Binding energy of fermion-antifermion system in $0^{-+}/1^{--}$ channels by BS and Schrödinger eq

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Outline

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   (1) ladder BS equation for 0^{-+} and 1^{--}

   (2) nonrelativistic reduction at leading order

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   (2) physical hadronic scale

4. Summary

Motivation

Schrödinger / Lippmann–Schwinger equations are widely used to discuss the hadronic molecules.

Usually, the potential in these equations is

(1) fitted from experimental data (cross section…).

(2) matched to the scattering amplitude in QFT

(3) reduded from the relativistic BS equation

(4) ………

usually (2,3) *give* similar results. Our question is

What is the difference between the BS and Sch eqs when the same interaction is used.

Two aspects

1. small mass exchange and weak coupling limit:

In atomic physics, the coupling is weak and the mass of the exchanged gauge boson is zero. The non-relativistic (NR) approximation works well. What will happen for other type interactions and how weak is good enough?

2. physical scale/coupling with FF:

What will happen for NR approximation when the physical meson scale is considered.

We take an equal-mass fermion–antifermion system in 0^{-+} and 1^{--} channels within ladder BS as example to discuss these two properties.

BS equation in relativistic QFT

BS wave function and BS vertex.

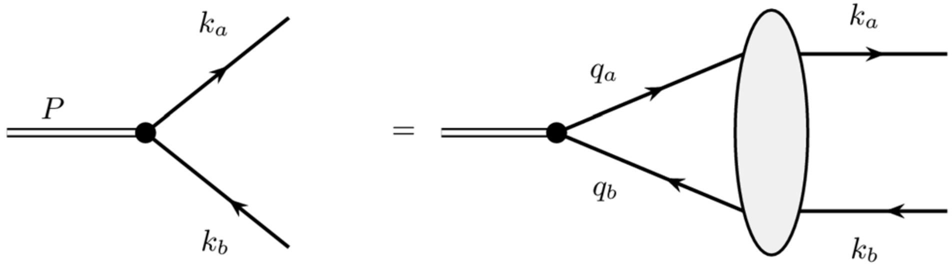
$$\chi(k,P) \equiv \int d^4x e^{ik\cdot x} \langle 0|\psi(x)\bar{\psi}(0)|P,..\rangle \equiv S(k_a)\,\Gamma(k,P)\,S(-k_b)$$

BS equation for the BS vertex

$$\Gamma(k,P) = \int \frac{d^4q}{(2\pi)^4}\, K_{\mathrm{BS}}(k,q)[S(q_a)\Gamma(q,P)S(-q_b)]$$

S: propagator of fermion

K_{BS}: interaction kernel.



In the rest frame: P = (M, 0), relative momenta k, q

$$k_a \equiv \frac{P}{2} + k, \qquad k_b \equiv \frac{P}{2} - k, \qquad q_a \equiv \frac{P}{2} + q, \qquad q_b \equiv \frac{P}{2} - q,$$

BS kernel in ladder approximation

In the weak coupling limit, thepoint-like interaction:

$$L_{\text{int}} \equiv g_S \bar{\Psi} \phi_S \Psi + i g_P \bar{\Psi} \gamma_5 \phi_P \Psi + g_V \bar{\Psi} \gamma^\mu V_\mu \Psi$$

the vertices and propagators

$$\Gamma_S^{\text{int}} = i g_S, \qquad D_S(\ell) \equiv \frac{i}{\ell^2 - m_S^2 + i\epsilon},$$

$$\Gamma_P^{\text{int}} = -g_P \gamma_5, \qquad D_P(\ell) \equiv \frac{i}{\ell^2 - m_P^2 + i\epsilon},$$

$$\Gamma_V^{\text{int},\mu} = i g_V \gamma^\mu, \qquad D_V^{\mu\nu}(\ell) \equiv \frac{-i(g^{\mu\nu} - (1-\xi)\ell^\mu \ell^\nu / m_V^2)}{\ell^2 - m_V^2 + i\epsilon}$$

the interaction kernels

$$K_{\text{BS}}^S(k,q)[A] \equiv \Gamma_S^{\text{int}} A \Gamma_S^{\text{int}} D_S(k-q),$$
$$K_{\text{BS}}^P(k,q)[A] \equiv \Gamma_P^{\text{int}} A \Gamma_P^{\text{int}} D_P(k-q),$$
$$K_{\text{BS}}^V(k,q)[A] \equiv \Gamma_V^{\text{int},\mu} A \Gamma_V^{\text{int},\nu} D_{\mu\nu}^V(k-q)$$

For vector particle, we take ξ = 1 as an approximation.

