The improved moment QCD sum rules

张锦鹏
中山大学物理学院

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张锦鹏，陈绪梁，陈伟

第九届强子谱和强子结构研讨会

QCD sum rules

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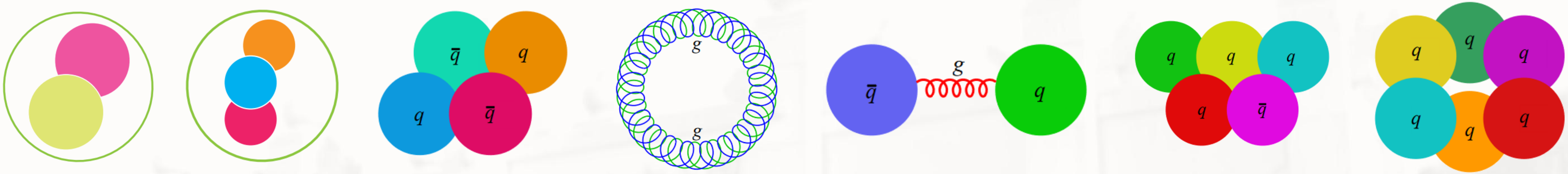
- QCD sum rules are widely used non-perturbative tools in hadron physics.

Nucl.Phys.B 147 (1979) 385-447
Nucl.Phys.B 147 (1979) 448-518
Phys.Rept. 127 (1985) 1



Annals Phys. 254 (1997) 328-396 *Phys.Rev.D 110 (2024) 11, 114008*

- Since 1979, several approaches have been developed: Monte Carlo-based method, variable continuum threshold, inverse problem, inverse matrix method, resonance sum rules, finite energy and Gaussian sum rules.

Phys.Rev.D 102 (2020) 114014
2211.13753[hep-th]

Phys.Rev.D 104 (2021) 11, 114017
Phys.Rev.D 106 (2022) 3, 034015
Chin.Phys.Lett. 41 (2024) 10, 101101
Eur.Phys.J.C 84 (2024) 10, 1105

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Nucl.Part.Phys.Proc. 324-329 (2023) 94-106

- Most commonly versions are Laplace and moment sum rules.

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Phys.Rept. 127 (1985) 1
hep-ph/0010175 [hep-ph]

- Laplace: based on Borel transform and “windows”—— OPE convergence and pole dominance
- Moment: take the n-th derivative of correlation function. Directly obtain the mass plateau.
- Those two methods have respective drawbacks.

Laplace QCD sum rules

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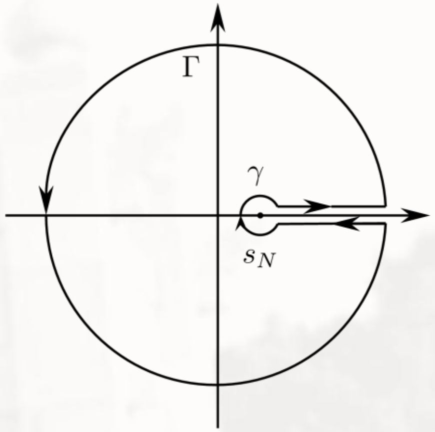
- It always starts from two-point correlation function: $\Pi(q^2) = \mathrm{i}\int \mathrm{d}^d x\, \mathrm{e}^{\mathrm{i}q\cdot x} \langle 0|\mathrm{T}[J(x)J^+(x)]|0\rangle$
- At quark-gluon level, it is computed via the OPE: $\Pi^{\mathrm{OPE}}(q^2) = \sum C_n(q^2,\mu)\langle \mathcal{O}_n(\mu)\rangle$
- On phenomenological side, $\Pi(q^2)$ is described by the dispersion relation



$$\Pi^{\mathrm{PH}}(q^2) = \int_{s_N}^{\infty} \mathrm{d}s \frac{(q^2)^n \rho^{\mathrm{PH}}(s)}{s^n(s-q^2-i0^+)} + \sum_{i=0}^{n-1} a_i (q^2)^i$$

- $\rho^{\mathrm{PH}}(s)$ is parametrized by "narrow resonance" assumption $\rho^{\mathrm{PH}}(s) = f_X^2\delta(s-m_X^2) + \rho^h(s)\theta(s-s_h)$
- The crucial step is quark-gluon duality $\rho^h(s) = \frac{1}{\pi}\mathrm{Im}\Pi^{\mathrm{OPE}}(s)\theta(s-s_0)$ or $\int_{s_h}^{\infty}\frac{\rho^h(s)\mathrm{d}s}{s-q^2} = \int_{s_0}^{\infty}\frac{\rho^{\mathrm{OPE}}(s)\mathrm{d}s}{s-q^2}$. This assumption removes the continuum $\rho^h(s)$ and introduces the duality parameter s_0.
- Borel transform is employed to improve the convergence of OPE, eliminate the subtraction terms…

$$\hat{B} = \lim_{\substack{-q^2, n\to\infty \\ -q^2/n = M_B^2}} \frac{1}{\Gamma(n+1)}(-q^2)^{n+1}\left(\frac{\mathrm{d}}{\mathrm{d}q^2}\right)^n$$

