Analytic Two-Loop Chiral Expression for the Pion–Nucleon Sigma Term

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Outline

1. Introduction

Importance and definition of $\sigma_{\pi N}$

Pion-nucleon sigma term ($\sigma_{\pi N}$) is a fundamental observable characterizing the internal scalar structure of the nucleon.

- ☞ The contribution of light-quark masses to the nucleon mass
- ☞ The density dependence of the chiral condensate in nuclear matter
- ☞ The dominant source of hadronic uncertainty in the elastic dark-matter–nucleon scattering crossing section
- ☞ . . .

- Nucleon scalar form factor (SFF) [J. Gasser and H. Leutwyler, Ann. Phys. (1984)]

$$\langle N(p')|\hat{m}(\bar{u}u+\bar{d}d)|N(p)\rangle = \bar{u}(p')u(p)\,\sigma(t) \underset{t=(p'-p)^2}{\longrightarrow} \boxed{\sigma_{\pi N} = \sigma(t=0)}$$

- Feynman–Hellmann Theorem (FHT) [Feynman, PR (1939)] [Hellmann, 1937]

$$\sigma_{\pi N} = \hat{m}\frac{\partial m_N}{\partial \hat{m}} = M_\pi^2 \frac{\partial m_N}{\partial M_\pi^2} + \cdots .$$

Status of $\sigma_{\pi N}$ in covariant BChPT

- Covariant BChPT with full EOMS scheme
  - One-loop $\sigma_{\pi N}$ is already well established [Alarcon et al., PRD (2012)] [Chen et al., PRD (2013)] [Alvarez-Ruso, PRD (2013)] [Ren et al., PRD (2015)] [Alarcon et al., PLB (2016)]
  - Two-loop $\sigma_{\pi N}$ has been extracted from the nucleon mass using FHT [Z.-R. Liang et al., arXiv:2508.11435 (2025)]
- Computation of loop integrals
  - Auxiliary mass flow (`AMFlow`) [X. Liu et al., PLB (2018), PRD (2022)], [R.-J. Huang et al, 2607.08477]
  - AmpRed [W. Chen, CPC (2025)]
  - Mellin–Barnes (MB) representation [Smirnov, PLB (1999)] [Tausk, PLB 469 (1999)]
  - Differential-equation (DE) method [Kotikov, PLB (1991) 314] [Henn, PRL (2013)]

  ......
- Our work
  - Derive the analytic expressions of one- and two-loop integrals using DEs
  - Obtain analytic $\sigma_{\pi N}$ directly from SFF at $\mathcal{O}(p^5)$ for the first time

Merits of analytic chiral expressions

- Explicit manifestation of chiral logarithms and threshold behavior
- Facilitates studies of the convergence of the chiral expansion
- Faster and more time-efficient than numerical methods
- Provides a rigorous chiral expression for lattice-QCD extrapolations

An analytic calculation of $\sigma_{\pi N}$ is essential!

2. Calculation of $\sigma_{\pi N}$ at leading two-loop order

Chiral effective Lagrangian

- Chiral operators relevant to our $\mathcal{O}(p^5)$ calculation:

$$\mathcal{L}_{\rm eff} = \mathcal{L}_{\pi\pi}^{(2)} + \mathcal{L}_{\pi\pi}^{(4)} + \mathcal{L}_{\pi N}^{(1)} + \mathcal{L}_{\pi N}^{(2)} + \mathcal{L}_{\pi N}^{(3)} + \mathcal{L}_{\pi N}^{(4)} \,.$$

$$\mathcal{L}_{\pi\pi}^{(2)} = \frac{F^2}{4}\langle u_\mu u^\mu + \chi_+\rangle \,, \qquad \mathcal{L}_{\pi\pi}^{(4)} = \frac{1}{8} l_4 \langle u_\mu u^\mu\rangle\langle\chi_+\rangle + \frac{1}{16}(l_3 + l_4)\langle\chi_+\rangle^2 \,,$$

$$\mathcal{L}_{\pi N}^{(1)} = \bar\psi(i\not{D} - m - \frac{1}{2} g \not{u}\gamma_5)\psi \,,$$