BS kernel in ladder approximation

strong-coupling case: dressed vertices by monopole FF

$$\Gamma_X^{\text{int}} \to \Gamma_X^{\text{int}} F_X(\ell^2), \qquad X \in \{S, P, V\}$$

with

$$F_X(\ell^2) \equiv \frac{m_X^2 - \Lambda^2}{\ell^2 - \Lambda^2 + i\epsilon}$$

Express BS vertex by scalar functions

For $0^{-+}/1^{--}$ states, we can write the BS vertices as

$$\Gamma(k,P)=\sum_b f_b(k_0,|\boldsymbol{k}|)B_b(k,P)$$

where

$$B^{0^-}_{1,2,3,4}(k)\equiv\left\{\gamma_5,\quad \gamma_5\frac{\not{P}}{M},\quad \gamma_5\frac{\not{k}_T}{M},\quad \gamma_5\frac{[\not{P},\not{k}_T]}{M^2}\right\}$$

$$\begin{aligned}
&B_1^\mu\equiv\gamma_T^\mu, && B_2^\mu\equiv\gamma_T^\mu\not{k}_T-k_T^\mu, && B_3^\mu\equiv\gamma_T^\mu\not{P}, && B_4^\mu\equiv\gamma_T^\mu[\not{k}_T,\not{P}]-2k_T^\mu\not{P},\\
&B_5^\mu\equiv k_T^\mu, && B_6^\mu\equiv k_T^\mu\not{k}_T, && B_7^\mu\equiv k_T^\mu\not{P}, && B_8^\mu\equiv k_T^\mu[\not{k}_T,\not{P}]
\end{aligned}$$

When C is fixed, the scalar functions have definite odd/even parity. So in the BS equation, $0^{--}/1^{-+}$ are also possible.

Express BS equation by scalar functions.

One can project the BS equation on some special Dirac matrix to get the equations for f. Finally, one have

$$\sum_b N_{ab}(k_0,|\boldsymbol{k}|)f_b(k_0,|\boldsymbol{k}|)=\sum_X\sum_b\int\frac{d^4q}{(2\pi)^4}G(q_0,|\boldsymbol{q}|;M)\,\bar{K}^{(X)}_{ab}(k,q,P)f_b(q_0,|\boldsymbol{q}|)$$

where

$$N_{ab}(k_0,|\boldsymbol{k}|)\equiv\mathcal{P}_a[B_b(k,P)]$$

$$\mathcal{P}_a[A]\equiv\frac{1}{\Omega_J}\int d\Omega_{\hat{k}}\,\mathrm{Tr}[B_a(k,P)A]$$

$$\bar{K}^{(X)}_{ab}(k,q,P)\equiv\mathcal{P}_a\left[K^X_{\mathrm{BS}}(k,q)\,[Q_+(q;P)B_b(q,P)Q_-(q;P)]\right]$$

$$G(q_0,|\boldsymbol{q}|;M)\equiv\frac{1}{(M/2+q_0)^2-|\boldsymbol{q}|^2-m^2+i\epsilon}\,\frac{1}{(M/2-q_0)^2-|\boldsymbol{q}|^2-m^2+i\epsilon}$$

$$Q_+(q;P)\equiv\not{q}_a+m,\qquad Q_-(q;P)\equiv\not{q}_b-m$$

Wick rotation

Direct numerical solution in Minkowski space is a little difficult. For a bound state, the Wick rotation of the BS equation does not change the binding energy.

