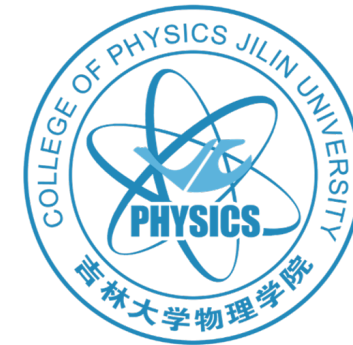


The Int'l Joint Workshop on the SM and Beyond 2026
Nanjing,
30th October 2026

Non-perturbative aspects of self-interacting dark matter

Takumi Kuwahara
Jilin U.



in collaboration

JHEP11(2023) 105, with A. Kamada, A. Patel

in preparation, with A. Kamada, P. Schmidt

contact:

kuwahara@jlu.edu.cn

takumi.kuwahara.hep@gmail.com

Self-interacting Dark Matter

Lambda CDM well explain

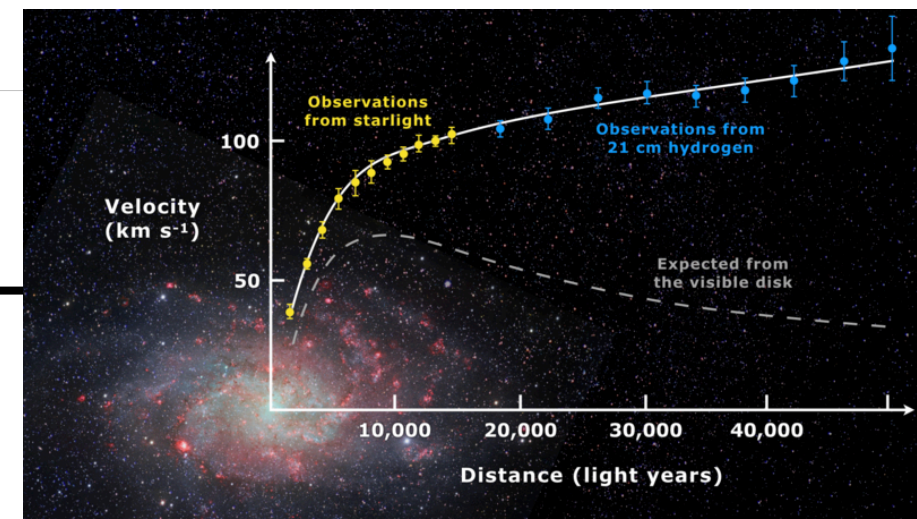
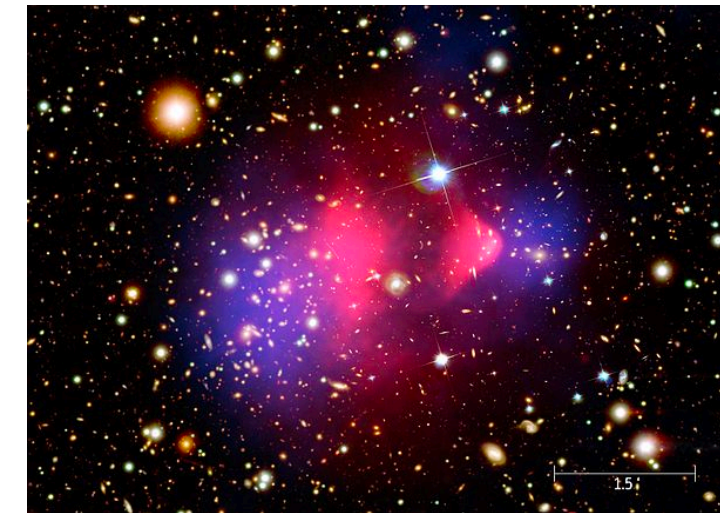
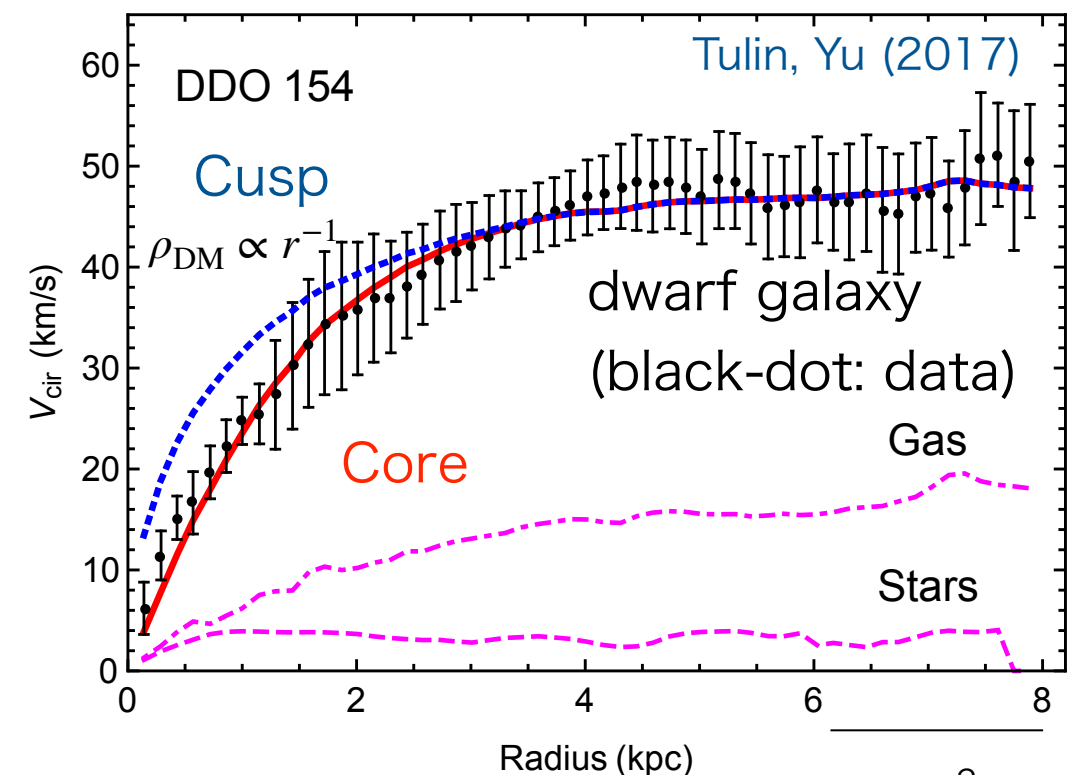
- large scale structure of the Universe
 - Galactic rotation curves
 - Galaxy cluster collision

Questioning

- small scale structure
 - core vs cusp problem of DM profile
 - diversity of galactic rotation curves
 - etc..