Laplace QCD sum rules

- We obtain the main equation $f_X^2 e^{-m_X^2/M_B^2} = \int_{s_N}^{s_0} \rho^{\mathrm{OPE}}(s) e^{-s/M_B^2} ds$ and hadron mass is extracted to be

$$m_X^2 = \frac{\int_{s_N}^{s_0} \rho^{\mathrm{OPE}}(s)\, s\, e^{-s/M_B^2} ds}{\int_{s_N}^{s_0} \rho^{\mathrm{OPE}}(s)\, e^{-s/M_B^2} ds}$$

- m_X depends on s_0 and M_B^2. They are constrained by convergence of OPE and pole contribution (PC).

- They are inherently subjective.

$$\left|\frac{\Pi_{D=6}(M_B^2)}{\Pi_{\mathrm{pert}}(M_B^2)}\right| \leqslant 25\%, \left|\frac{\Pi_{D=8}(M_B^2)}{\Pi_{\mathrm{pert}}(M_B^2)}\right| \leqslant 10\%.$$

$$\frac{m_Q^{12}\langle g_s^2 G^2\rangle \int_{36}^{\infty} e^{-\tilde{s}/\tilde{M}_B^2} \rho^{\langle g_s^2 G^2\rangle}(\tilde{s}) d\tilde{s}}{m_Q^{16}\int_{36}^{\infty} e^{-\tilde{s}/\tilde{M}_B^2} \rho^{\mathrm{pert}}(\tilde{s}) d\tilde{s}} \leq 30\%$$

$$\mathrm{CVG} \equiv \left|\frac{\Pi_-^{12+13}(\infty, M_B^2)}{\Pi_-(\infty, M_B^2)}\right| \leq 5\%$$

$$\mathrm{CVG} \equiv \left|\frac{\Pi_-^{6+7+8}(\infty, M_B^2)}{\Pi_-(\infty, M_B^2)}\right| \leq 10\%$$

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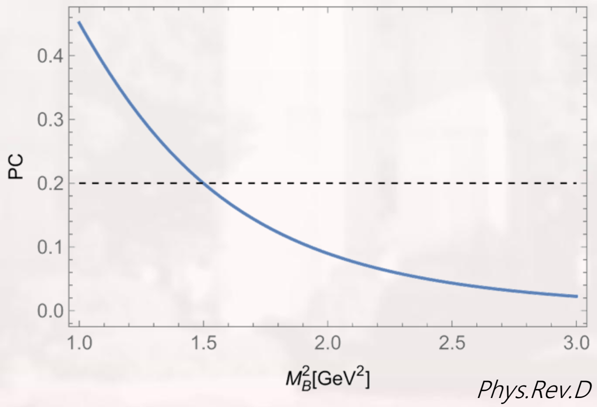
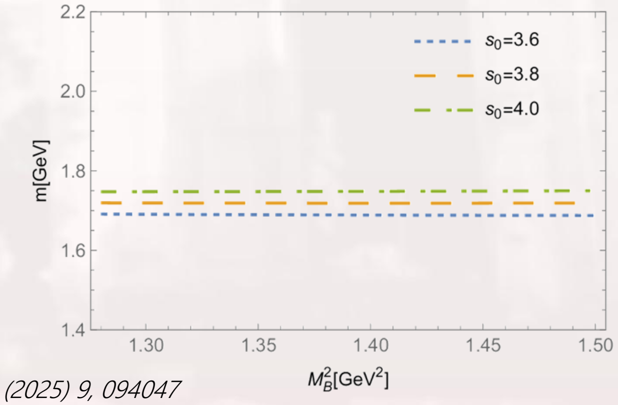
Phys.Rev.D 113 (2026) 11, 114015
Chin.Phys.C 50 (2026) 083106

- Definition of pole contribution remains questionable

$$L_k(s_0, M_B^2) = \int_{s_N}^{s_0} \rho^{\mathrm{OPE}}(s) s^k e^{-s/M_B^2}\, ds + \Pi^{\mathrm{NS}}(M_B^2) \qquad \mathrm{PC}(s_0, M_B^2) := \frac{L_0(s_0, M_B^2)}{L_0(\infty, M_B^2)}$$

Within Borel window, the mass is stable, whereas the PC varies considerably. If PC were physical, it should not be sensitive to M_B^2.