$$\mathcal{L}_{\pi N}^{(2)} = \bar\psi\left\{c_1\langle\chi_+\rangle - \frac{c_2}{4m^2}\langle u^\mu u^\nu\rangle(D_\mu D_\nu + h.c.) + \frac{c_3}{2}\langle u_\mu u^\mu\rangle - \frac{c_4}{4}\gamma^\mu\gamma^\nu[u_\mu, u_\nu]\right\}\psi \,,$$

$$\mathcal{L}_{\pi N}^{(3)} = \bar\psi\left\{\frac{d_{16}}{2}\gamma^\mu\gamma_5\langle\chi_+\rangle u_\mu + i\frac{d_{18}}{2}\gamma^\mu\gamma_5[D_\mu, \chi_-]\right\}\psi \,,$$

$$\mathcal{L}_{\pi N}^{(4)} = \bar\psi\left\{e_{38}\langle\chi_+\rangle\langle\chi_+\rangle + \frac{e_{115}}{4}\langle\chi_+^2 - \chi_-^2\rangle - \frac{e_{116}}{4}(\langle\chi_-^2\rangle - \langle\chi_-\rangle^2 + \langle\chi_+^2\rangle - \langle\chi_+\rangle^2)\right\}\psi \,,$$

$$u_\mu = i(u^\dagger\partial_\mu u - u\partial_\mu u^\dagger) \,, \quad u = \exp(i\frac{\phi^a\tau^a}{2F}) \,, \quad D_\mu = \partial_\mu + \Gamma_\mu \,, \quad \Gamma_\mu = \frac{1}{2}(u^\dagger\partial_\mu u + u\partial_\mu u^\dagger) \,,$$

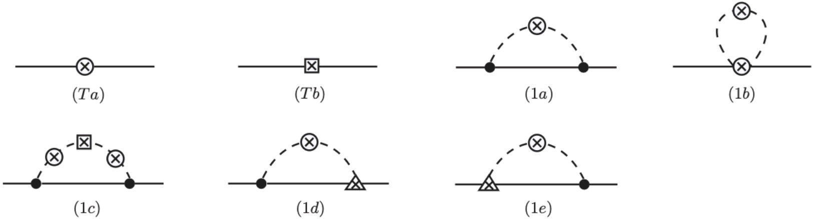
$$\chi_\pm = u^\dagger\chi u^\dagger \pm u\chi^\dagger u \,, \quad \chi = 2Bs = 2B\hat m \,, \quad \hat m = (m_u + m_d)/2 \,.$$

Feynman diagrams

- # of diagrams up to $\mathcal{O}(p^5)$:

$$2\ (\text{tree}) + 10\ (\text{one-loop}) + 26\ (\text{two-loop}) = 38$$

- Tree-level and one-loop diagrams



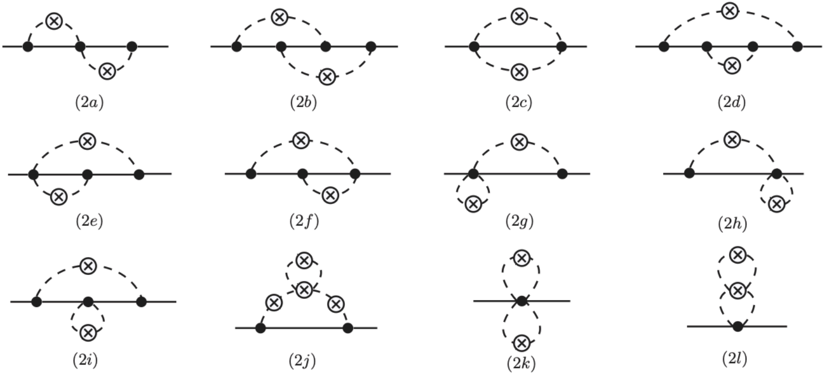
☞ "$\times$" denotes a scalar-current insertion

☞ filled circles: $\mathcal{O}(p^1)$; open circles: $\mathcal{O}(p^2)$; triangles: $\mathcal{O}(p^3)$; squares: $\mathcal{O}(p^4)$

☞ solid line: nucleon; dashed line: pion

Feynman diagrams

- Two-loop diagrams



☞ Symmetry factors for the two-loop diagrams

(2a)	(2b)	(2c)	(2d)	(2e)	(2f)	(2g)	(2h)	(2i)	(2j)	(2k)	(2l)
1	1	2	1	1	1	2	2	2	2	4	4