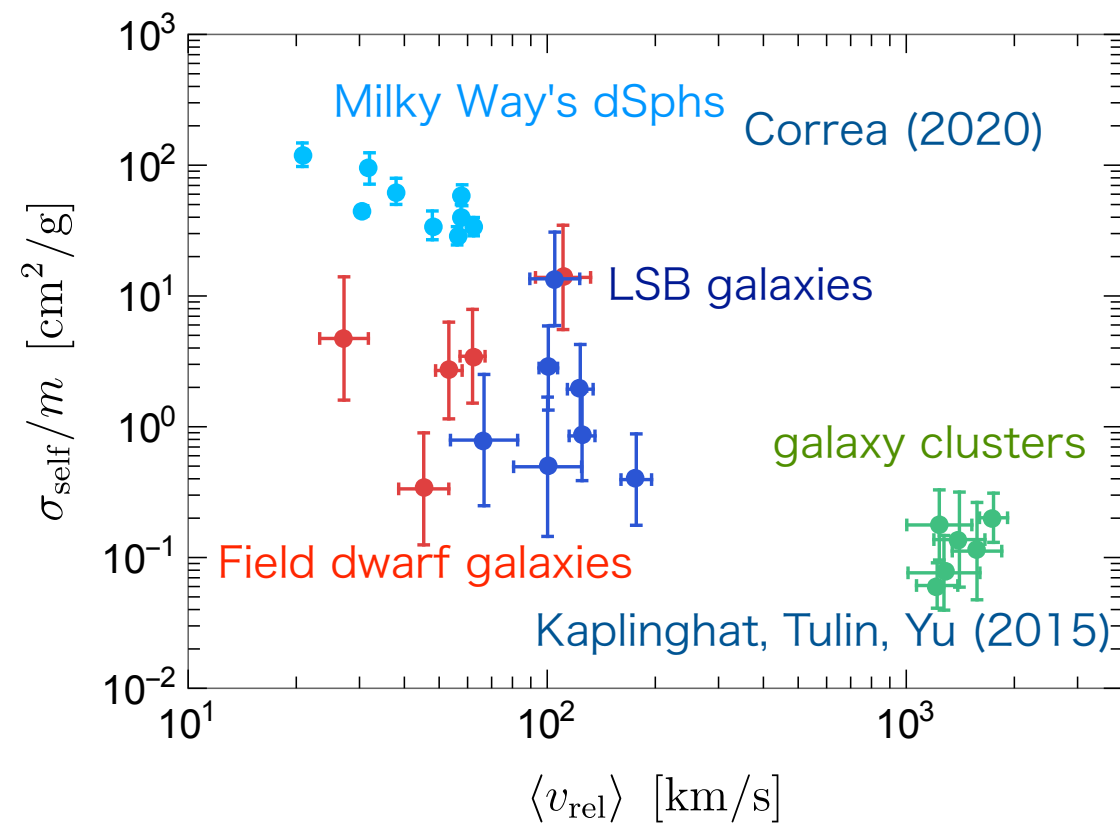
Self-interaction of DM

would change the small scale structure



Velocity-dependent self-interaction

Different scale indicates different size of self interaction

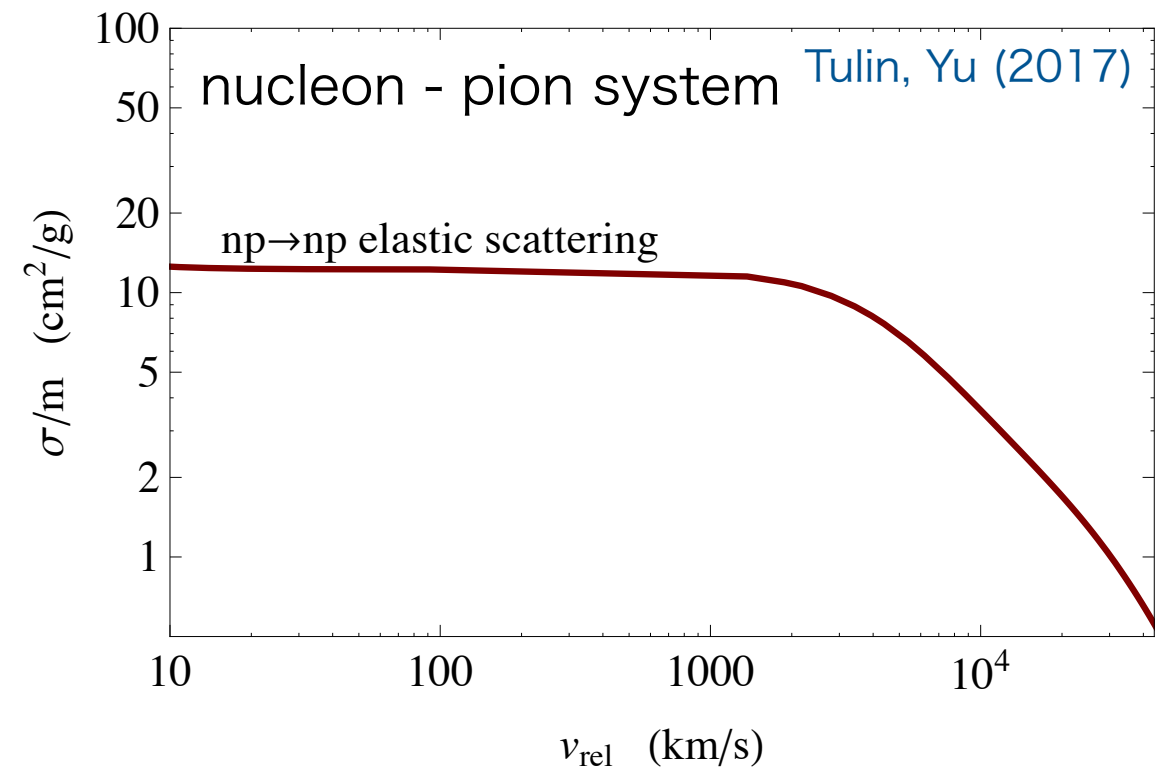
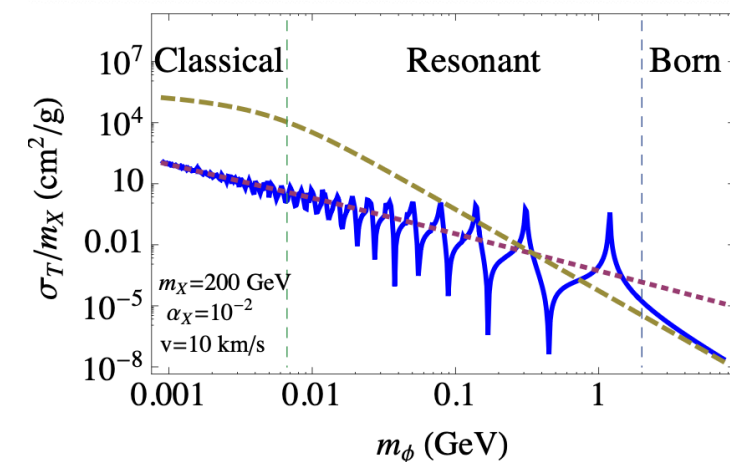


Kamada, Kim, TK (2020)

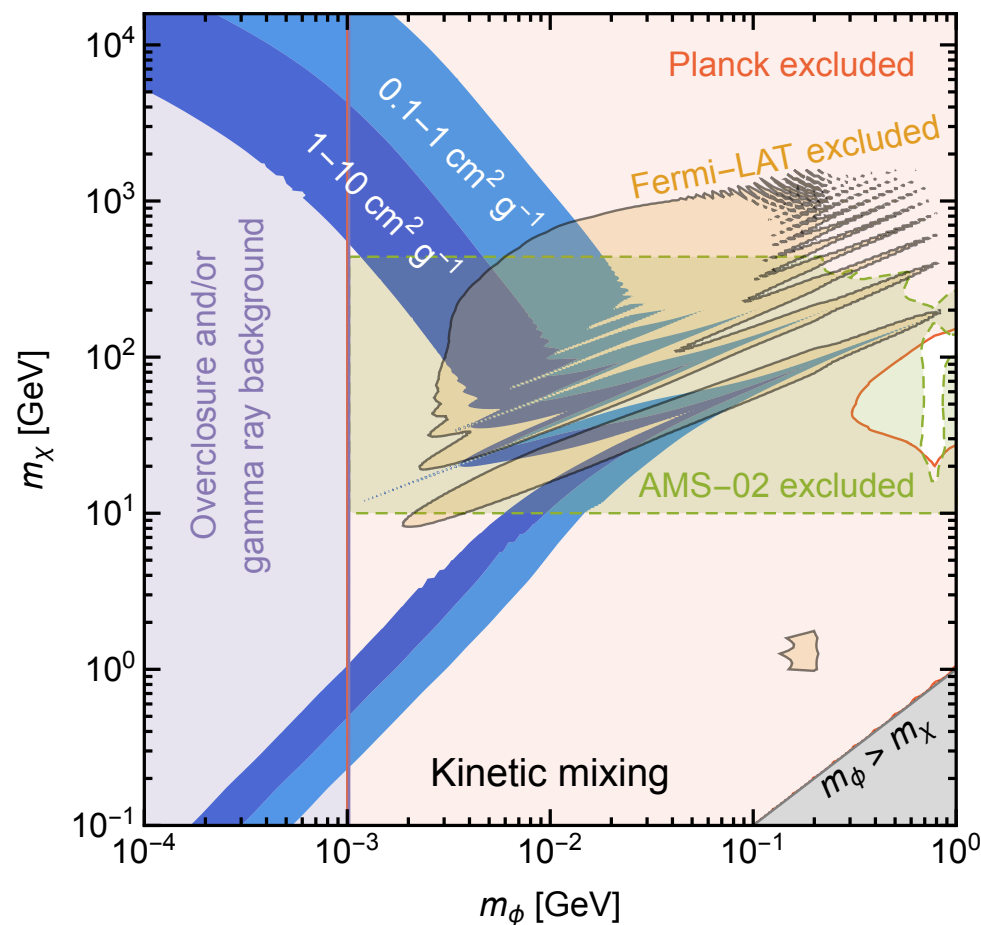
light-mediator + (sub-)GeV DM provides velocity dependent self-interaction

Tulin, Yu, Zurek (2013)

requires treatment beyond Born approximation



Coincidence of scattering and annihilation



Bringmann, Kahlhoefer, Schmidt-Hoborg, Walia (2016)

s-wave dominated scattering and annihilation

ϕ kinematically mixes with photon ($\phi \rightarrow \text{SM} + \text{SM}$)

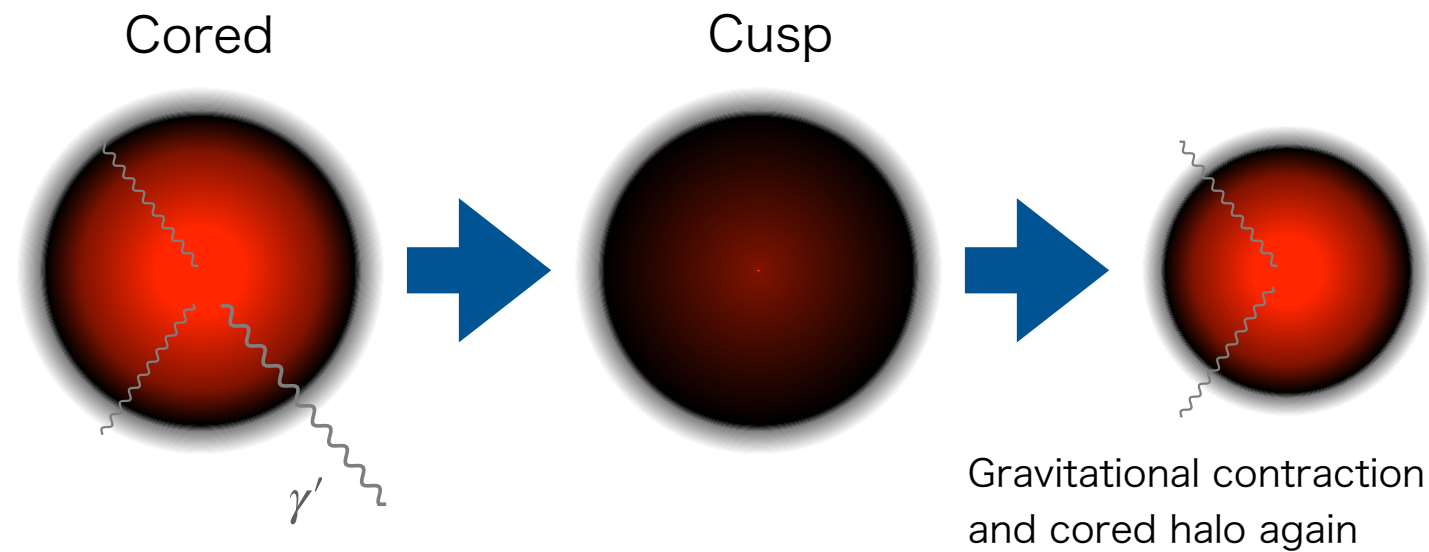
Resonantly enhancement points of self-scattering
= enhancing annihilation process excluded by

- CMB (Planck) $v/c \simeq 10^{-8}-10^{-7}$
- γ -ray signals (Fermi-LAT) $v/c \simeq 10^{-4}-10^{-3}$