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Moment QCD sum rules

- It defines moments by taking derivatives of $\Pi(q^2)$

$$M_n(Q_0^2) = \frac{1}{\Gamma(n+1)}\left(-\frac{\mathrm{d}}{\mathrm{d}Q^2}\right)^n \Pi(Q^2)|_{Q^2=Q_0^2}$$

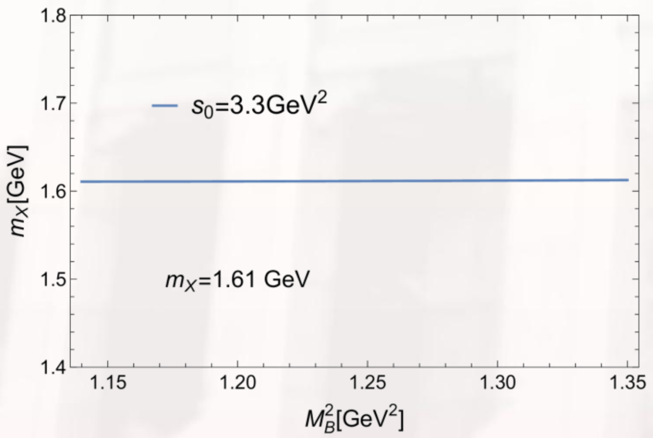
- The phenomenological side also adopts "narrow resonance"

$$M_n^{\mathrm{PH}}(Q_0^2) = \frac{f_X^2}{(m_X^2+Q_0^2)^{n+1}}[1+\delta_n(s_h,Q_0^2)] \qquad \delta_n(s_h,Q_0^2) = \frac{(m_X^2+Q_0^2)^{n+1}}{f_X^2}\int_{s_h}^{\infty}\frac{\rho^h(s)}{(s+Q_0^2)^{n+1}}\mathrm{d}s$$

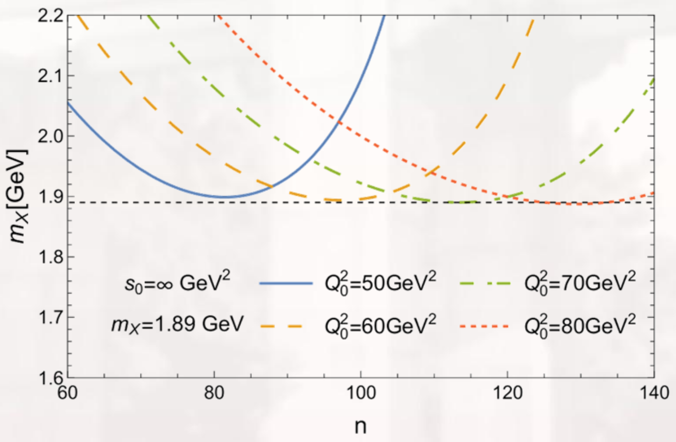
- Consider the ratio of moments $r^{\mathrm{PH}}(n,Q_0^2) = \frac{M_n^{\mathrm{PH}}(Q_0^2)}{M_{n+1}^{\mathrm{PH}}(Q_0^2)} = (m_X^2+Q_0^2)\frac{1+\delta_n(s_h,Q_0^2)}{1+\delta_{n+1}(s_h,Q_0^2)}$ $\qquad r^{\mathrm{OPE}}(n,Q_0^2) = \frac{M_n^{\mathrm{OPE}}(Q_0^2)}{M_{n+1}^{\mathrm{OPE}}(Q_0^2)}$
- For sufficiently large n, one expects that $\delta_n(s_h,Q_0^2) \approx \delta_{n+1}(s_h,Q_0^2) \quad \Rightarrow \quad r^{\mathrm{PH}}(n,Q_0^2) = m_X^2 + Q_0^2$
- The mass is extracted to be $m_X = \sqrt{r^{\mathrm{OPE}}(n,Q_0^2) - Q_0^2}$
- Moment method yields a larger mass than Laplace sum rules.
- Ignorance of $\rho^h(s)$ prevents the determination of coupling f_X.

Improved moment sum rules —— core idea

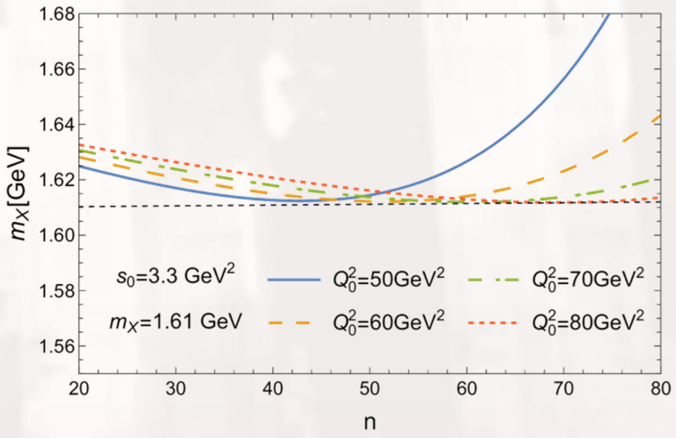
- Approximation $\delta_n(s_h, Q_0^2) \approx \delta_{n+1}(s_h, Q_0^2)$ acts only on the phenomenological side.
- $(m_X^2 + Q_0^2)\frac{1+\delta_n(s_h,Q_0^2)}{1+\delta_{n+1}(s_h,Q_0^2)} = \frac{M_n^{\mathrm{OPE}}(Q_0^2)}{M_{n+1}^{\mathrm{OPE}}(Q_0^2)}$ $\qquad M_n^{\mathrm{OPE}}(Q_0^2) = \int_{s_N}^{\infty}\frac{\rho^{\mathrm{OPE}}(s)}{(s+Q_0^2)^{n+1}}ds + M_n^{\mathrm{NS}}(Q_0^2)$
- On the OPE side, integration over the $\rho^{\mathrm{OPE}}(s)$ extends to ∞. It inevitably receives contributions from higher excited states, which naturally pushes the mass upward.
- Using the same upper limit of integration s_0 eliminates the discrepancy between two methods.