BChPT result of $\sigma_{\pi N}$

★ Take $s = \hat{m}$ as the external source:

$$2Bs = 2B\hat{m} = M_\pi^2.$$

- The chiral expression of $\sigma_{\pi N}$

$$\boxed{\sigma_{\pi N} = -4c_1 M_\pi^2 - 4e_m M_\pi^4 + \sigma_{1-\text{loop}} + \sigma_{2-\text{loop}}}\,, \qquad e_m = e_{115} + e_{116} + 8e_{38}.$$

- For instance, diagram (1a)

$$\sigma^{(1a)}_{1-loop} = \frac{3ig_A^2 M_\pi^2 \kappa}{16F^2 m}\left(8M_\pi^2 m^2 J_{21} + 2M_\pi^2 J_{20} + 8m^2 J_{11} + 2J_{10} - 2J_{2-1}\right)$$

Definitions of loop integrals

- One-loop integral

$$J_{\nu_1\nu_2} = \int \frac{\mathrm{d}^d \ell}{\mathrm{i}\pi^{d/2}} \frac{1}{[\ell^2 - M_\pi^2 + i0^+]^{\nu_1}[(\ell+p)^2 - m^2 + i0^+]^{\nu_2}} ,$$

- Two-loop integral

$$I_{\nu_1\nu_2\nu_3\nu_4\nu_5} = \int \frac{\mathrm{d}^d \ell_1}{\mathrm{i}\pi^{d/2}} \frac{\mathrm{d}^d \ell_2}{\mathrm{i}\pi^{d/2}} \frac{1}{\mathcal{D}_1^{\nu_1}\mathcal{D}_2^{\nu_2}\mathcal{D}_3^{\nu_3}\mathcal{D}_4^{\nu_4}\mathcal{D}_5^{\nu_5}} ,$$

$$\mathcal{D}_1 = \ell_1^2 - M_\pi^2 + i0^+ , \quad \mathcal{D}_2 = \ell_2^2 - M_\pi^2 + i0^+ , \quad \mathcal{D}_3 = (\ell_1 + \ell_2 + p)^2 - m^2 + i0^+ ,$$
$$\mathcal{D}_4 = (\ell_1 + p)^2 - m^2 + i0^+ , \quad \mathcal{D}_5 = (\ell_2 + p)^2 - m^2 + i0^+ .$$

Number of irreducible scalar products: $N = L \times E + L(L+1)/2$

IBP reduction

- Integration By Parts (IBP) [Chetyrkin and Tkachov, NPB (1981)][Smirnov, JHEP (2008)]

$$\boxed{\int \cdots \int \mathrm{d}^d\ell_1 \mathrm{d}^d\ell_2 \cdots \frac{\partial}{\partial k_i}\left(p_j \frac{1}{D_1^{a_1}\cdots D_n^{a_n}}\right) = 0}$$

- Master integrals for $\sigma_{\pi N}$

$$J_{\text{Master}} = \{J_{01}, J_{10}, J_{11}\}$$

$$I_{\text{Master}} = \{I_{00101},\ I_{00111},\ I_{01101},\ I_{10001},\ I_{10101},\ I_{10111},\ I_{11000},\ I_{11001},\ I_{11011},\ I_{11100},\ I_{11101},\ I_{11111},\ I_{21100}\}$$

- Diagram (1a) after reduction

$$\sigma^{(1a)}_{1-loop} = \frac{3ig_A^2 M_\pi^2 m\kappa}{4F^2\left(4m^2 - M_\pi^2\right)}\left\{2\left[(d-2)M_\pi^2 - 2(d-1)m^2\right] J_{11} + 2(d-2)J_{01} - (d-2)J_{10}\right\}$$