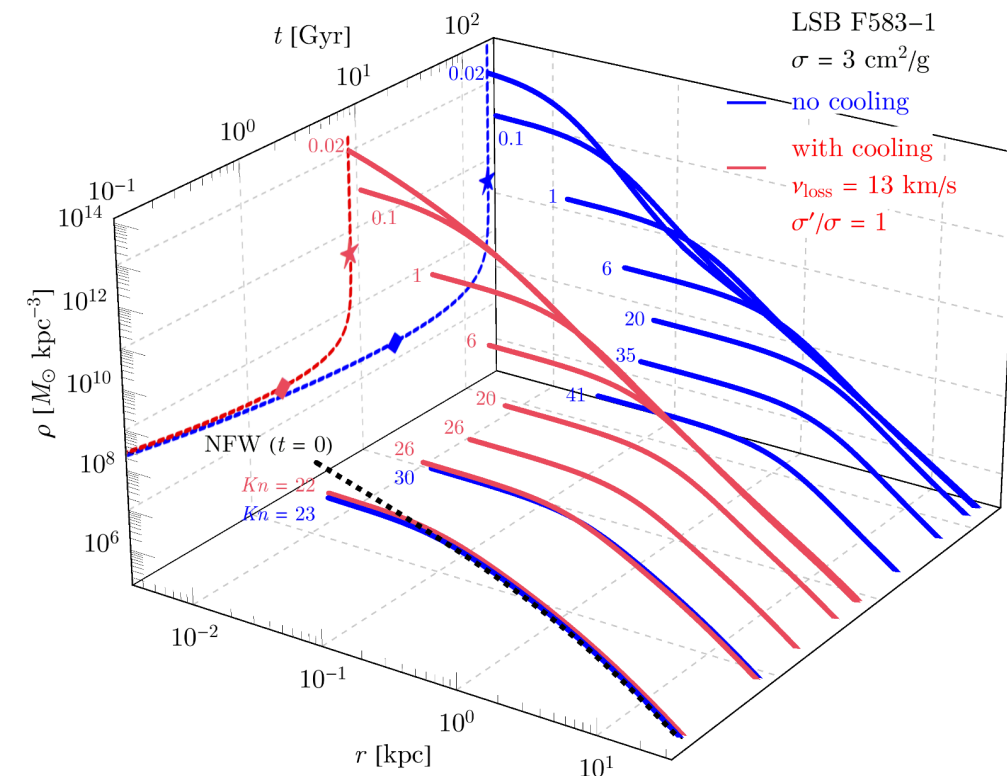
Remaining candidate for SIDM

- p-wave or higher partial-waves (v -dependent)
- asymmetric dark matter (no/less annihilation pair)
- etc.

Gravothermal Collapse



self-gravitating + self-interacting system



Essig, McDermott, Yu, Zhong (2018)

The presence of dissipative (cooling) process
is a key to determine the collapse timescale

Energy loss by

- Compton scattering with mediator particles
- Bremsstrahlung of mediator particles
- and so on

Beyond Born Approximation

2-body DM system is suffered from re-scattering process due to low velocity

To account for the process, solve Schrödinger equation for 2-body DM

$$\left[-\frac{1}{2\mu} \nabla^2 + V_\chi(r) \right] \psi_k(x) = E \psi_k(x) \quad V_\chi(r) = -\frac{\alpha e^{-m_\phi r}}{r}$$

relative energy: $E = \frac{k^2}{2\mu}$
reduced mass: $\mu = \frac{m_\chi}{2}$

Scattering state (Energy-eigenstate) at large r

$$\psi_k(x) \rightarrow e^{ikz} + f(k, \theta) \frac{e^{ikr}}{r}$$

in-coming plane wave scattering amplitude out-going spherical wave

Scattering cross section

$$\frac{d\sigma}{d\Omega} = |f(k, \theta)|^2$$

partial-wave decomposition

$$\psi_k(x) = \sum_{\ell} \frac{1}{k} e^{\frac{i}{2}\ell\pi + i\delta_{\ell}} (2\ell + 1) R_{k,\ell}(r) P_{\ell}(\cos \theta)$$

asymptotic behavior of the radial wave-function $R_{k,\ell}(r)$

$$R_{k,\ell}(r) \rightarrow \frac{1}{r} \sin \left(kr - \frac{\ell\pi}{2} + \delta_{\ell}(k) \right)$$

Scattering phase shift: δ_{ℓ}

Normalization

$\delta_{\ell} \rightarrow 0$ at high k : potential term negligible

scattering amplitude and cross section

$$f(k, \theta) = \sum_{\ell} (2\ell + 1) f_{\ell}(k) P_{\ell}(\cos \theta), \quad f_{\ell}(k) \equiv \frac{e^{2i\delta_{\ell}(k)} - 1}{2ik} = \frac{1}{k \cot \delta_{\ell} - ik}$$

$$\sigma = \sum_{\ell} \frac{4\pi}{k^2} (2\ell + 1) \sin^2 \delta_{\ell}(k)$$

partial-wave cross section saturates the Unitarity bound @ $\delta_{\ell}(k) = \frac{\pi}{2}$

Sommerfeld Enhancement

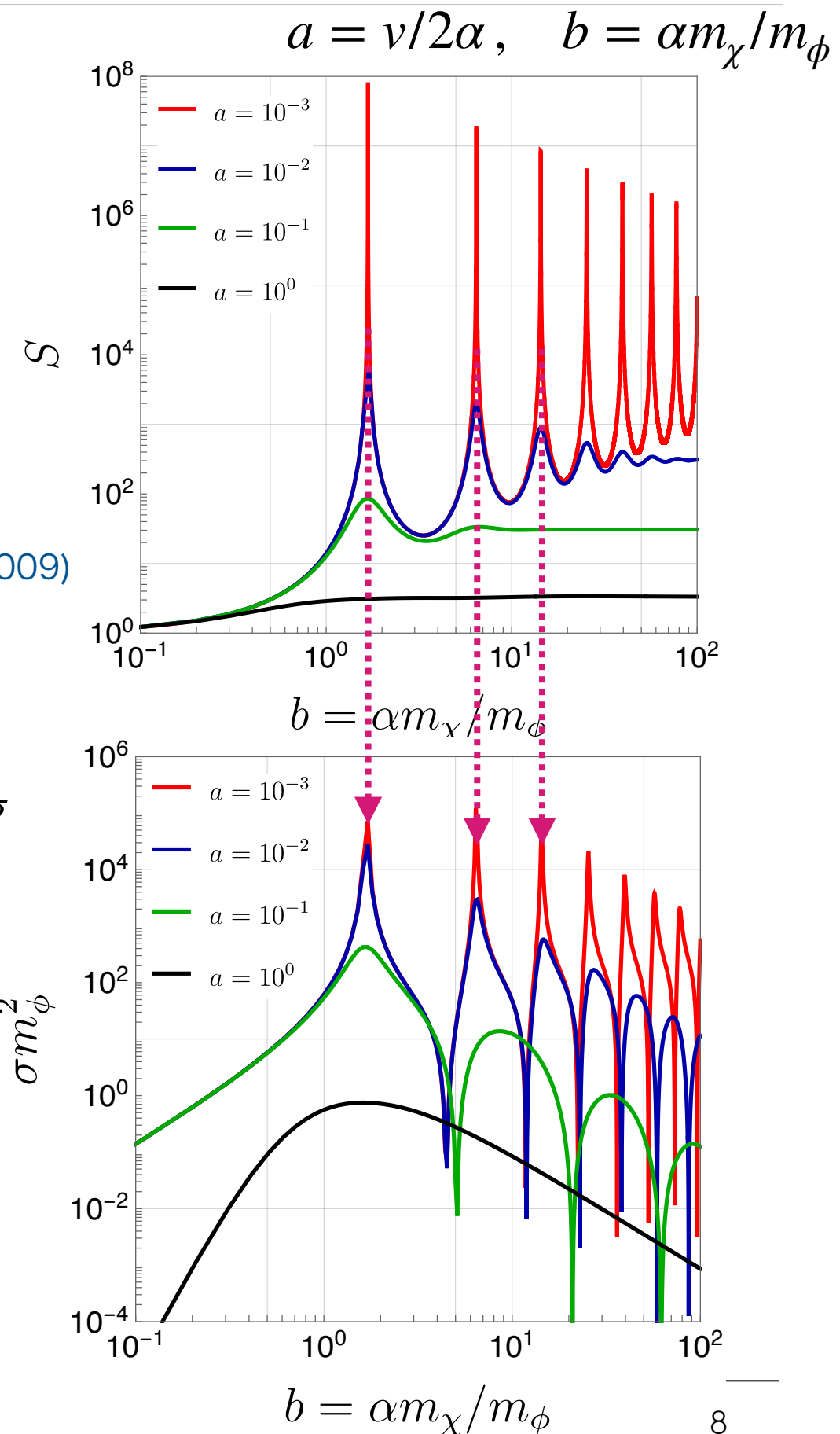
Wave-function is distorted near origin, resulting in the enhancement of annihilation

$$S_\ell(k^2) = \frac{\sigma_{\text{ann},\ell} v}{(\sigma_{\text{ann},\ell} v)_0} = \lim_{r \rightarrow 0} \frac{|R_{k,\ell}(r)|^2}{|R_{k,\ell}^{(0)}(r)|^2}$$

Iengo (2009), Cassel (2009)

Strong Correlation

- same wave-function determines the behavior of S and σ
- not surprising to have a correlation
- direct connection
-> **Watson's final/initial state theorem**