Laplace sum rules



Moment sum rules with $s_0 = \infty$



Moment sum rules with $s_0 = 3.3\mathrm{GeV}^2$

Improved moment sum rules —— core idea

- s_0 originates from quark-hadron duality, which must be considered in moment sum rules.
- Then moment is divided into two parts: $\widetilde{M}_n^{\mathrm{OPE}}(Q_0^2) = \int_{s_N}^{s_0} \frac{\rho^{\mathrm{OPE}}(s)}{(s+Q_0^2)^{n+1}} \mathrm{d}s + M_n^{\mathrm{NS}}(Q_0^2)$ $\quad M'^{\mathrm{OPE}}_n(Q_0^2) = \int_{s_0}^{\infty} \frac{\rho^{\mathrm{OPE}}(s)}{(s+Q_0^2)^{n+1}} \mathrm{d}s$
- $(m_X^2 + Q_0^2)\frac{1+\delta_n(s_h,Q_0^2)}{1+\delta_{n+1}(s_h,Q_0^2)} = \frac{M_n^{\mathrm{OPE}}(Q_0^2)}{M_{n+1}^{\mathrm{OPE}}(Q_0^2)} \implies (m_X^2 + Q_0^2)\frac{1+\delta_n(s_h,Q_0^2)}{1+\delta_{n+1}(s_h,Q_0^2)} = \frac{\widetilde{M}_n^{\mathrm{OPE}}(Q_0^2)+M'^{\mathrm{OPE}}_n(Q_0^2)}{\widetilde{M}_{n+1}^{\mathrm{OPE}}(Q_0^2)+M'^{\mathrm{OPE}}_{n+1}(Q_0^2)}$
- In order to maintain a uniform formalism, we introduce the definition $\delta'_n(s_0, Q_0^2) = \frac{M'^{\mathrm{OPE}}_n(Q_0^2)}{\widetilde{M}_n^{\mathrm{OPE}}(Q_0^2)}$
- The mass can be expressed as $(m_X^2 + Q_0^2)\frac{1+\delta_n(s_h,Q_0^2)}{1+\delta_{n+1}(s_h,Q_0^2)} = \frac{\widetilde{M}_n^{\mathrm{OPE}}(Q_0^2)}{\widetilde{M}_{n+1}^{\mathrm{OPE}}(Q_0^2)} \frac{1+\delta'_n(s_0,Q_0^2)}{1+\delta'_{n+1}(s_0,Q_0^2)}$
- Quark-hadron duality implies $\int_{s_h}^{\infty} \frac{\rho^h(s)\mathrm{d}s}{s-q^2} = \int_{s_0}^{\infty} \frac{\rho^{\mathrm{OPE}}(s)\mathrm{d}s}{s-q^2} \implies \delta_n(s_h, Q_0^2) = \delta'_n(s_0, Q_0^2)$
- Suppose that on the phenomenological side we have $\delta_n(s_h, Q_0^2) \approx \delta_{n+1}(s_h, Q_0^2)$. Then on the OPE side, we have $\delta'_n(s_0, Q_0^2) \approx \delta'_{n+1}(s_0, Q_0^2)$. Meanwhile, one can use $\delta'_n(s_0, Q_0^2) \approx \delta'_{n+1}(s_0, Q_0^2)$ as a posterior condition to constrain s_0.

$$m_X^2 + Q_0^2 = \frac{\widetilde{M}_n^{\mathrm{OPE}}(s_0, Q_0^2)}{\widetilde{M}_{n+1}^{\mathrm{OPE}}(s_0, Q_0^2)}$$

Dependence conditions

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- Evidently, one condition is insufficient to constrain 3 parameters (s_0, Q_0^2, n)

$$m_X^2 + Q_0^2 = \frac{\widetilde{M}_n^{\mathrm{OPE}}(s_0, Q_0^2)}{\widetilde{M}_{n+1}^{\mathrm{OPE}}(s_0, Q_0^2)}$$

- Quark-hadron duality implies that the continuum can be approximately described by the OPE. Therefore, the duality parameter s_0 must be physical.
- The conventional approach requires the mass to be independent of $(M_B^2 = Q_0^2/n, Q_0^2, n)$.

s_0 is physical.
(Q_0^2, n) are unphysical.