$$k_E \equiv (k_0^E, \boldsymbol{k}) \equiv (-ik_0, \boldsymbol{k}), q_E \equiv (q_0^E, \boldsymbol{q}) \equiv (-iq_0, \boldsymbol{q})$$

$$\sum_b N_{ab}(ik_0^E, |\boldsymbol{k}|) f_b(ik_0^E, |\boldsymbol{k}|) = \sum_X \sum_b \int_{-\infty}^{\infty} dq_0^E \int_0^{\infty} d|\boldsymbol{q}|\, |\boldsymbol{q}|^2 G_E(q_0^E, |\boldsymbol{q}|; M) \times \mathcal{K}_{ab}^{(X)}(k_0^E, |\boldsymbol{k}|; q_0^E, |\boldsymbol{q}|, M) f_b(iq_0^E, |\boldsymbol{q}|)$$

$$\mathcal{K}_{ab}^{(X)}(k_0^E, |\boldsymbol{k}|; q_0^E, |\boldsymbol{q}|, M) \equiv \frac{i}{(2\pi)^4} \int d\Omega_{\hat{q}}\, \bar{K}_{ab}^{(X)}(k, q, P)$$

From BS eq to Schrödinger eq

To reduce the BS equation to the Schrödinger form, we apply the following two approximations

(1): 3D reduction

$$G(q,P) \approx -iG_{NR}(\boldsymbol{q},E_b)\delta(q_0), G_{NR}(\boldsymbol{q},E_b) \equiv \frac{\pi}{2m^2}\frac{1}{-E_b-\boldsymbol{q}^2/(2\mu)+i0}$$

(2): static approximation

$$Q_+(q)\Gamma Q_-(q) \approx -4m^2 P_+ \Gamma P_-$$

$E_b = 2m - M, \quad \mu = m/2$

From BS eq to Schrödinger eq

Finally, we can get a Schrödinger-like equation

$$g_J(|\boldsymbol{k}|) = \int \frac{d^3\boldsymbol{q}}{(2\pi)^3} \widetilde{V}_{\rm cent}(|\boldsymbol{k}-\boldsymbol{q}|) \frac{g_J(|\boldsymbol{q}|)}{-E_b - \boldsymbol{q}^2/(2\mu) + i0}, \qquad J \in \{0^{-+}, 1^{--}\}$$

where

$$\widetilde{V}_{\rm cent}(\ell) \equiv -\frac{g_S^2}{\ell^2 + m_S^2} - \frac{g_V^2}{\ell^2 + m_V^2}, \qquad \ell \equiv |\boldsymbol{k}-\boldsymbol{q}|, \qquad \widetilde{V}_P = 0 \quad \text{at } O(v^0)$$

$$g_{0^{-+}}(|\boldsymbol{k}|) \equiv f_1^{0^{-+}}(0, |\boldsymbol{k}|) - f_2^{0^{-+}}(0, |\boldsymbol{k}|),$$
$$g_{1^{--}}(|\boldsymbol{k}|) \equiv f_1^{1^{--}}(0, |\boldsymbol{k}|) - M f_3^{1^{--}}(0, |\boldsymbol{k}|)$$

BS equation vs. Schrödinger equation

Short summary

$$\sum_b N_{ab}(ik_0^E, |\boldsymbol{k}|) f_b(ik_0^E, |\boldsymbol{k}|) = \sum_X \sum_b \int_{-\infty}^{\infty} dq_0^E \int_0^{\infty} d|\boldsymbol{q}|\, |\boldsymbol{q}|^2 G_E(q_0^E, |\boldsymbol{q}|; M)$$
$$\times \mathcal{K}_{ab}^{(X)}(k_0^E, |\boldsymbol{k}|; q_0^E, |\boldsymbol{q}|, M) f_b(iq_0^E, |\boldsymbol{q}|)$$

$$g_J(|\boldsymbol{k}|) = \int \frac{d^3\boldsymbol{q}}{(2\pi)^3} \widetilde{V}_{\mathrm{cent}}(|\boldsymbol{k}-\boldsymbol{q}|) \frac{g_J(|\boldsymbol{q}|)}{-E_b - \boldsymbol{q}^2/(2\mu) + i0}, \qquad J \in \{0^{-+}, 1^{--}\}$$