Watson's theorem

in- and out-states

Energy-eigenstates for interacting system

$$H = H_0 + V$$

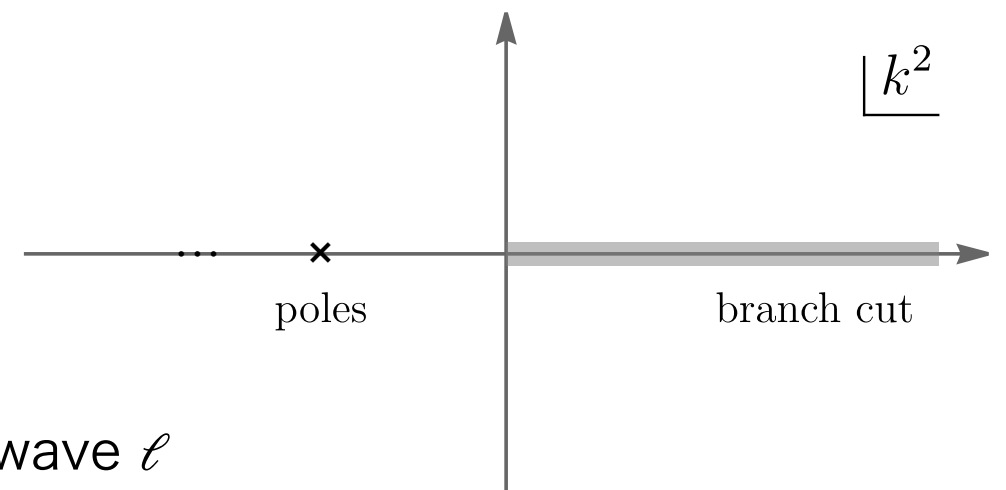
$$|\Psi_\alpha^\pm\rangle = |\Phi_\alpha\rangle + \frac{1}{E - H_0 \pm i\epsilon} V |\Psi_\alpha^\pm\rangle$$

Lippmann-Schwinger equation

Annihilation amplitude (in-state): $\Gamma_\ell(k^2 + i\epsilon) \equiv \langle 0 | \Theta_\chi | \Psi_\ell^+ \rangle$

inserting out-state complete set

$$\begin{aligned} \Gamma_\ell(k^2 + i\epsilon) &= \langle 0 | \Theta_\chi | \Psi_\ell^- \rangle \langle \Psi_\ell^- | \Psi_\ell^+ \rangle \\ &= e^{2i\delta_\ell} \Gamma_\ell(k^2 - i\epsilon) \end{aligned}$$



discontinuity is given by S-matrix elements for partial-wave ℓ

$$e^{2i\delta_\ell} = \langle \Psi_\ell^- | \Psi_\ell^+ \rangle$$

Watson's theorem and Omnès solution

Assuming that the amplitude is real ($f(x^*) = [f(x)]^*$): (time-inversion invariance)

$$\Gamma_\ell(k^2 + i\epsilon) = e^{2i\delta_\ell} \Gamma_\ell(k^2 - i\epsilon) = e^{2i\delta_\ell} \Gamma_\ell(k^2 + i\epsilon)^*$$

Formal solution is given by **Omnès solution**

$$\Gamma_\ell(k^2) = F_\ell(k^2) e^{\omega_\ell(k^2)}, \quad \omega_\ell(k^2) = \frac{1}{\pi} \int_0^\infty \frac{\delta_\ell(p)}{p^2 - k^2} dp^2$$

- Poles represented by $F_\ell(k^2)$

- Discontinuity by $e^{\omega_\ell(k^2)}$

$$F_\ell(k^2) = \prod_{\kappa_b} \frac{k^2}{k^2 + \kappa_b^2}$$

$$\frac{1}{p^2 - k^2 \mp i\epsilon} = P \frac{1}{p^2 - k^2} \pm i\pi \delta(p^2 - k^2)$$

$$\omega(k^2 \pm i\epsilon) = \omega(k^2) \pm i\delta(k)$$

Sommerfeld Enhancement

[Kamada, TK, Patel \(2023\)](#)

Ratio of amplitude square with and without phase-shift (potential term)

$$S_\ell(k^2) = \frac{|\Gamma_\ell(k^2)|^2}{|\Gamma_\ell^{(0)}(k^2)|^2} = |\Gamma_\ell(k^2)|^2$$

Bound states

positions of bound states = poles of $F_\ell(k^2)$

$$F_\ell(k^2) = \prod_{\kappa_b} \frac{k^2}{k^2 + \kappa_b^2}$$

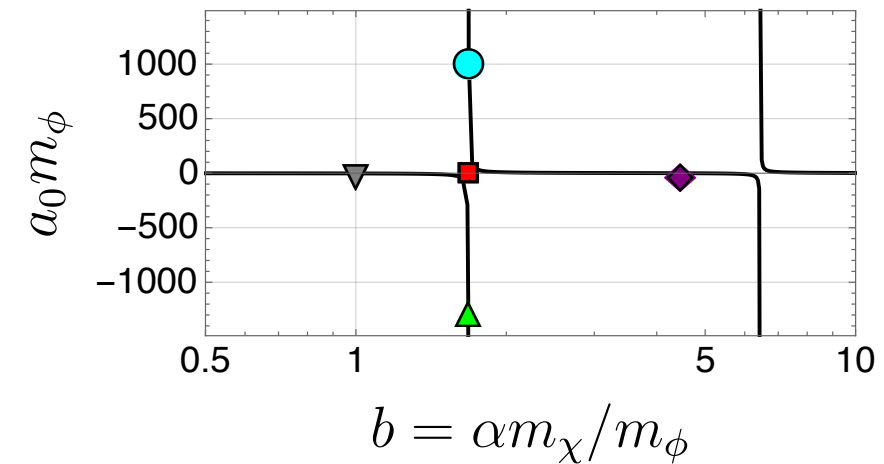
- shallow (nearly zero-energy) states

utilizing the effective range theory

finding a pole of scattering amplitude at $k = i\kappa_b$

$$f_\ell(k) = \frac{1}{k \cot \delta_\ell - ik}$$

this method is valid as far as $|\kappa_b| < r_{e,\ell}^{-1}$



$$k^{2\ell+1} \cot \delta_\ell(k) \simeq -\frac{1}{a_\ell^{2\ell+1}} + \frac{1}{2r_{e,\ell}^{2\ell-1}} k^2$$

$$\kappa_b = \frac{1}{r_{e,0}} \left(1 - \sqrt{1 - \frac{2r_{e,0}}{a_0}} \right) \quad \text{for } \ell = 0$$

$$\kappa_b^2 \simeq -\frac{2r_{e,\ell}^{2\ell-1}}{a_\ell^{2\ell+1}} \quad \text{for } \ell \geq 1$$

Comparison of two different methods

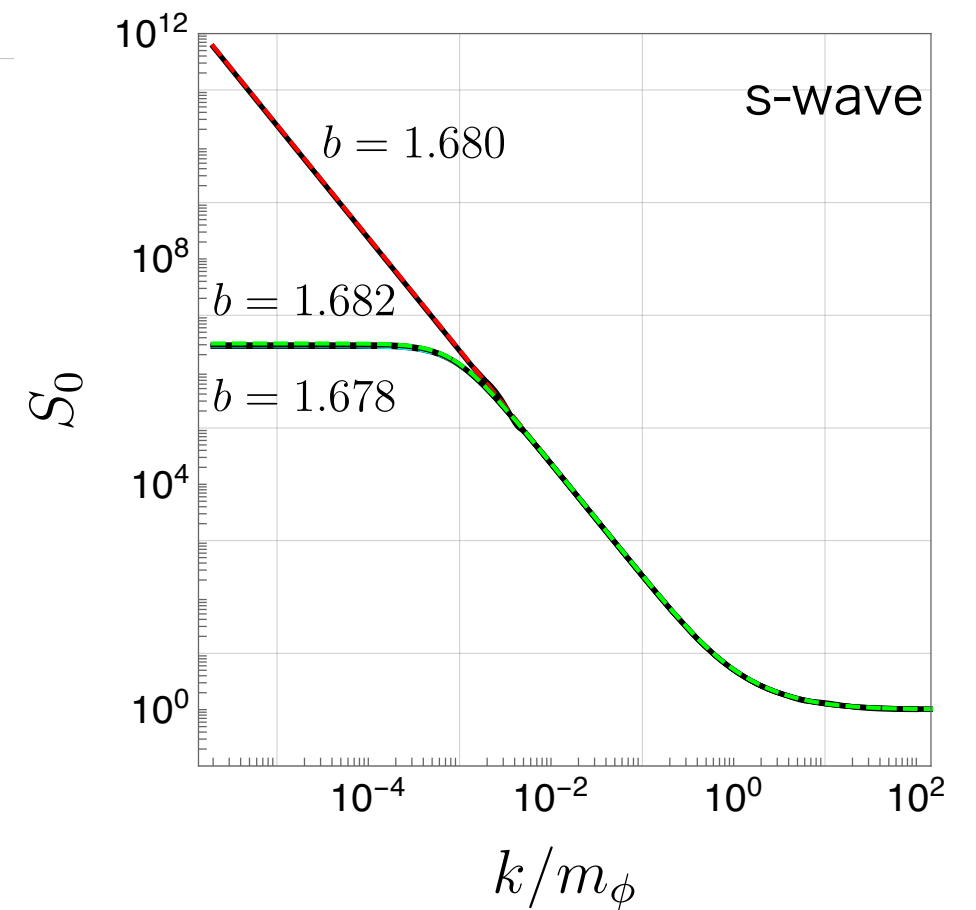
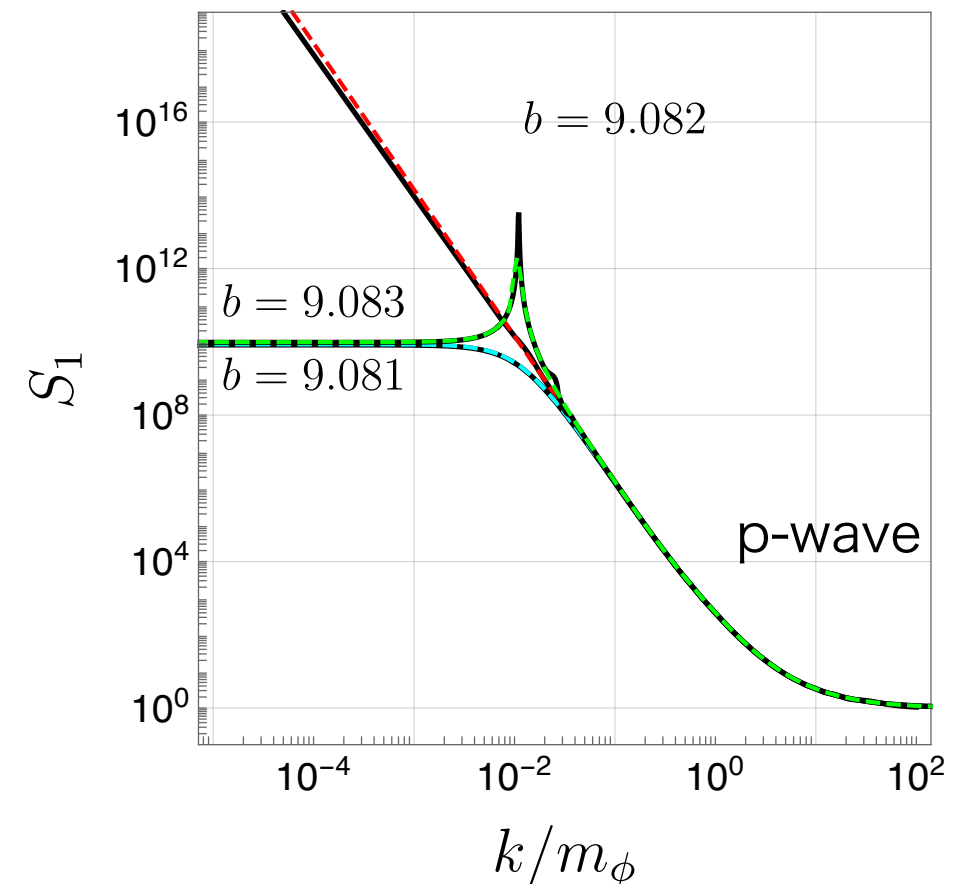
Kamada, TK, Patel (2023)