We define the dependence conditions

$$\frac{\partial m_X^2}{\partial Q_0^2} < \epsilon_1 \qquad \frac{\partial m_X^2}{\partial n} < \epsilon_2 \ \mathrm{GeV}^2$$

For simplicity, we impose $\epsilon_1 = \epsilon_2 = \epsilon$

- The dependence conditions and the posterior condition $\delta'_n(s_0, Q_0^2) \approx \delta'_{n+1}(s_0, Q_0^2)$ naturally give three constraints.

General procedure

- With the approximation and dependence condition, we can establish a basic procedure.
  1. Select a trial range $[n_0, n_1]$ and ϵ. Examine for various s_0 whether there exist a set of Q_0^2 satisfies the dependence condition.
  2. Define the approximate function $\eta(s_0, Q_0^2, n) = \frac{\delta'_n(s_0,Q_0^2)}{\delta'_{n+1}(s_0,Q_0^2)}$. Mass formula requires $\eta(s_0, Q_0^2, n) = 1$. It is realized only in the limit $n \to \infty$: $\lim_{n\to\infty} \frac{\delta'_n(s_0,Q_0^2)}{\delta'_{n+1}(s_0,Q_0^2)} = 1$.
  3. Within $[n_0, n_1]$, we can find the minimum value $\eta_{\min} > 1$ and corresponding s_0. Shift $[n_i, n_{i+1}]$, we obtain a sequence $(\eta_{\min}^i, s_0^i)$
  4. Fit function $s_0(\eta_{\min})$ and extrapolate to $\eta_{\min} = 1$.

The determined s_0 is free from subjectivity.

- Some Remarks:
  - ϵ can be smaller. It requires trial. $(10^{-3} \to \frac{10^{-3}}{2} \to \frac{10^{-3}}{4} \to 10^{-4} \to \ldots)$
  - $\eta(s_0, Q_0^2, n)$ is a monotonically decreasing function of n. So $\eta > 1$ for all $n > 0$.

Numerical analysis and applications

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- We present a specific example. A tetraquark $J = u_a^T C\gamma_\mu\gamma_5 d_b[\bar{d}_a\gamma_\mu C\bar{s}_b^T + (a \leftrightarrow b)]$

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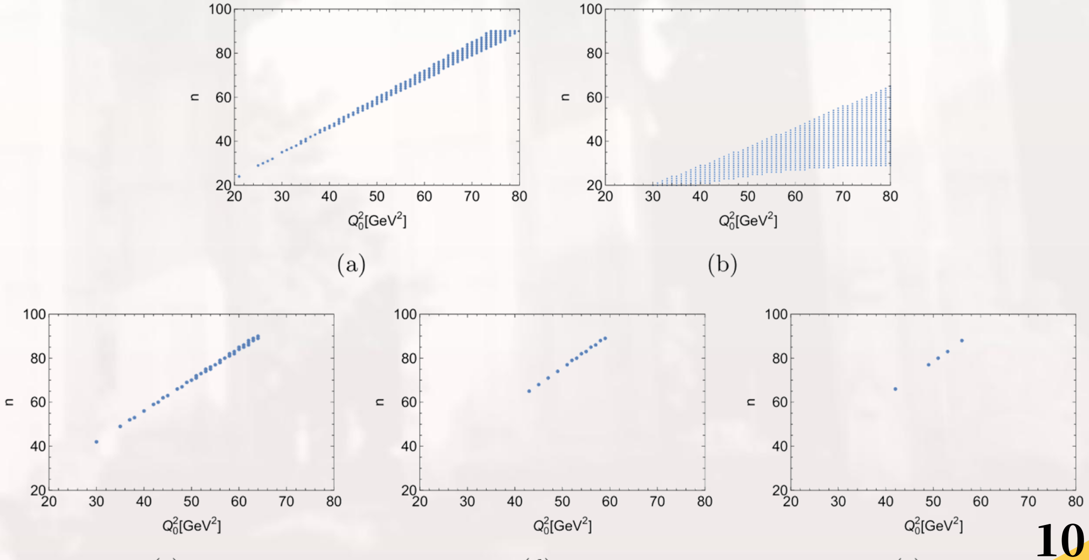
$$\Pi_{P_6}^{\mathrm{OPE}}(q^2) = \ln\left(\frac{-q^2}{\mu^2}\right)\left[\frac{-q^8}{7680\pi^6} - \left(\frac{m_s\langle\bar{s}s\rangle}{48\pi^4} - \frac{m_s\langle\bar{q}q\rangle}{48\pi^4} + \frac{5\langle g^2G^2\rangle}{3072\pi^6}\right)q^4 - \left(\frac{7g^2\langle\bar{s}s\rangle^2}{1296\pi^4}\right.\right.$$
$$\left.+\frac{7g^2\langle\bar{q}q\rangle^2}{432\pi^4} - \frac{113\langle g^3fG^3\rangle}{13824\pi^6} - \frac{m_s\langle g\bar{q}G\sigma q\rangle}{128\pi^4} + \frac{7m_s\langle g\bar{s}\sigma Gs\rangle}{384\pi^4} - \frac{\langle\bar{q}q\rangle^2}{6\pi^2} + \frac{\langle\bar{q}q\rangle\langle\bar{s}s\rangle}{6\pi^2}\right)q^2$$
$$\left.-\left(\frac{5m_s\langle g^2G^2\rangle\langle\bar{s}s\rangle}{768\pi^4} - \frac{7m_s\langle g^2G^2\rangle\langle\bar{q}q\rangle}{1152\pi^4} - \frac{\langle\bar{q}q\rangle\langle g\bar{q}\sigma Gq\rangle}{24\pi^2} + \frac{\langle g\bar{s}\sigma Gs\rangle\langle\bar{q}q\rangle}{48\pi^2} + \frac{\langle g\bar{q}\sigma Gq\rangle\langle\bar{s}s\rangle}{48\pi^2}\right)\right]$$

