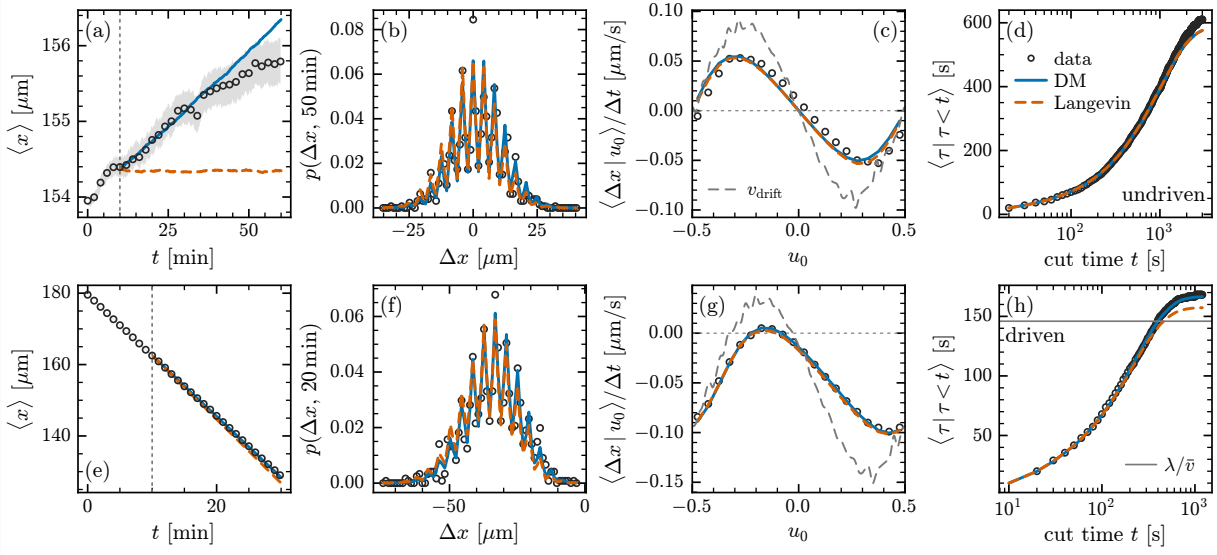
$(r_t,\ r_{t+\delta t})$, $\delta t = 10$ s, first 10 min only
condition: the phase in the potential
no potential, no D, no drive is given

Prediction

iterate from the positions at 10 min
300 steps undriven, 120 driven

Experiment: Dieball et al., Soft Matter 21, 2170 (2025); data: Zenodo 10.5281/zenodo.13908799.
Langevin reference of the original study: overdamped, $U/k_B\,T = -\ln p_{\mathrm{eq}}$ from the undriven density, $D = 0.04\,\mu\mathrm{m}^2/\mathrm{s}$, drift $-v_0$ under driving.

Transport in the colloid experiment



top: undriven, bottom: driven. dotted line: end of training

Driven within 1%, Langevin 6% off

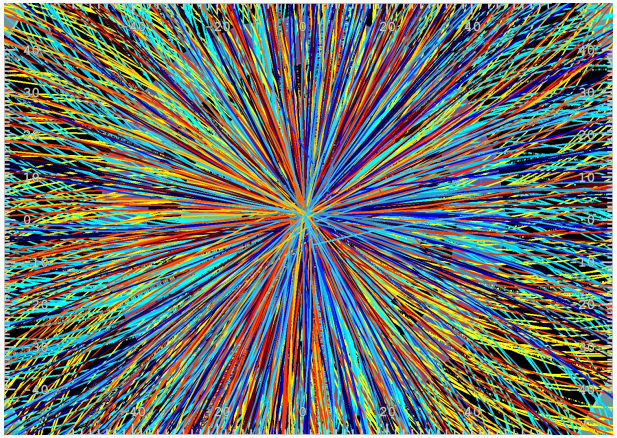
Undriven drift in (a) is learned, the equilibrium Langevin has none

Passage time first arrival at $x_\mathrm{i} \pm \lambda$ undriven, $x_\mathrm{i} - \lambda$ driven

driven	$\bar{v}_x$ [μm/s]	passage [s]
data	$-0.0282(3)$	$168(4)$
learned kernel	-0.0284	166.5
Langevin	-0.0298	157.3

(a,e) mean position. (b,f) net displacement over 50 and 20 min. (c,g) displacement rate over 10 s vs phase. (d,h) paths that cross before t, 10 s grid. 32 replicas per start.