Near zero-energy resonance

black solid-lines $S_\ell(k^2) = \lim_{r \rightarrow 0} \frac{|R_{k,\ell}(r)|^2}{|R_{k,\ell}^{(0)}(r)|^2}$

colored dashed-lines $S_\ell(k^2) = |\Gamma_\ell(k^2)|^2$

- Our method precisely reproduces $S_\ell(k^2)$ from phase shift
- Even a resonance in p-wave is precisely reproduced



Mediator Emission

Bremsstrahlung Process

relative momentum of DMs: p to p'
emission mediator: k

Vector emission from two-DM system

(χ_1 and χ_2 with charge q_1 and q_2 : e.g. DM-DM or DM- $\overline{\text{DM}}$)

One of key processes to release DM energy in DM halo (e.g. gravothermal process)

Differential cross section

$$\frac{d\sigma_{\text{brems}}}{d\omega_k} = \frac{4|\mathbf{k}||\mathbf{p}'|}{|\mathbf{p}|} \frac{(q_1 - q_2)^2 g^2}{(\varepsilon' - \varepsilon)^4} \left[-\frac{1}{3} \mathbf{k}^2 + (\varepsilon' - \varepsilon)^2 \right] W_{2,p'p}$$

dipole approximation:

$$W_{2,p'p} \equiv (4\pi)^{-4} \int d\Omega_{\hat{\mathbf{p}'}} \mathbf{W}_{p'p} \cdot \mathbf{W}_{p'p}^*$$

$$\mathbf{W}_{p'p} \equiv \int d^3r \phi_p^*(\mathbf{r}) \nabla_r V_\chi(r) \phi_{p'}(\mathbf{r})$$

$\varepsilon - \varepsilon'$: relative energy difference

g : coupling vector mediator and DMs

Amplitude square

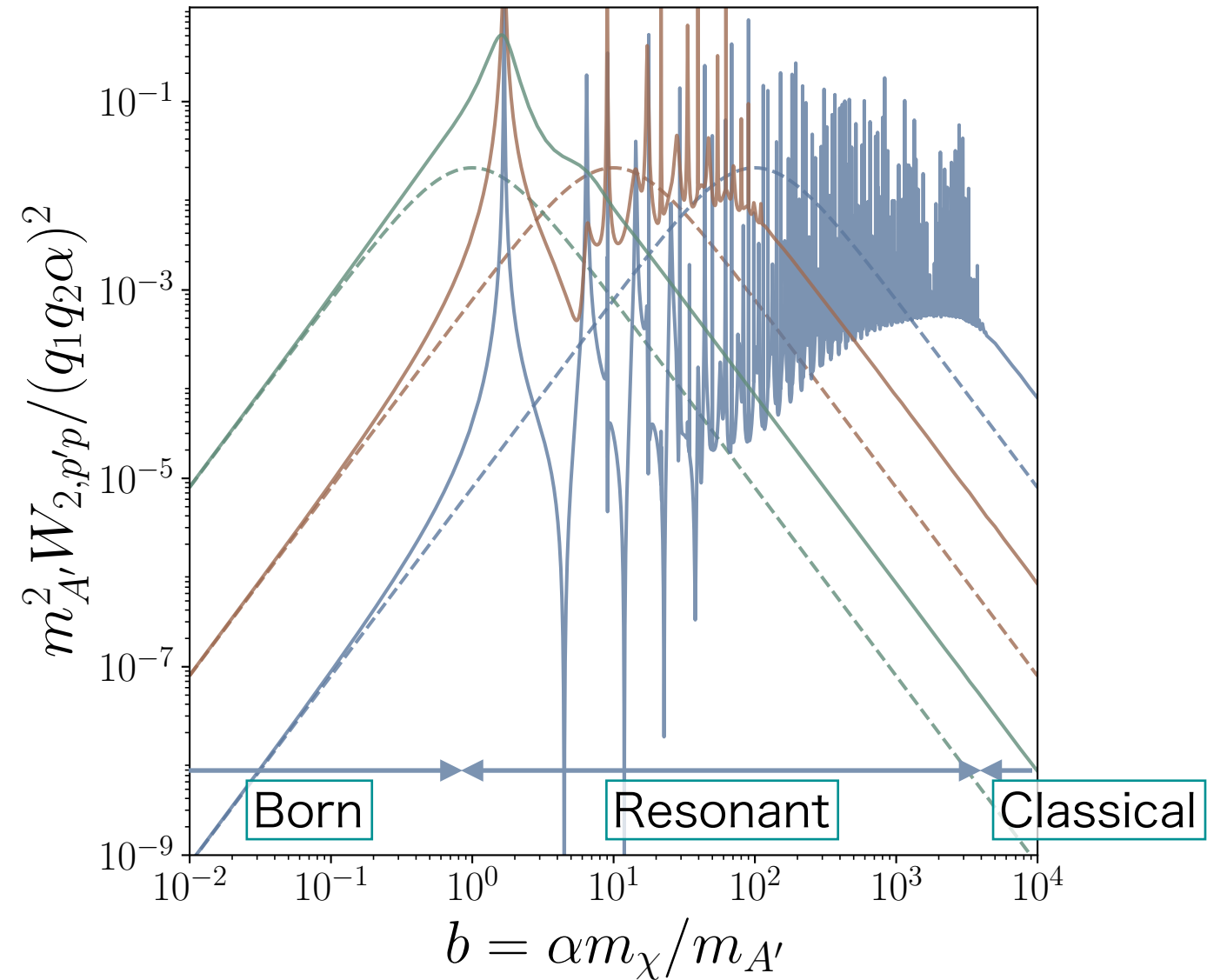
Kamada, TK, Schmidt, in prep

$$\frac{d\sigma_{\text{brems}}}{d\omega_k} = \frac{16 |\mathbf{k}| |\mathbf{p}'| \mu^2 (q_1 - q_2)^2 g^2}{|\mathbf{p}| m_\chi^2 (\epsilon' - \epsilon)^4} \left[-\frac{1}{3} \mathbf{k}^2 + (\epsilon' - \epsilon)^2 \right] W_{2,p'p}$$

partial-wave expansion

$$W_{2,p'p} = (4\pi)^{-1} \sum_{\ell} \left\{ (\ell + 1) |I_{\ell+}(p', p)|^2 + \ell |I_{\ell-}(p', p)|^2 \right\}$$

$$I_{\ell\pm}(p', p) = e^{\mp \frac{i\pi}{2} + i[\delta_{\ell}(p) + \delta_{\ell\pm 1}(p')]} \int r^2 dr R_{p', \ell\pm 1}(r) V'_\chi(r) R_{p, \ell}(r)$$



resonant regime:

due to zero-energy bound states

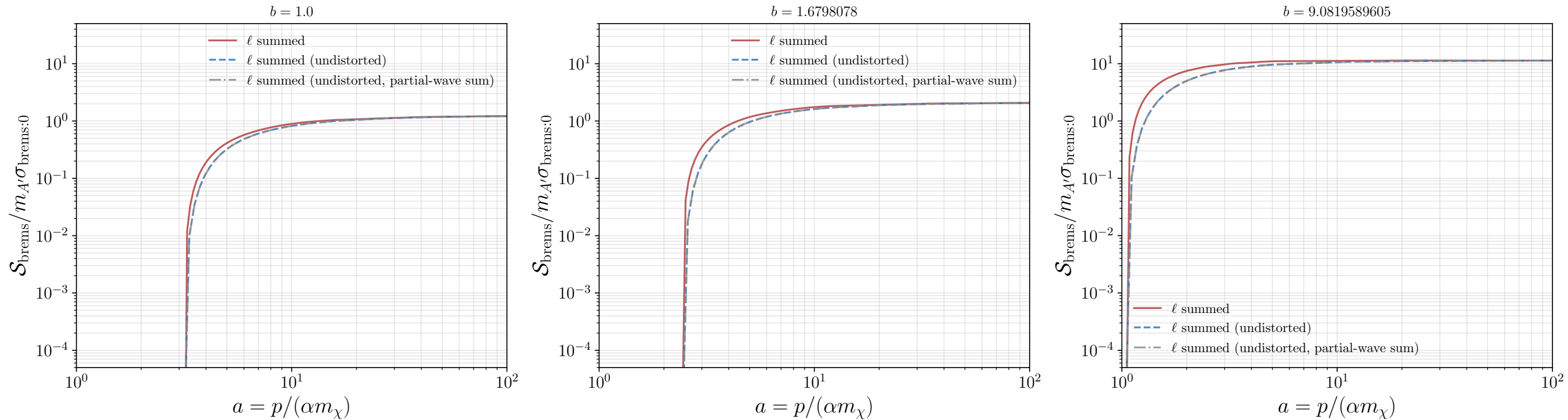
of initial/final two DM states

dipole emission is more complicated: two different partial-waves

$$a^{(\cdot)} = p^{(\cdot)} / \alpha m_\chi$$

$$a'/a = 0.1 \text{ and } a = 1.0, 0.1, 0.01$$

Energy weighted cross section Kamada, TK, Schmidt, in prep



A key ingredient for gravothermal collapse: energy-loss via emitting mediator particles

energy-weighted cross section

$$\mathcal{S}_{\text{brems}} = \int d\omega_k \omega_k \frac{d\sigma_{\text{brems}}}{d\omega_k}$$

$$\sigma_{\text{brems}:0} \equiv \frac{16\alpha^3}{3\mu^2} q_1^2 q_2^2 (q_1 - q_2)^2$$

enhanced cross section nearly at kinematical threshold: $|\mathbf{k}| \geq 0$ i.e. $a \geq (\alpha b)^{-1/2}$