$$m_q(2\ \mathrm{GeV}) = 0\ \mathrm{MeV},\ m_s(2\ \mathrm{GeV}) = 93.5 \pm 0.8\ \mathrm{MeV},$$
$$\langle\bar{q}q\rangle = -(0.24\pm0.01)^3\ \mathrm{GeV}^3, \langle\bar{s}s\rangle = (0.8\pm0.1)\langle\bar{q}q\rangle,$$
$$\langle g\bar{q}\sigma Gq\rangle = (0.8\pm0.2)\langle\bar{q}q\rangle\ \mathrm{GeV}^2,\ \langle g\bar{s}\sigma Gs\rangle = (0.8\pm0.2)\langle g\bar{q}\sigma Gq\rangle,$$
$$\langle\alpha_s G^2\rangle = (6.35\pm0.35)\times10^{-2}\ \mathrm{GeV}^4,\ \langle g^3fG^3\rangle = 1.2\langle\alpha_s G^2\rangle\ \mathrm{GeV}^2$$

- We first test the dependence conditions for $s_0 = 3,4,5,6,7\ \mathrm{GeV}^2$ with $\epsilon = 10^{-3}$. (Corresponding to Figs. (a)–(e))
- Too large and too small s_0 reduce the parameter space.
- With $\epsilon = 10^{-3}/4 \quad\rightarrow\quad s_0 < 5\ \mathrm{GeV}^2$.
- Then, we select a trial range $n \in [60,80]$



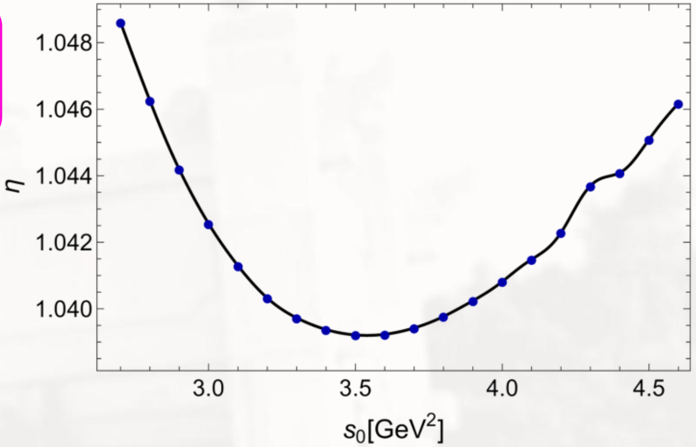
(a) (b)

(c) (d) (e)

Numerical analysis and applications

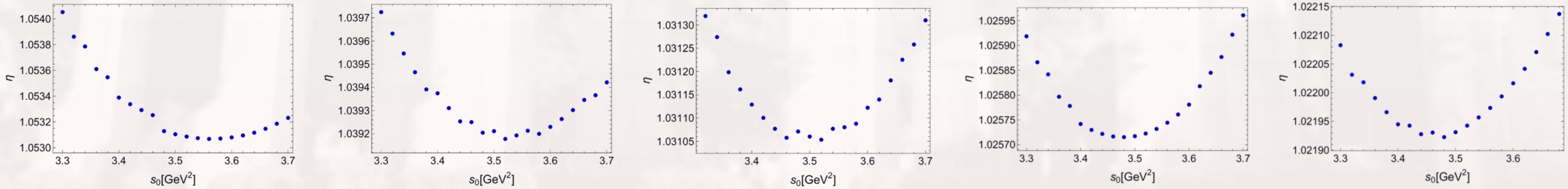
- With $\epsilon = 10^{-3}/4$, $n \in [60,80]$, and $\Delta s_0 = 1/10\ \mathrm{GeV}^2$. We examine the optimal approximation value $\eta_{\min}$.
- The optimal $\eta_{\min} = 1.03921$ is achieved at $s_0^* = 3.5\ \mathrm{GeV}^2$.