Numerical method

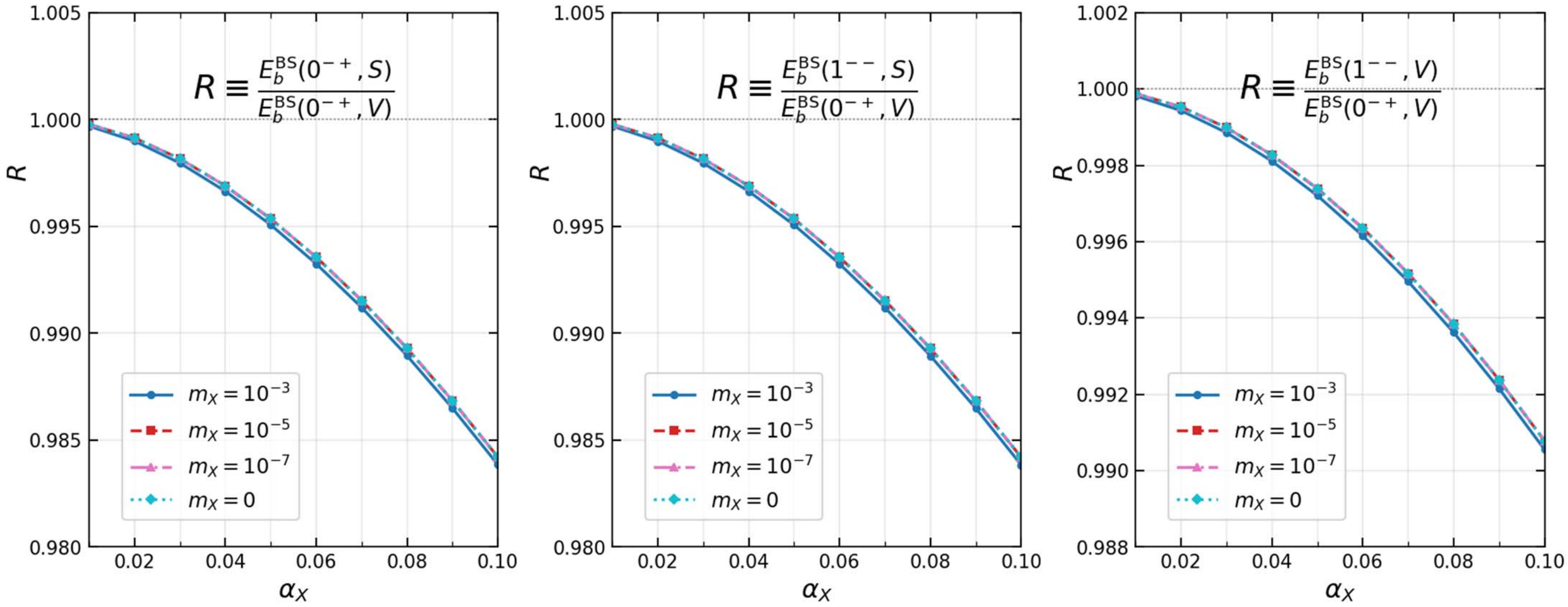
1. We discretize directly on the Euclidean relative energy q_0^E and the magnitude $|q|$ .The integrations are evaluated by Gauss--Legendre quadrature after a mapping.

2. for m_X=0, the analytical expression for a small region integral is applied.