Outlook: stochastic hydrodynamics near the QCD critical point



ALICE event display, CERN

Statics where is the critical point?

sign problem at $\mu_B \neq 0$, lattice, FRG

Beam energy scan

net-proton cumulants, $\kappa_3 \propto \xi^{4.5}$, $\kappa_4 \propto \xi^7$

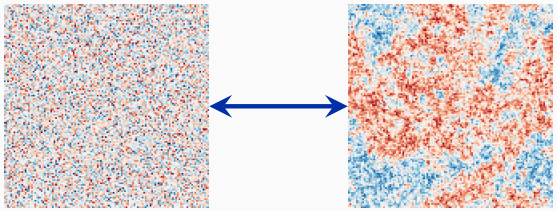
Dynamics the fireball never equilibrates

lifetime ~ 10 fm/c, $\tau \sim \xi^z$, ξ freezes at 2–3 fm

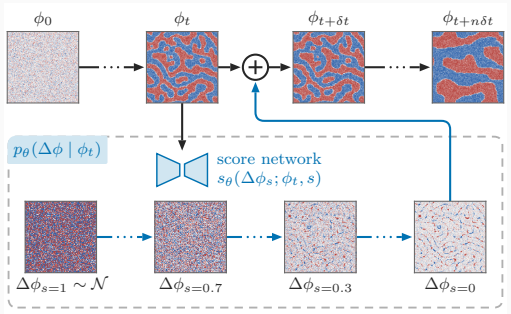
1	**Model H**	n coupled to momentum $\boldsymbol{\pi}$, $z \simeq 3$
2	**UrQMD**	hadrons in a box, autonomous the degrees of freedom change
3	**Stochastic hydro**	long steps, lattices beyond the training size

Critical endpoint: QCD with FRG, Fu, Pawlowski, Rennecke, PRD 101, 054032 (2020).
Non-Gaussian cumulants: Stephanov, PRL 102, 032301 (2009). **Beam energy scan:** STAR, PRL 126, 092301 (2021). **Critical slowing down:** Berdnikov, Rajagopal, PRD 61, 105017 (2000).
Model H: Son, Stephanov, PRD 70, 056001 (2004). **UrQMD:** Bass et al., Prog. Part. Nucl. Phys. 41, 255 (1998); Bleicher et al., J. Phys. G 25, 1859 (1999).