$$\boxed{\eta(s_0, Q_0^2, n) = \frac{\delta'_n(s_0, Q_0^2)}{\delta'_{n+1}(s_0, Q_0^2)}}$$



- If one is not satisfied with the precision $s_0^* = 3.5 \pm 0.1\ \mathrm{GeV}^2$. We can reduce Δs_0.
- Then we take $\Delta s_0 = 1/50\ \mathrm{GeV}^2$ and get $\eta_{\min} = 1.03918$ at $s_0^* = 3.52\ \mathrm{GeV}^2$.

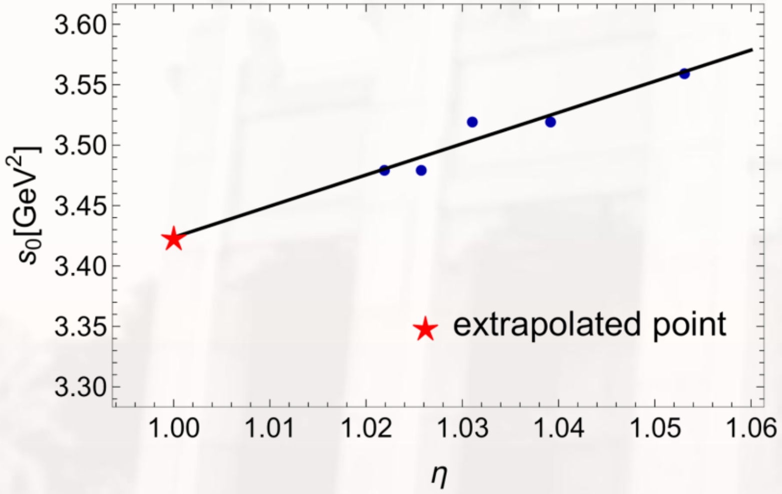
- Repeat analysis for several intervals: $n \in [40,60], [80,100], [100,120], [120,140]$.
- Increasing the n-interval decreases the $\eta_{\min}$ and causes a slight shift in s_0^*.



Numerical analysis and applications

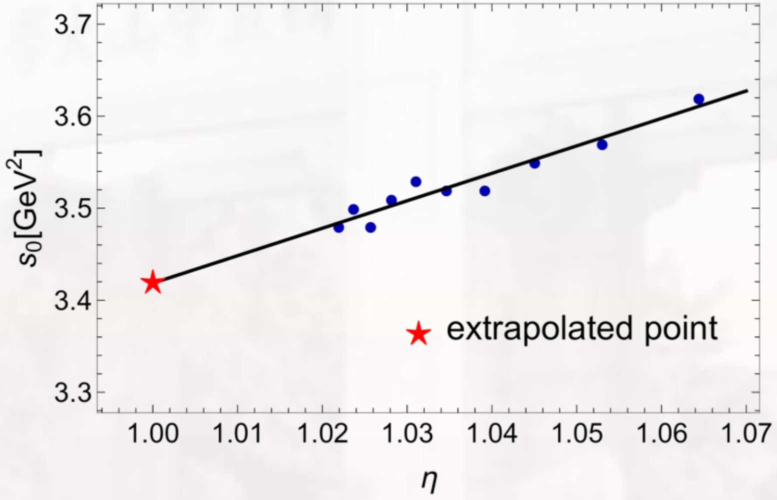
- Plot the minima of these curves.
- Perform a linear fit and extrapolate s_0 to $\eta = 1$.
- To assess the robustness of the fit, we can perform more precise calculations.

$s_0(\eta = 1) = 3.42 \pm 0.03\ \mathrm{GeV}^2$



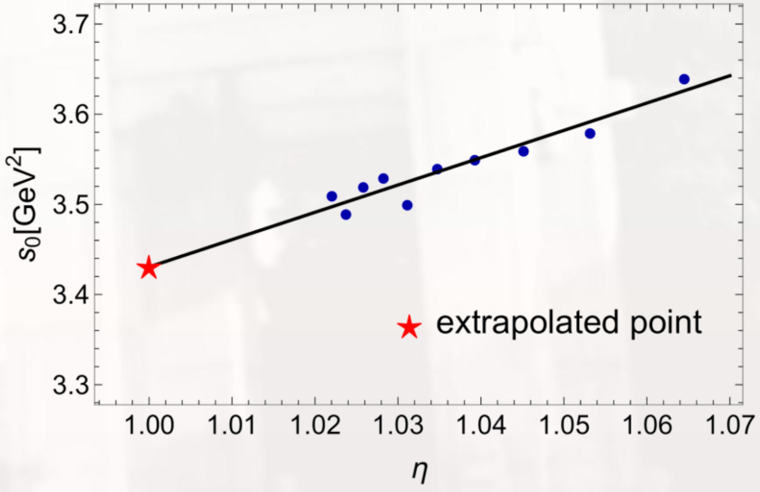
$\Delta n = 20, \Delta s_0 = \frac{1}{50}\ \mathrm{GeV}^2, \epsilon = \frac{10^{-3}}{4}$