3. Rewrite the equation as aneigenvalue problem in λ: physical bound state $\Leftrightarrow$ $\lambda(E_b) = 1$

Results in small mass, weak coupling limit

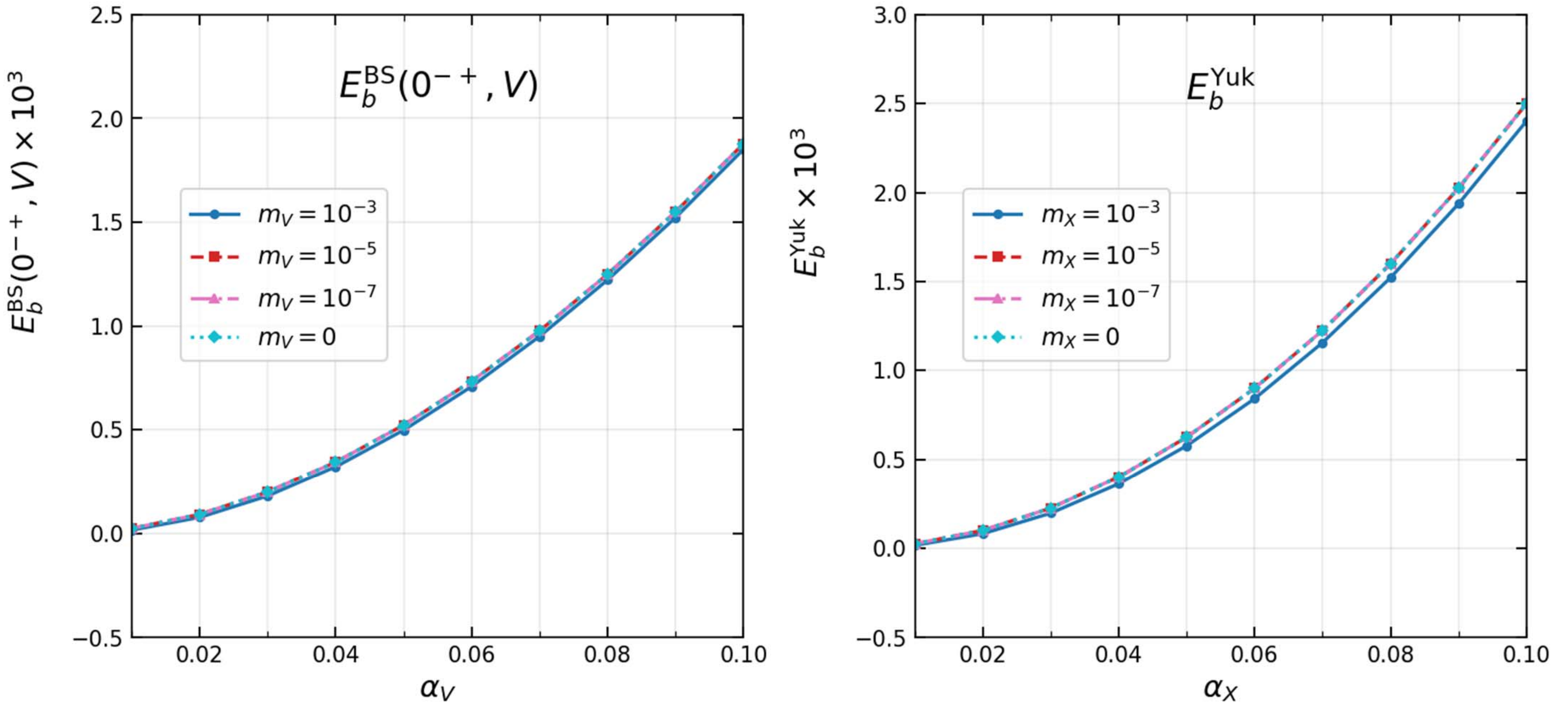
1: spin splitting in BS spectrum



The spin splitting in BS spectrum is very small.