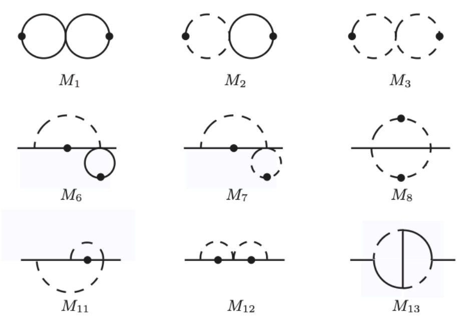
3. Analytic master integrals from DEs

Master integrals

- The choice of master integrals is not unique:

$$\boldsymbol{I}_{\text{Master}} = \{I_{00101},\ I_{00111}, \cdots,\ I_{21100}\}^T, \quad \boldsymbol{M}_{\text{Master}} = \{M_1, M_2, \cdots, M_{13}\}^T, \quad \cdots$$

- $\boldsymbol{M}_{\text{Master}}$ are more convenient to construct DEs than $\boldsymbol{I}_{\text{Master}}$



$$\begin{aligned} M_1 &= c\, I_{00022}\,, \\ M_2 &= c\, I_{20002}\,, \\ M_3 &= c\, I_{22000}\,, \\ M_4 &= c\, m^2 I_{00122}\,, \\ M_5 &= c\,(1-\epsilon) I_{10102}\,, \\ &\vdots \\ M_{13} &= c\, m^2 (1-2\epsilon)\epsilon I_{11111}\,. \end{aligned}$$

DEs for M_{master}

- DEs in terms of $\boldsymbol{M}$:

$$\frac{\mathrm{d}}{\mathrm{d}x}\boldsymbol{M}(x,\epsilon) = A(x,\epsilon)\boldsymbol{M}(x,\epsilon)\ , \qquad x = \frac{M_\pi^2}{m^2}$$

  - ☞ A is a 13×13 matrix
  - ☞ Equations of M_i:

$$\frac{\mathrm{d}M_1}{\mathrm{d}x} = 0\ ,$$

$$\vdots$$

$$\frac{\mathrm{d}M_5}{\mathrm{d}x} = \boxed{\frac{M_1 - 2M_5}{x(x-4)}} + \frac{\epsilon}{x(x-4)}\left[(M_2 - 2M_5)x - 2M_1 + 4M_5\right]\ ,$$

$$\vdots$$

☹ **Terms of various orders in ϵ arise!**

Canonical DEs

- Canonical DEs:

$$\mathrm{d}\boldsymbol{F}(x;\epsilon) = \epsilon\, \mathrm{d}\widetilde{A}(x)\, \boldsymbol{F}(x;\epsilon) \qquad \boldsymbol{F} = \{F_1, F_2, \cdots, F_{13}\}^T$$

☞ Canonical basis

$$F_1 = M_1\ ,\ \cdots\ ,\ F_5 = \frac{1}{2}\sqrt{\frac{x-4}{4x}}(M_1 - 2M_5)\ ,\ \cdots$$

☞ Proof: F_5 as an example ($\Delta \equiv M_1 - 2M_5$, $\mu(x) \equiv \sqrt{\frac{x-4}{x}}$)

$$\frac{\mathrm{d}\Delta}{\mathrm{d}x} = -\frac{2}{x(x-4)}\Delta + \mathcal{O}(\epsilon)\ , \quad \frac{\mathrm{d}}{\mathrm{d}x}\left[\mu(x)\cdot\Delta\right] = \mu(x)\frac{\mathrm{d}\Delta}{\mathrm{d}x} + \frac{\mathrm{d}\mu}{\mathrm{d}x}\Delta = 0\ ,$$

$$\rightarrow \quad F_5 = \frac{1}{2}\sqrt{\frac{x-4}{x}}(M_1 - 2M_5)$$

Rationalizion

- Rationalizes the algebraic root into a rational function of z.

$$x = y^2, \quad y = \frac{2(4z^2+1)}{4z^2-1} \quad \longrightarrow \quad \sqrt{\frac{x-4}{x}} = \frac{\sqrt{y^2-4}}{y} = \frac{4z}{4z^2+1}.$$

- $\mathit{Letters} = \{z,\ 2z-1,\ 2z+1,\ 4z^2+1,\ 4z^2+3,\ 12z^2+1,\ 16z^4+24z^2+1\}$

$$\frac{\mathrm{d}F_5}{\mathrm{d}z} = \epsilon\left(\frac{8}{4z^2-1}, \frac{8}{1-4z^2}, 0, 0, -\frac{2(16z^4+24z^2+1)}{z-16z^5}, 0, 0, 0, 0, 0, 0, 0, 0\right)\boldsymbol{F}$$