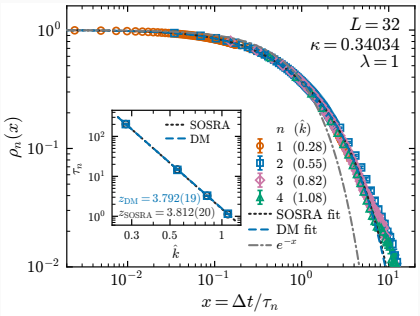
Summary



A diffusion model learns the drift of stochastic quantization

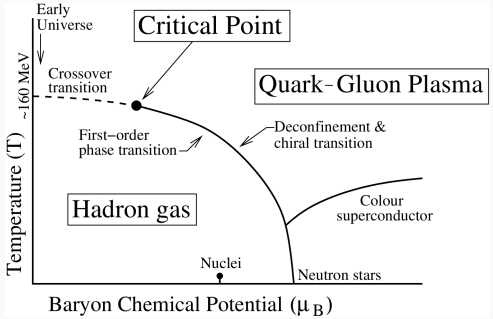


A conditional diffusion model learns the transition kernel



Long-time universality emerges from short-time pairs

$$\delta t \ll \tau \sim \xi^z$$



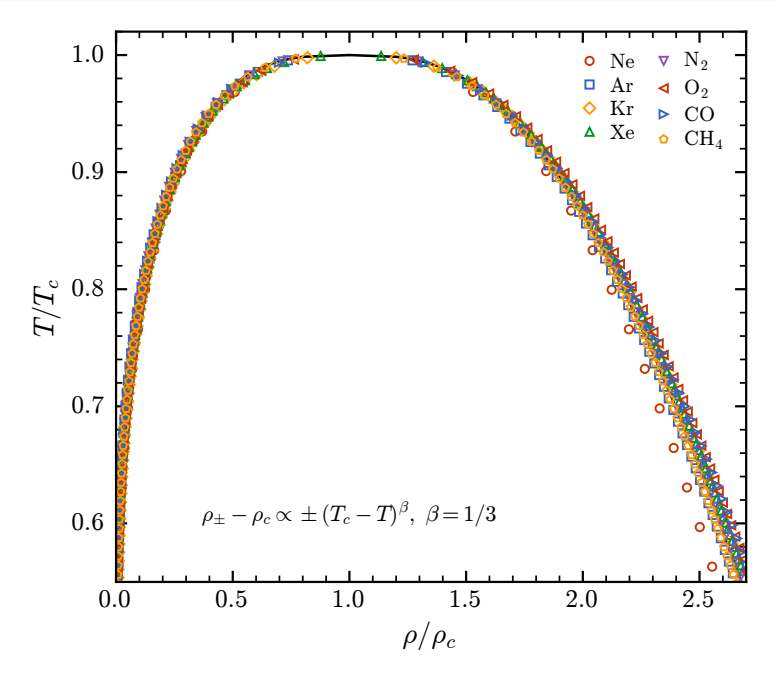
Next, stochastic hydrodynamics near the QCD critical point

from microscopic $(n, \boldsymbol{\pi})$ pairs

Thank you!

Backup

Universality



eight fluids, one coexistence curve

experimental data

	β	γ	ν
fluids	0.32–0.33	1.23–1.25	0.62–0.63
mixtures	0.32–0.33	1.23–1.25	0.62–0.64
magnets	0.32–0.33	1.24–1.25	0.62–0.64
3D Ising	0.3265	1.2372	0.6301

Only d and symmetry matter

fluids, mixtures, uniaxial magnets

and the QCD critical end point:

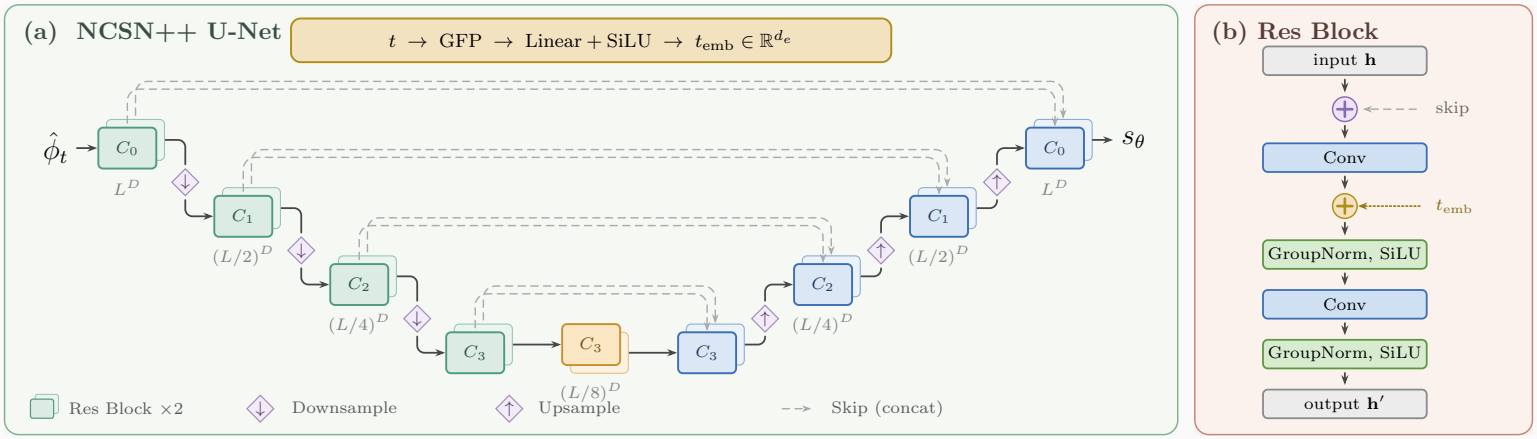
all 3D Ising

cell membranes: 2D Ising, same story

Lattice ϕ^4 is the minimal representative of the class

Coexistence curves: Guggenheim, J. Chem. Phys. 13, 253 (1945); data from the NIST Chemistry WebBook, line $\rho_\pm/\rho_c = 1 + \frac{3}{4}(1 - T/T_c) \pm \frac{7}{4}(1 - T/T_c)^{1/3}$. **Exponents:** Pelissetto, Vicari, Phys. Rept. 368, 549 (2002).

Score network and reference configurations



Physics built in

periodic lattice: circular padding

translation symmetry: convolutions

nothing else, not even $\phi \to -\phi$

Architecture

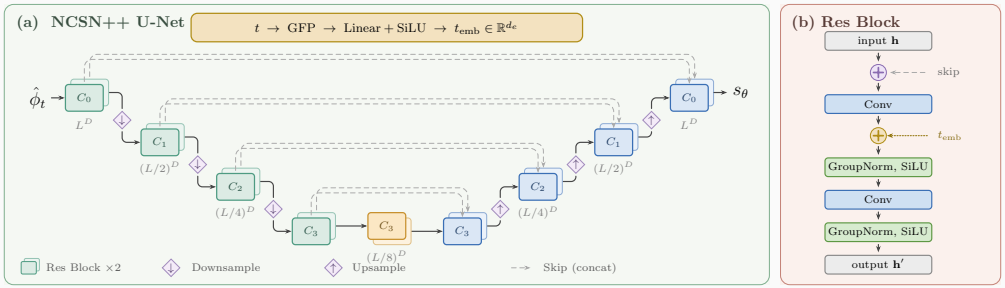
U-Net, $L \to L/8$: reaches the infrared

skip connections keep the ultraviolet

no dense layer: any L, same weights

NCSN++: Song et al., ICLR (2021). The noise level t enters every Res Block through an embedding, panel (b). $\kappa_c = 0.27088$ (2D, $L = 128$), 0.19225 (3D, $L = 64$) from Binder crossings; 3D: Hasenbusch, J. Phys. A 32, 4851 (1999).

Score network and reference configurations



Physics built in

periodic lattice: circular padding

translation symmetry: convolutions

nothing else, not even $\phi \to -\phi$

Physics learned

the action $S[\phi]$, never given

the $\phi \to -\phi$ symmetry

critical correlations, which phase

Architecture

U-Net, $L \to L/8$: reaches the infrared

skip connections keep the ultraviolet

no dense layer: any L, same weights

Reference configurations

HMC with Wolff reflections

κ below, at and above κ_c

10240 / 2560 configs, all it ever sees

NCSN++: Song et al., ICLR (2021). The noise level t enters every Res Block through an embedding, panel (b). $\kappa_c = 0.27088$ (2D, $L = 128$), 0.19225 (3D, $L = 64$) from Binder crossings; 3D: Hasenbusch, J. Phys. A 32, 4851 (1999).