Summary

- Dark-sector models for (sub-)GeV DM
 - ✓ self-interaction (small-scale problems) and annihilation into light mediator particle (indirect detection)
 - ✓ strong correlation of resonant enhancement
- **Beyond Born approximation** is sometimes required to evaluate physical processes
 - velocity dependent self-scattering cross section (**small-scale structure**)
 - annihilation, Sommerfeld enhancement (**indirect signals**)
 - mediator emission, e.g. Bremsstrahlung (**gravothermal collapse**)
- A theoretical relation between σ_{self} and S : Watson's theorem and Omnès function
 - astrophysical observations of self-scattering cross section in different velocity scales -> Sommerfeld enhancement factor

Backup Slides

More on Scattering

Effective Range Theory

Bethe (1949), Blatt, Jackson (1949)

invented for nucleon-nucleon scattering

Analytic property of wave-function determines δ_ℓ at $k = 0$

$$k^{2\ell+1} \cot \delta_\ell(k) \simeq -\frac{1}{a_\ell^{2\ell+1}} + \frac{1}{2r_{e,\ell}^{2\ell-1}} k^2$$

valid as far as $|k| \lesssim |r_{e,\ell}|^{-1}$ (k^4 and higher terms negligible)

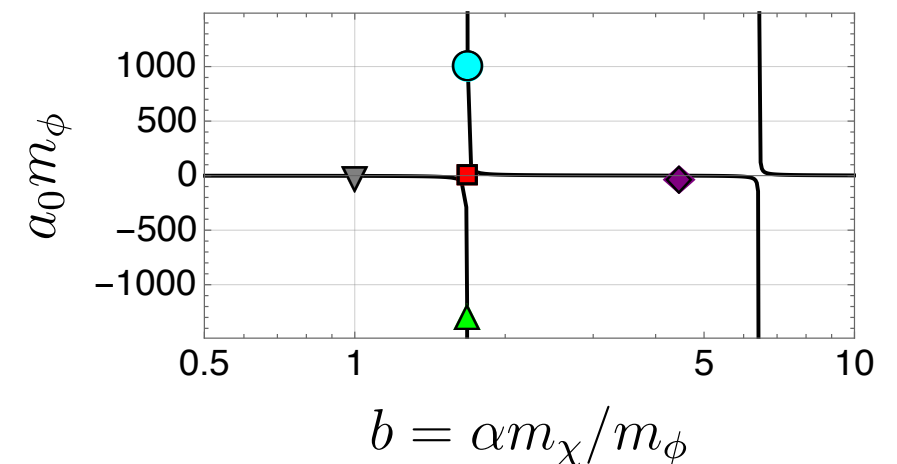
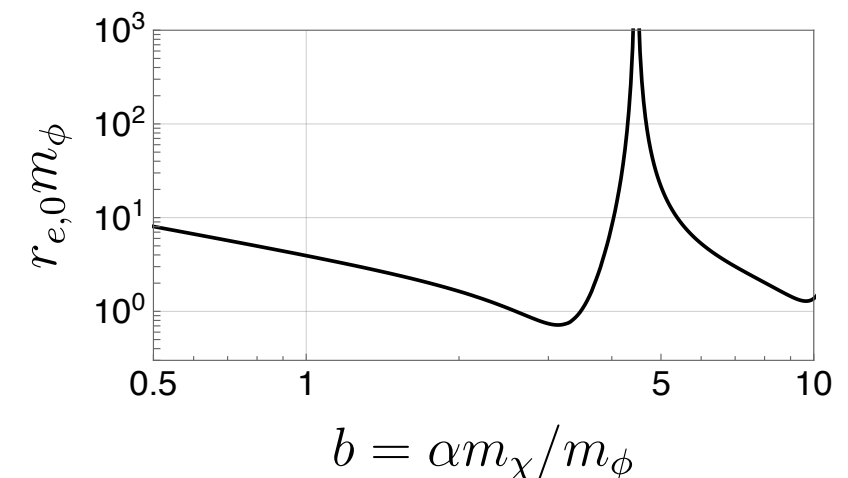
a_ℓ : scattering length

$r_{e,\ell}$: effective range

zero-energy states/shallow states

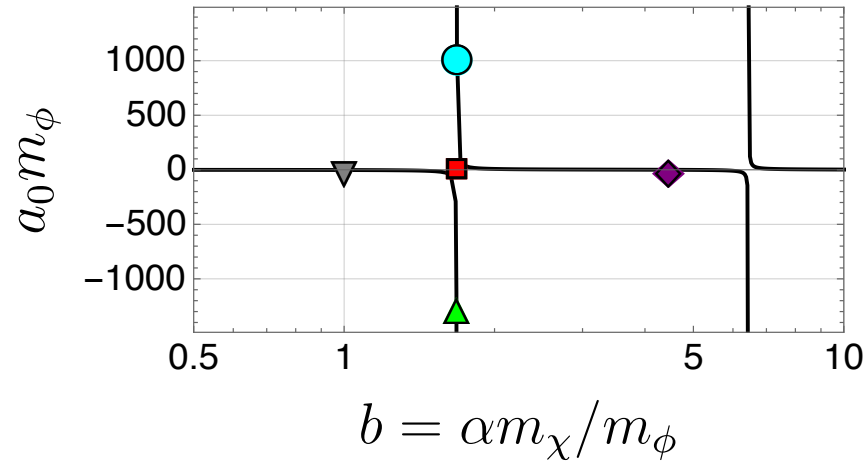
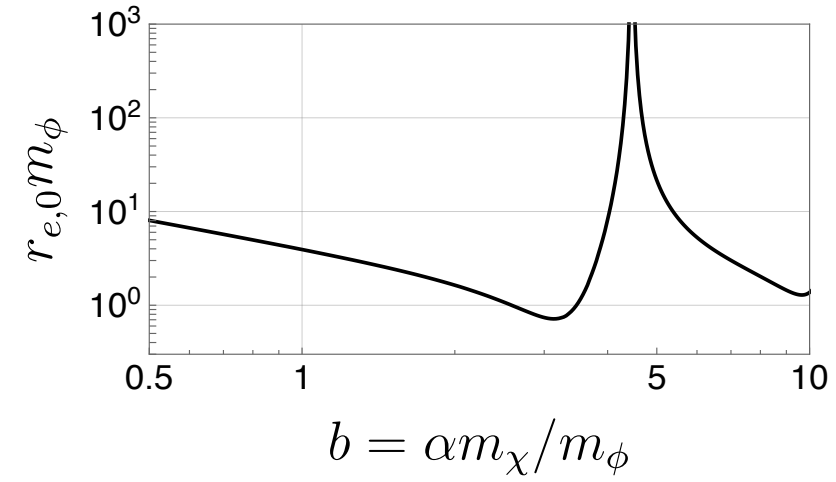
scattering amplitude may have poles at $k = i\kappa_b$

- zero-energy resonance ($|a_\ell| \rightarrow \infty$ and $\kappa_b \rightarrow 0$)
- bound state ($\kappa_b > 0$): normalizable state $e^{-\kappa_b r}$
- virtual levels ($\kappa_b < 0$): non-normalizable state $e^{-\kappa_b r}$