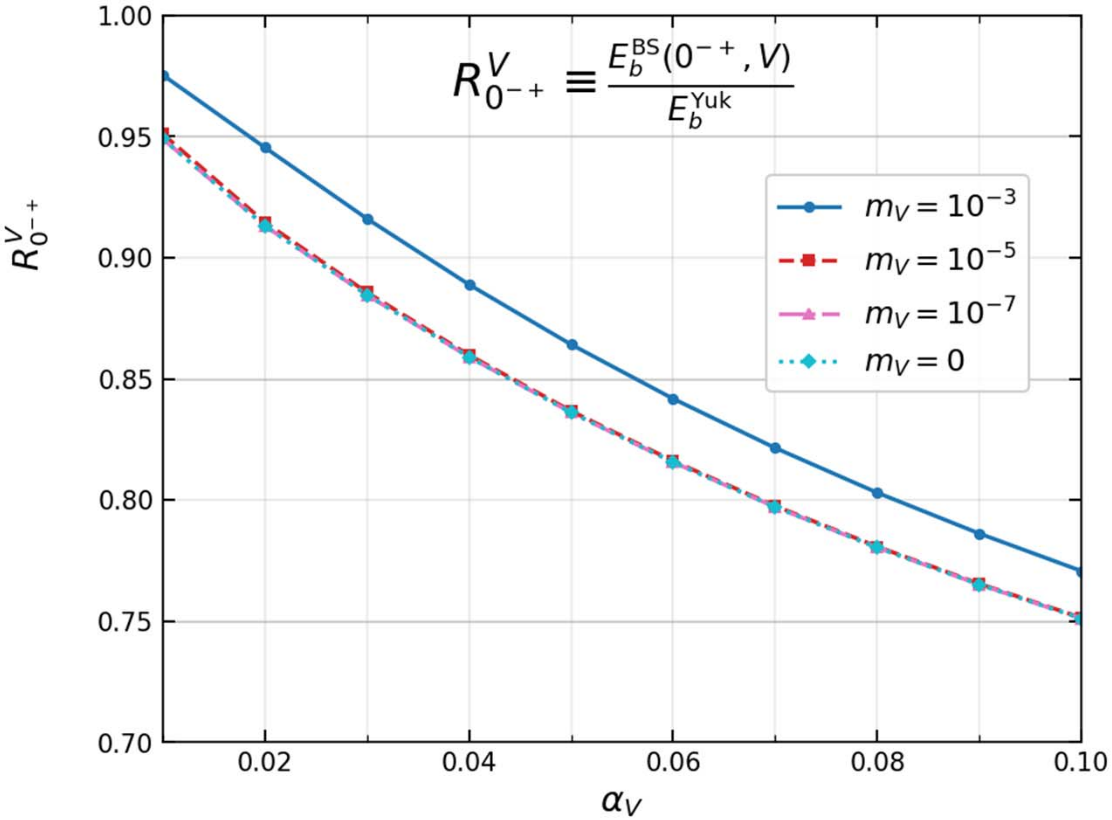
Results in small mass, weak coupling limit

2: absolute values of BS spectrum and Yukawa



Results in small mass, weak coupling limit

3.BS/Sch



The difference between the BS spectrum and NR spectrum is as large as $O(\alpha)$.

In hydrogen, the relativistic correction is at order $O(\alpha^2)$.

Results in small mass, weak coupling limit

1: why is it so different from hydrogen case?

QED is a gauge theory. For a bound state, Coulomb gauge is a natural choice, where at leading order, the instantaneous approximation is exact.