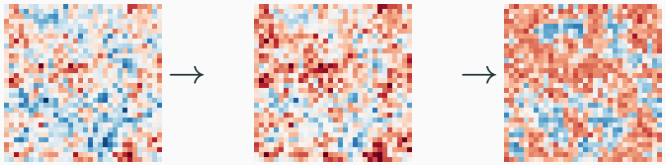
Training data for the kernel

Critical kernel $\kappa = \kappa_c,\ L = 32$

Gaussian field χ with propagator

$$\langle \chi_k \chi_{-k} \rangle = \frac{1}{Zk^2 + m^2}, \quad \phi_0 = M + \chi$$

Z sets the amplitude, m^2 the correlation length, M the mean, all random over a broad range



χ $\quad$ $\phi_0 = M + \chi$ $\quad$ $\phi_{\delta t}$

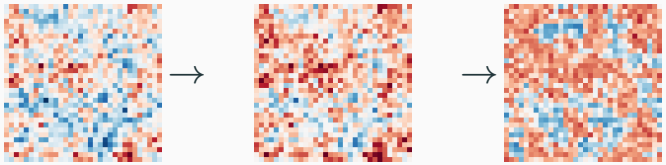
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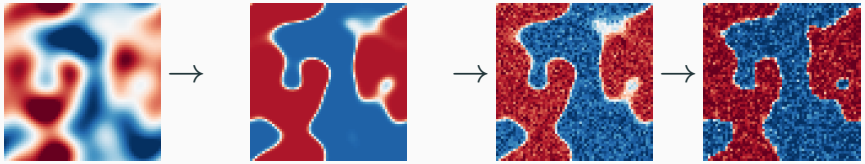
Coarsening kernel $\kappa = 0.85$, $L \le 64$

g Gaussian with $\langle g(\boldsymbol{x})\, g(0) \rangle = e^{-\boldsymbol{x}^2/2\ell^2}$

$$\phi_0 = A \tanh\!\left(\tfrac{\ell}{w}\, g\right) + \eta\, \xi$$

domains $\pm A$ of size ℓ, walls of width w, white noise ξ of level η

ℓ, w, A, η random over a broad range



g $\quad$ $A \tanh(\frac{\ell}{w} g)$ $\quad$ ϕ_0 $\quad$ $\phi_{\delta t}$

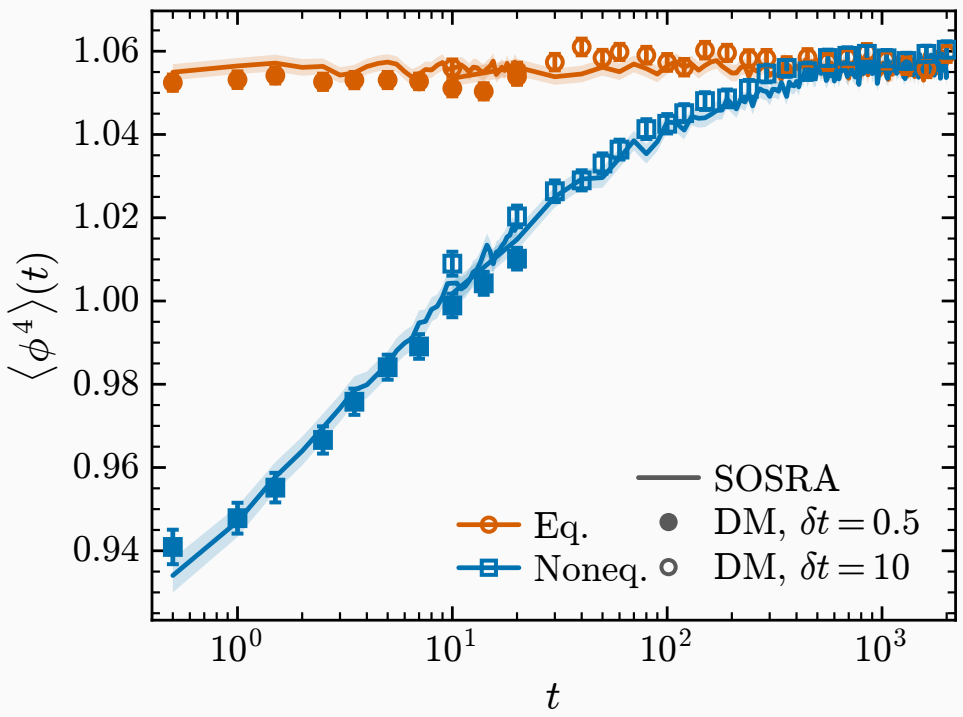
Target Model B (SOSRA) run from ϕ_0 over $\delta t = 10$, $\Delta\phi = \phi_{\delta t} - \phi_0$