$$f_\ell(k) = \frac{1}{k \cot \delta_\ell - ik}$$

s-wave scattering

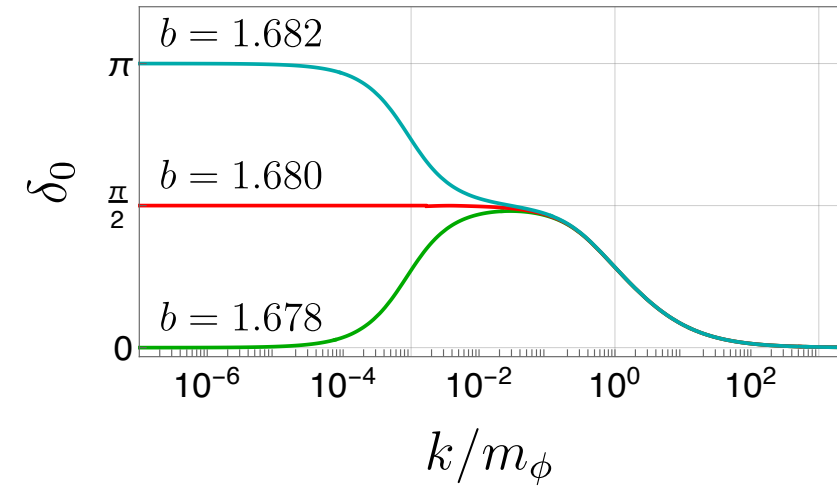


phase shift

$$k \cot \delta_0(k) \simeq -\frac{1}{a_0} + \frac{r_{e,0}}{2} k^2$$

$$\delta_0(k) \rightarrow 0, \pi \pmod{2\pi} \quad |a_0|^{-1} \neq 0$$

$$\delta_0(k) \rightarrow \frac{\pi}{2} \pmod{\pi} \quad |a_0|^{-1} = 0$$

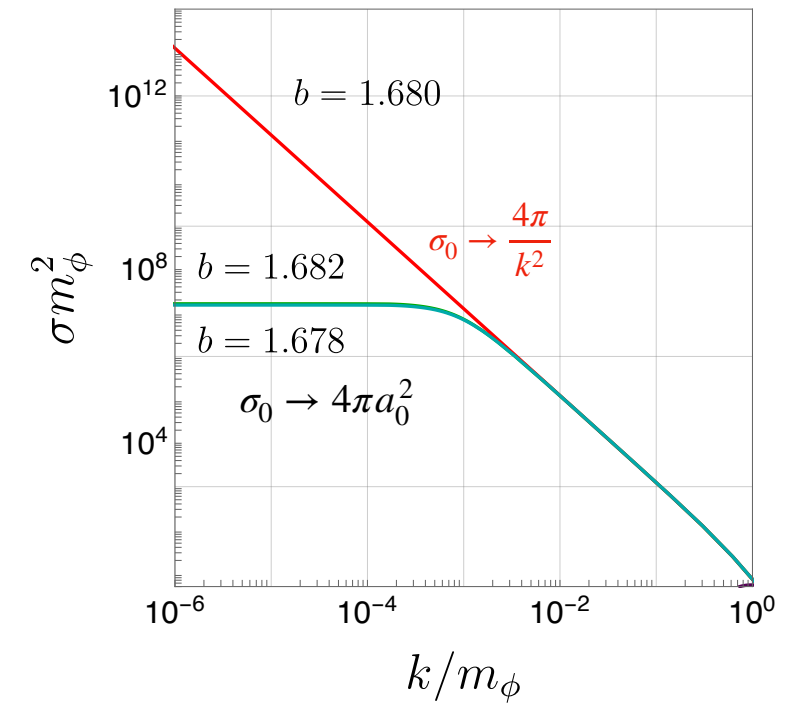


bound state

$$f_0(k) = \frac{1}{k \cot \delta_\ell - ik} \sim \left(-\frac{1}{a_0} - ik \right)^{-1} \quad \text{a pole at } k = i\kappa_b = ia_0^{-1}$$

$$a_0 > 0: \text{ bound state } \psi_b(x) \rightarrow \frac{e^{-\kappa_b r}}{r}$$

$a_0 < 0$: virtual level (non-normalizable bound state)



scattering cross section

$$\sigma_0 = \frac{4\pi}{k^2} \sin^2 \delta_0(k)$$

p-wave scattering

phase shift

$$k^3 \cot \delta_1(k) \simeq -\frac{1}{a_1^3} + \frac{1}{2r_{e,1}}k^2$$

$$\delta_0(k) \rightarrow 0, \pi \pmod{2\pi}$$

resonances

$$a_1 < 0 \text{ \& } r_{e,1} < 0$$

quick change of phase shift by $\sim \pi$ @ $k^2 = 2r_{e,1}/a_1^3$

sharply saturate the unitarity bound

bound state

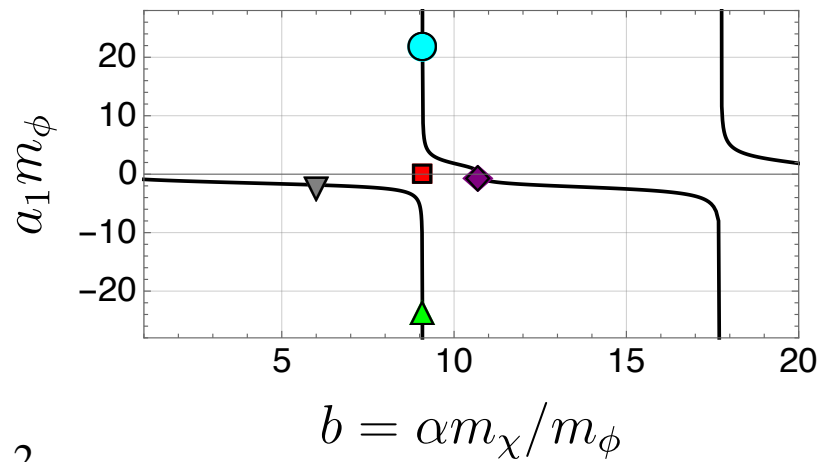
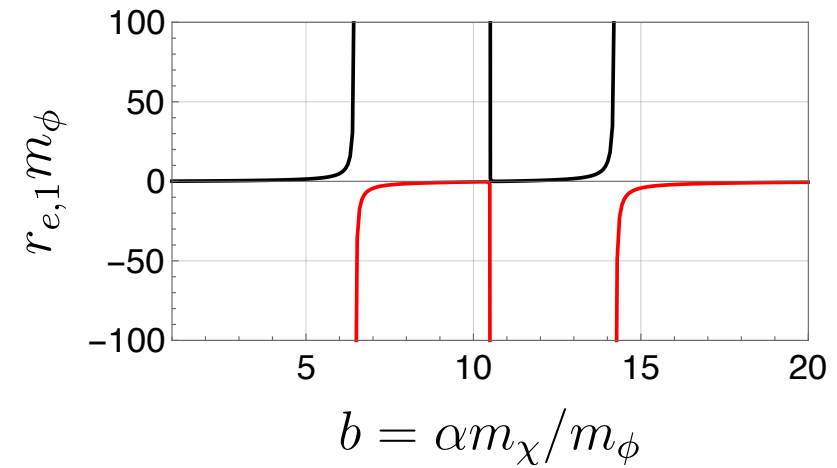
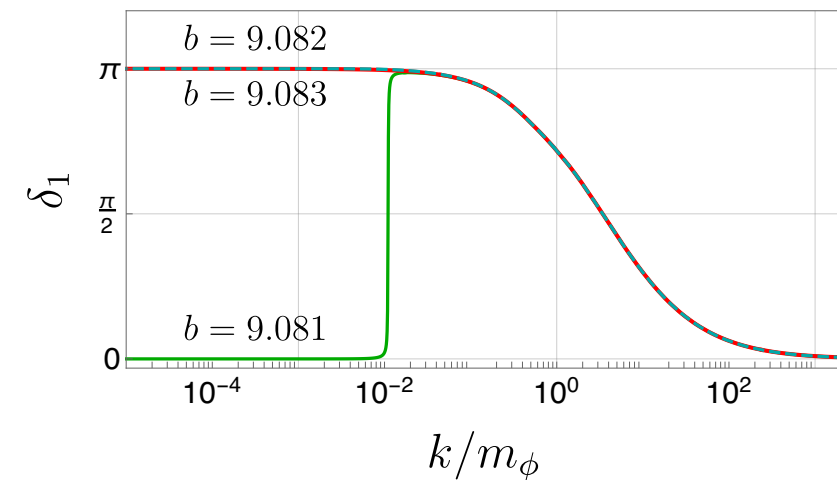
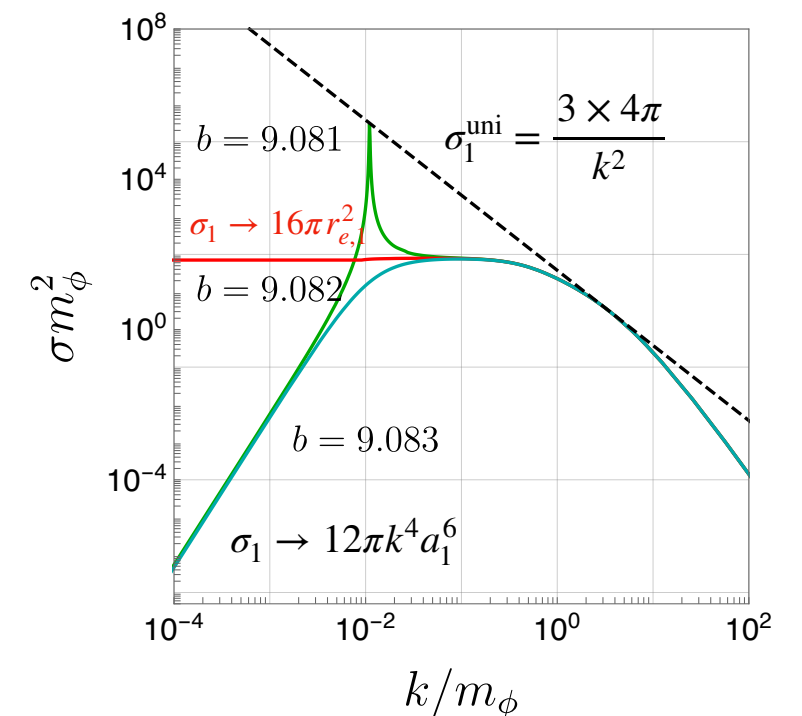
$$f_1(k) \sim k^2 \left(-\frac{1}{a_1^3} + \frac{k^2}{2r_{e,1}} - ik^3 \right)^{-1}$$

poles at $k = i\kappa_b = \pm i\sqrt{-\frac{2r_{e,1}}{a_1^3}}$

$a_1 > 0 \text{ \& } r_{e,1} < 0$: bound state and virtual level

scattering cross section

$$\sigma_1 = \frac{3 \times 4\pi}{k^2} \sin^2 \delta_1(k)$$



Levinson's theorem

Levinson (1949)

Weinberg, Lectures on QM (2015)