- Further factorization gives 13 linear letters $z - z_i^0$:

$$\mathrm{d}\log(4z^2+1) = \mathrm{d}\log(z^2+\frac{1}{4}) = \mathrm{d}\log(z+\frac{i}{2}) + \mathrm{d}\log(z-\frac{i}{2}),$$

$$\begin{aligned}\mathrm{d}\log(16z^4+24z^2+1) &= \mathrm{d}\log\left(z-\frac{i}{2}\sqrt{3-2\sqrt{2}}\right) + \mathrm{d}\log\left(z+\frac{i}{2}\sqrt{3-2\sqrt{2}}\right)\\ &\quad + \mathrm{d}\log\left(z-\frac{i}{2}\sqrt{3+2\sqrt{2}}\right) + \mathrm{d}\log\left(z+\frac{i}{2}\sqrt{3+2\sqrt{2}}\right).\end{aligned}$$

Solving canonical DEs

- Canonical DEs with respect to z:

$$\boxed{\mathrm{d}\boldsymbol{F}(z;\epsilon) = \epsilon\, \mathrm{d}\widetilde{A}(z)\, \boldsymbol{F}(z;\epsilon)}\,, \qquad \mathrm{d}\widetilde{A} = \sum_{i=1}^{13} A_i'\, \mathrm{d}\log(z - z_i^0)\,.$$

☞ **Recurrence relation:**

$$\boldsymbol{F}(z;\epsilon) = \sum_{k\geq 0} \boldsymbol{F}^{(k)}(z)\epsilon^k \longrightarrow \boxed{\mathrm{d}\boldsymbol{F}^{(k)}(z) = \mathrm{d}\widetilde{A}\, \boldsymbol{F}^{(k-1)}(z)\,, \quad \boldsymbol{F}^{(-1)}(z) = 0\,.}$$

$$\mathcal{O}(\epsilon^0)\text{: } \mathrm{d}\boldsymbol{F}^{(0)}(z) = 0 \longrightarrow \boldsymbol{F}^{(0)}(z) = (c_0(1), \cdots, c_0(13))^T$$
$$\mathcal{O}(\epsilon^1)\text{: } \mathrm{d}\boldsymbol{F}^{(1)}(z) = \mathrm{d}\widetilde{A}\, \boldsymbol{F}^{(0)}(z) \longrightarrow \boldsymbol{F}^{(1)}(z) = \int \mathrm{d}\widetilde{A}(z)\boldsymbol{F}^{(0)}(z)\mathrm{d}z + (c_1(1), \cdots, c_1(13))^T$$

☞ **Boundary condition:** the integration constants are fixed by `AMFlow`.

GPL solutions to the canonical DEs

- Solutions in terms of GPLs:

$$
\begin{aligned}
F_1 =& 1\,, \\
F_2 =& 1 + 2\epsilon\left[G_2(z) + G_3(z) - G_4(z) - G_5(z) - \log(2)\right] + 2\epsilon^2\left[2G_{2,3}(z) + 2G_{3,2}(z)\right. \\
& -2G_{2,4}(z) - 2G_{4,2}(z) - 2G_{2,5}(z) - 2G_{5,2}(z) + 2G_{2,2}(z) - \log(4)G_2(z) \\
& -2G_{3,4}(z) - 2G_{4,3}(z) - 2G_{3,5}(z) - 2G_{5,3}(z) + 2G_{3,3}(z) - \log(4)G_3(z) \\
& \left.+2G_{4,5}(z) + 2G_{5,4}(z) + 2G_{4,4}(z) + \log(4)G_4(z) + 2G_{5,5}(z) + \log(4)G_5(z) + \log^2 2\right] \\
& \vdots
\end{aligned}
$$

  - ☞ Normalized to F_1
  - ☞ $G_{a_1,\cdots,a_n}(x)$ are GPLs:

$$
G_{a_1,\cdots,a_n}(x) = G(x_{a_1},\cdots,x_{a_n},x) = \int_0^x \frac{\mathrm{d}t}{t - x_{a_1}} G(x_{a_2},\cdots,x_{a_n},t)\,.
$$