The point the kernel learns the rule of motion, not trajectories

every test is an extrapolation: other ensembles, longer times, larger lattices

Z log-uniform in $[0.05, 10]$, $m^2 \in [0, 0.8]$, $M \in [-0.9, 0.9]$; ℓ log-uniform in $[2, L/3]$, $w \in [0.7, 3]$, $A \in [0.8, 1.8]$, $\eta \in [0.05, 0.35]$, $1/8$ pure noise; 51200 pairs per L.

Fourth order moment



Fourth moment

$\langle\phi^4\rangle$ over sites and trajectories

Same evolution

2D, $L = 32$, critical point

$\delta t = 10$, 200 steps to $t = 2000$

$\delta t = 0.5$, 40 steps to $t = 20$

Initial ensembles

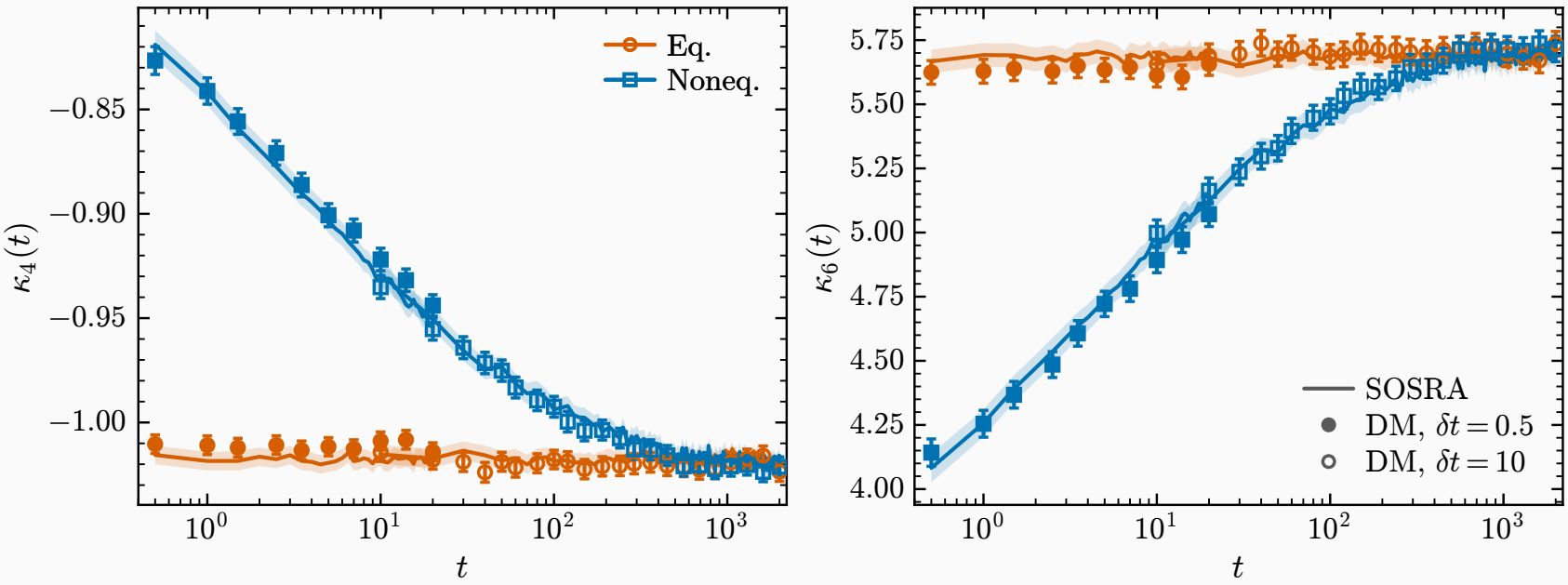
HMC equilibrium

Gaussian quench, off the scale at $t = 0$ with $\langle\phi^4\rangle = 5.3$

Against SOSRA

within 1% at all times

Site cumulants



$\kappa_n = \langle \phi_x^n \rangle_c$ over sites and trajectories

Same evolution

2D, $L = 32$, critical point

$\delta t = 10$, 200 steps

$\delta t = 0.5$, 40 steps

Site distribution

two peaks, $\kappa_4 < 0$

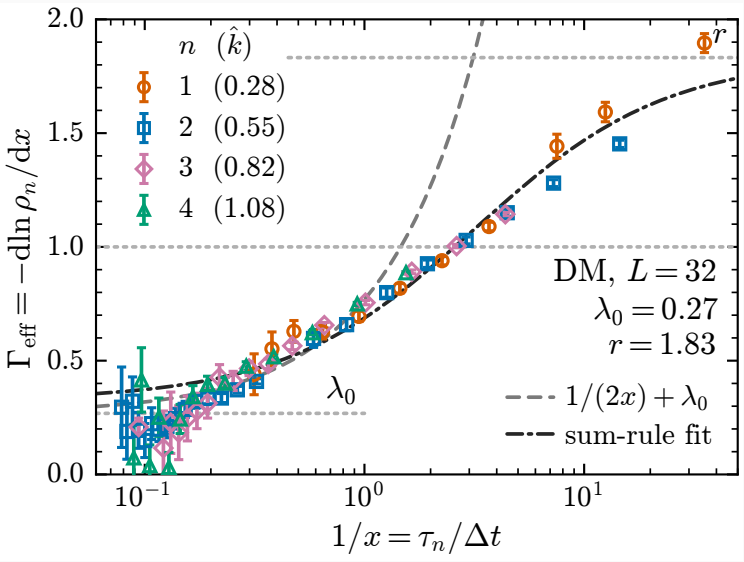