Connection b/w the behavior of phase shift and the number of bound states

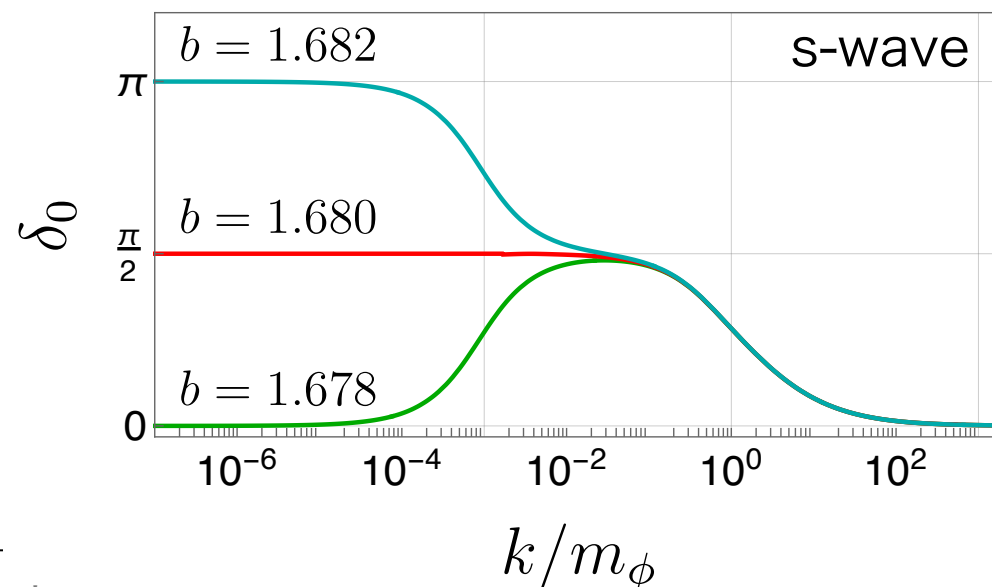
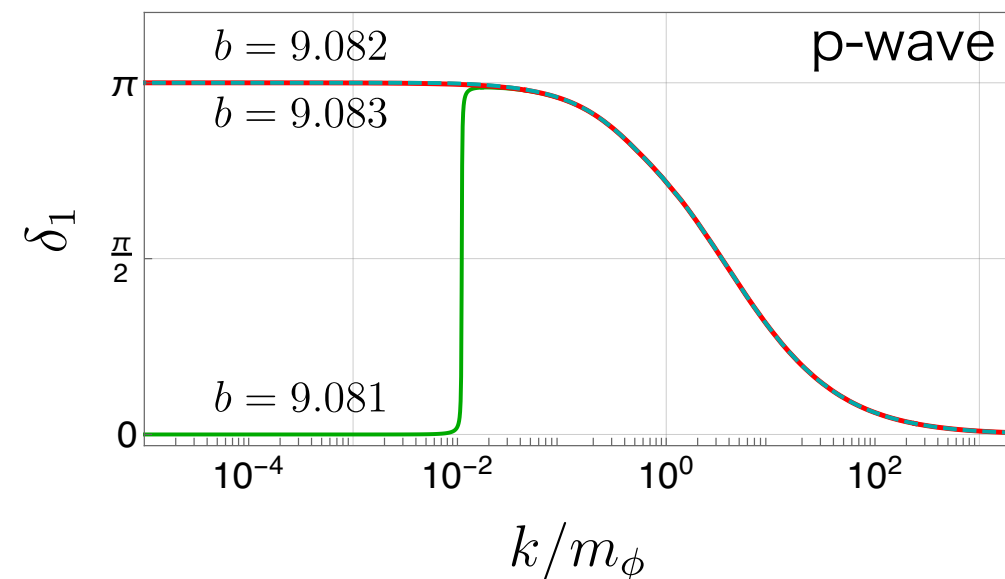
$$\delta_\ell(k=0) - \delta_\ell(\infty) = \left(n_\ell + \frac{1}{2}N \right) \pi$$

$N = 1$: on s-wave zero-energy resonance

$N = 0$: other case

- a portion of scattering states has negative energy in the presence of potential
- total number of states unchanged
- negative energy states converted into bound states

near zero-energy first resonance



Phase Shift and Effective Range Theory

at sufficiently large $r = a$

$$\tan \delta_\ell(k) = \frac{kaj'_\ell(ka) - \beta_\ell j_\ell(ka)}{kan'_\ell(ka) - \beta_\ell n_\ell(ka)}, \quad \beta_\ell = \frac{a\chi'_{k,\ell}(a)}{\chi_{k,\ell}(a)} - 1$$

$$\chi_{k,\ell} = rR_{k,\ell} \rightarrow \sin \left(kr - \frac{\ell\pi}{2} + \delta_\ell(k) \right)$$

The asymptotic behavior of Spherical Bessel for $ka \ll 1$

$$j_\ell(ka) \simeq \frac{1}{(2\ell+1)!!} (ka)^\ell, \quad n_\ell(ka) \simeq -(2\ell-1)!! (ka)^{-\ell-1}$$

At low energy, the phase shift behaves as

$$\tan \delta_\ell(k) \propto -\frac{(ka)^{2\ell+1}}{(2\ell+1)!!(2\ell-1)!!}$$

Therefore, the phase shift approaches to a constant value $a_\ell^{2\ell+1} \equiv -\lim_{k \rightarrow 0} \frac{\tan \delta_\ell(k)}{k^{2\ell+1}}$

and can be expanded in k^2

$$k^{2\ell+1} \cot \delta_\ell(k) \simeq -\frac{1}{a_\ell^{2\ell+1}} + \frac{1}{2r_{2,\ell}^{2\ell-1}} k^2 + O(k^4)$$

Levinson's theorem

Levinson (1949)

$$\delta_\ell(k=0) - \delta_\ell(\infty) = \left(n_\ell + \frac{1}{2}N \right) \pi$$

Rough sketch

Weinberg, Lectures on QM (2015)

Consider a system enclosed in a sphere (radius R)

$$k_n R - \frac{\ell \pi}{2} + \delta_\ell(k_n) = n\pi$$

wave-function vanishes at $r = R$

discretized momentum k_n

$$\frac{1}{\pi} [\delta_\ell(k_n) - \delta_\ell(k_{n'})] + \frac{1}{\pi} (k_n - k_{n'}) R = \underline{N_\ell(k_n)}$$

number of scattering states with $k_{n'} \leq k < k_n$

number of scattering states in the absence of potential

The variation of #(scattering state) due to the potential term

$$\Delta N_\ell(k_n) = \frac{1}{\pi} [\delta_\ell(k_n) - \delta_\ell(0)]$$

Total Hilbert space unchanged:

the scattering state with negative energy in the presence of potential convert into the bound state.

$$N_b = -\Delta N_\ell(\infty) = \frac{1}{\pi} [\delta_\ell(0) - \delta_\ell(\infty)]$$

More on Emission

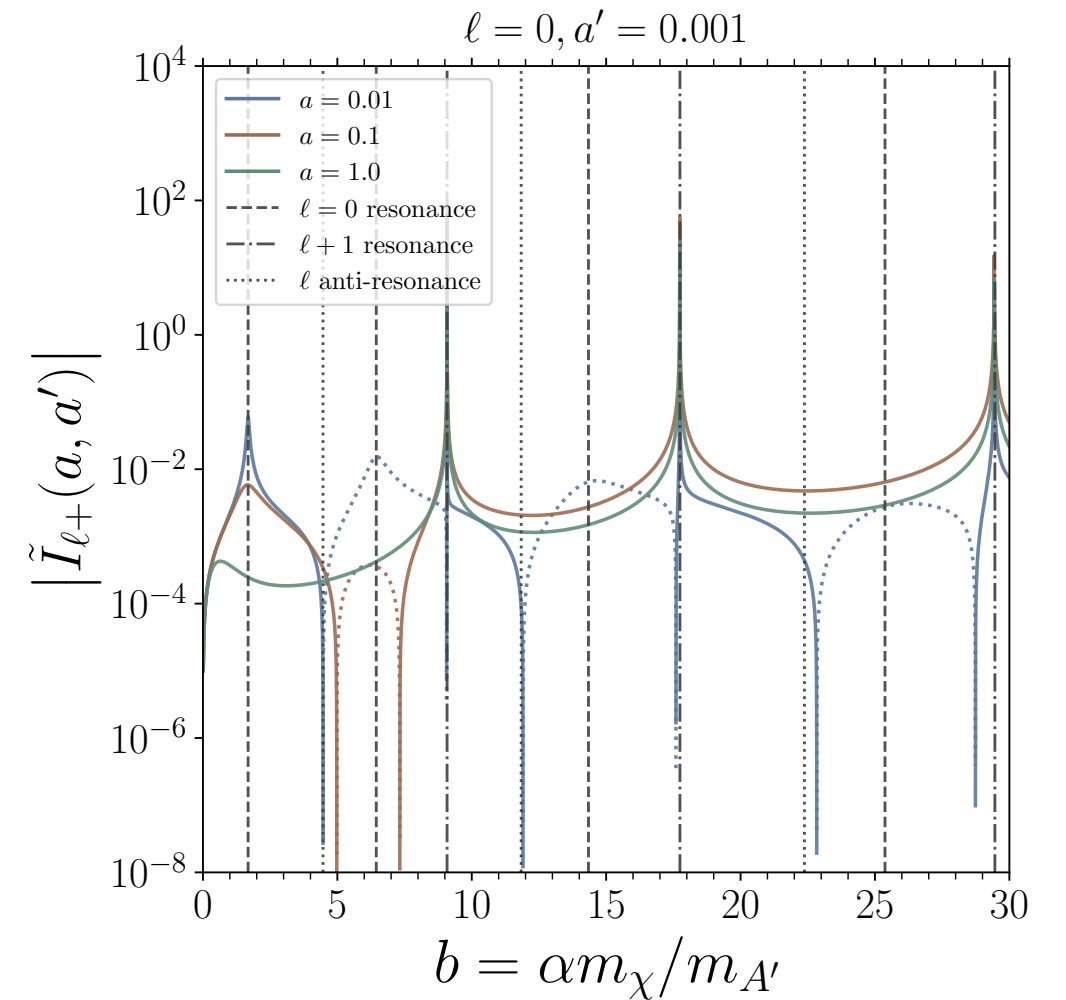
Multiple Resonance

$$W_{2,p'p} = (4\pi)^{-1} \sum_{\ell} \left\{ (\ell + 1) |I_{\ell+}(p', p)|^2 + \ell |I_{\ell-}(p', p)|^2 \right\}$$

$$I_{\ell\pm}(p', p) = e^{\mp \frac{i\pi}{2} + i[\delta_{\ell}(p) + \delta_{\ell\pm 1}(p')]} \int r^2 dr R_{p', \ell\pm 1}(r) V'_{\chi}(r) R_{p, \ell}(r)$$

Dipole emission is characteristic: both ℓ and $\ell + 1$
leads to complicated structure in resonant regime

Scalar emission: no change of partial waves



Cross section