Analytic expression of $\sigma_{\pi N}$

$$\boxed{\text{GPLs}} \longrightarrow \begin{pmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \\ F_5 \\ F_6 \\ F_7 \\ F_8 \\ F_9 \\ F_{10} \\ F_{11} \\ F_{12} \\ F_{13} \end{pmatrix} \longrightarrow \begin{pmatrix} M_1 \\ M_2 \\ M_3 \\ M_4 \\ M_5 \\ M_6 \\ M_7 \\ M_8 \\ M_9 \\ M_{10} \\ M_{11} \\ M_{12} \\ M_{13} \end{pmatrix} \longrightarrow \begin{pmatrix} I_{00101} \\ I_{00111} \\ I_{01101} \\ I_{10001} \\ I_{10101} \\ I_{10111} \\ I_{11000} \\ I_{11001} \\ I_{11011} \\ I_{11100} \\ I_{11101} \\ I_{11111} \\ I_{21100} \end{pmatrix} \longrightarrow \boxed{\sigma_{\pi N}}$$

4. Numerical examination

Parameter setup and GPLs

- **Values of parameters** [Yao et al., PRD (2017)] [Aoki et al., EPJC (2022)] [Siemens et al., PLB (2017)] [Chen et al., PRD (2013)]

LECs	Values	Error
e_m [GeV^{-3}]	6.831	–
c_1 [GeV^{-1}]	-1.39	–
c_2 [GeV^{-1}]	3.35	± 0.03
c_3 [GeV^{-1}]	-5.70	± 0.06
d_{16} [GeV^{-2}]	-0.83	± 0.03
d_{18} [GeV^{-2}]	-0.56	± 1.42
F [GeV]	0.0867	–
g_A	1.13	–
$10^3 \ell_3^r$	1.21	± 1.01
$10^3 \ell_4^r$	1.17	± 2.85

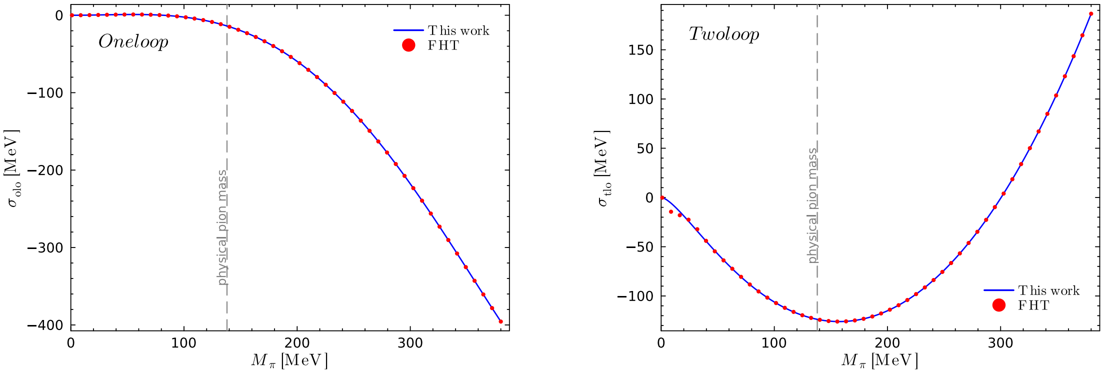
- **GPLs:** `GiNaC`, `PolyLogTools`, `handyG` [Vollinga et al., CPC2005][Duhr et al., JHEP2019][Naterop et al., CPC2020]
- HandyG.jl is used for fast, high-precision direct evaluation.

Numerical comparion

- Numerical evaluation of I_{11111}: `handyG` versus `AMFlow`

Metric / Tool	`AMFlow` (Mathematica)	`handyG` (Julia)
Result	-4.105975573122 $806\ldots$	-4.105975573122 $695\ldots$
Time	197.954 s	**0.011221 s**
Environment	Apple M1	Apple M1

Comparison between the analytic and FHT results



- One-loop and two-loop 1PI contributions to $\sigma_{\pi N}$;
- The agreement indicates the correctness of the analytic results.

5. Summary

Summary

1. Analytic two-loop expression at $\mathcal{O}(p^5)$ for $\sigma_{\pi N}$ in relativistic BChPT is obtained;
2. Evaluated numerically using the HandyG.jl package;
3. The analytic result agree precisely with that obtained by FHT.

Thank you very much for your patience!