the quench relaxes to it

Against SOSRA

κ_4, κ_6 within 2%

at all times

The scaling function



decay rate of the learned kernel's ρ_n

Asymptotics

$x \to 0$: $f \simeq 1 - rx$, $\Gamma_{\rm eff} \to r$

$x \to \infty$: $f \simeq (ax)^{-b}\, e^{-\lambda_0 x}$

$\Gamma_{\rm eff} \simeq \lambda_0 + b/x$, $b \simeq \frac{1}{2}$

The rate runs with the time scale

$\Gamma_{\rm eff} = -\,d\ln\rho_n/dx$, $x = \Delta t/\tau_n$

free field: eigenmodes, $\Gamma_{\rm eff} = 1$

here ϕ_n mixes many relaxation modes,

$\Gamma_{\rm eff}$ runs from the bare r to the gap λ_0

Short-time rate from the statics

fluctuation–dissipation at $\Delta t \to 0^+$

$-\rho_n'(0^+) = \Gamma \hat{k}_n^2\, T/G_n$, $\Gamma = T = 1$

$r = \tau_n \hat{k}_n^2/G_n \simeq 1.8$, τ_n measured

one r for all modes $\Leftrightarrow$ $\tau_n \propto G_n/\hat{k}_n^2$

Long-time rate: a gap

$\Gamma_{\rm eff} \to \lambda_0 \simeq 0.27$ from the tail, $x \geq 2$

dashed: $\lambda_0 + 1/(2x)$, $b = \frac{1}{2}$

the sum-rule fit gives 0.33 instead,

the crossover is approximate