$s_0(\eta = 1) = 3.42 \pm 0.02\ \mathrm{GeV}^2$



$\Delta n = 10, \Delta s_0 = \frac{1}{100}\ \mathrm{GeV}^2, \epsilon = \frac{10^{-3}}{4}$

$s_0(\eta = 1) = 3.43 \pm 0.03\ \mathrm{GeV}^2$



$\Delta n = 10, \Delta s_0 = \frac{1}{100}\ \mathrm{GeV}^2, \epsilon = 10^{-4}$

Discussion: Comparison with Laplace sum rules

- We wish to clarify, if the results are minimally dependent on M_B^2, the two methods yield similar results.

$\chi^2(s_0) = \sum_{i=1}^{N}\left[\frac{m_X(s_0,M_{B,i}^2)}{\bar{m}_X(s_0)} - 1\right]^2$ is defined to achieve it.

Chin.Phys.Lett. 42 (2025) 070201
JHEP 03 (2026) 087
2512.16637 [hep-ph]
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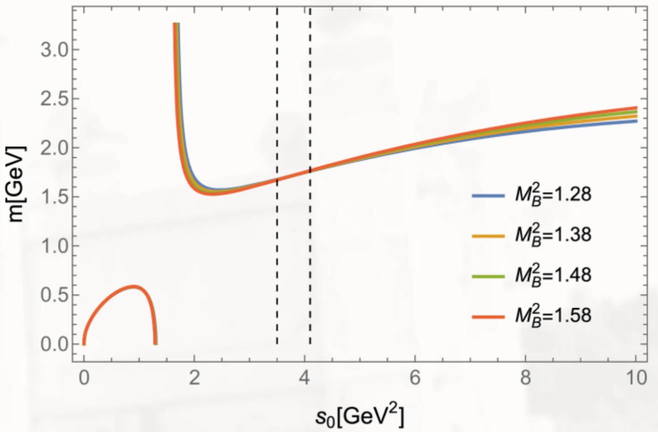


(b)

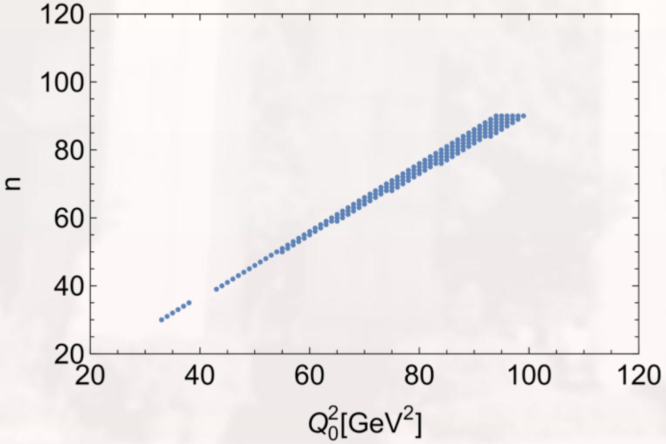
$$\left|\frac{\Pi_{D=6}(M_B^2)}{\Pi_{\text{pert}}(M_B^2)}\right| \leqslant 25\%, \left|\frac{\Pi_{D=8}(M_B^2)}{\Pi_{\text{pert}}(M_B^2)}\right| \leqslant 10\%.$$

$$\text{PC}(s_0, M_B^2) \geq 10\%$$

In the previous analysis, the Borel window is determined to be $1.14\ \text{GeV}^2 \leq M_B^2 \leq 1.35\ \text{GeV}^2$

  - $s_0(\eta = 1) = 3.42 \pm 0.03\ \text{GeV}^2$ is consistent with $s_0 = 3.3 \pm 0.2\ \text{GeV}^2$.

  - With $s_0 = 3.42\ \text{GeV}^2$ and $\epsilon = \frac{10^{-3}}{4}$, we identify the space of (Q_0^2, n). The ratio $\frac{Q_0^2}{n} \approx 1.1$ is very close to the Borel window $1.14\ \text{GeV}^2 \leq M_B^2 \leq 1.35\ \text{GeV}^2$.



Summary

- In this work, we present an improved moment sum rules (IMSR).
- The dependence conditions and the posterior condition $\delta'_n(s_0, Q_0^2) \approx \delta'_{n+1}(s_0, Q_0^2)$ naturally give three constraints.
- The results of the IMSR are essentially consistent with those of the Laplace sum rule with minimal parameter dependence.
- The IMSR no longer requires the convergence and the pole contribution (PC). The final results are free from subjectivity.

Thank you!