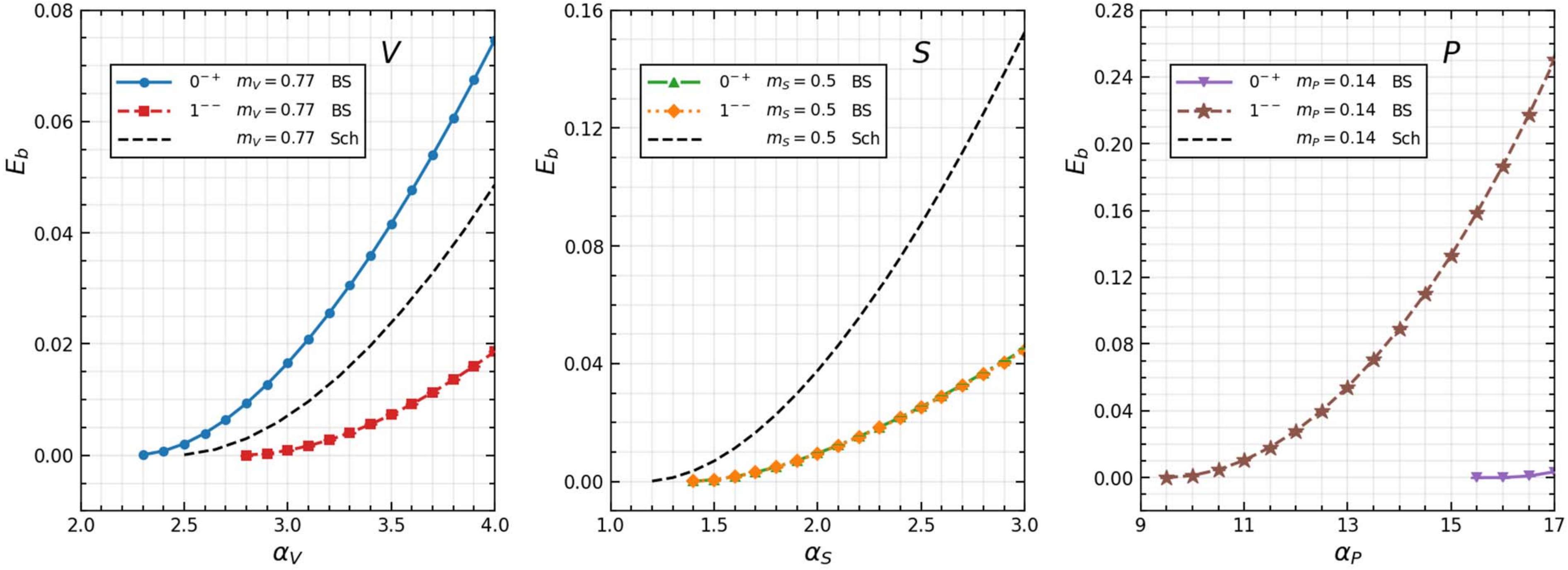
Feynman gauge vs. Coulomb gauge

crossed-ladder diagrams are important in Feynman gauge.

2: for massless scalar-exchange, there is no gauge freedom, the BS-Sch difference still exists.

How the crossed-ladder corrections change the results is an open question.

Results with larger masses and form factors



$$E_b^{\rm BS}(0^{-+}, V) > E_b^{\rm Yuk}(V) > E_b^{\rm BS}(1^{--}, V),$$
$$E_b^{\rm Yuk}(S) > E_b^{\rm BS}(0^{-+}, S) \simeq E_b^{\rm BS}(1^{--}, S),$$
$$E_b^{\rm BS}(1^{--}, P) \gg E_b^{\rm BS}(0^{-+}, P), \qquad E_b^{\rm Yuk}(P) \sim 0$$

m = 1.00 GeV, Λ = 2.00 GeV (monopole)

m_S = 0.50 GeV, m_P = 0.14 GeV, m_V = 0.77 GeV

Results with larger masses and form factors

Numerical ratios E^{BS}/E^{Sch}

Exchange	α_X	$E_b^{\rm BS}(0^{-+})$	$E_b^{\rm BS}(1^{--})$	$E_b^{\rm Yuk}$	$R^X_{0^{-+}}$	$R^X_{1^{--}}$
V, $m_V = 0.77$ GeV	3.0	0.016582	0.000879	0.007045	2.354	0.125
V, $m_V = 0.77$ GeV	4.0	0.074666	0.018627	0.048661	1.534	0.383
S, $m_S = 0.50$ GeV	2.0	0.009595	0.009491	0.037669	0.255	0.252
S, $m_S = 0.50$ GeV	3.0	0.044998	0.044494	0.152545	0.295	0.292
P, $m_P = 0.14$ GeV	10.0	–	0.001246	–	–	–
P, $m_P = 0.14$ GeV	17.0	0.003500	0.250349	–	–	–

In the Wick–Cutkosky model, crossed-ladder contributions reduce the difference.

Whether crossed-ladder contributions can reduce the large BS-Yukawa difference found here remains an open question.

Summary

1. light mass exchange, weak coupling:
 the BS-Sch difference is at order $O(\alpha)$ especially for scalar exchange.

2. physical masses and couplings with FF:
 scalar exchange: nearly degenerate
 vector exchange: favors 0^{-+} with a large splitting
 pseudoscalar exchange: prefers 1^{--}

3. crossed-ladder contribution may be is important.

Thanks!

Any comments and discussions are welcome.