NLO relativistic and QCD corrections to prompt J/ψ pair photoproduction at future e^+e^- colliders

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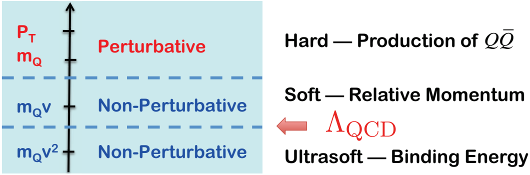
Outline

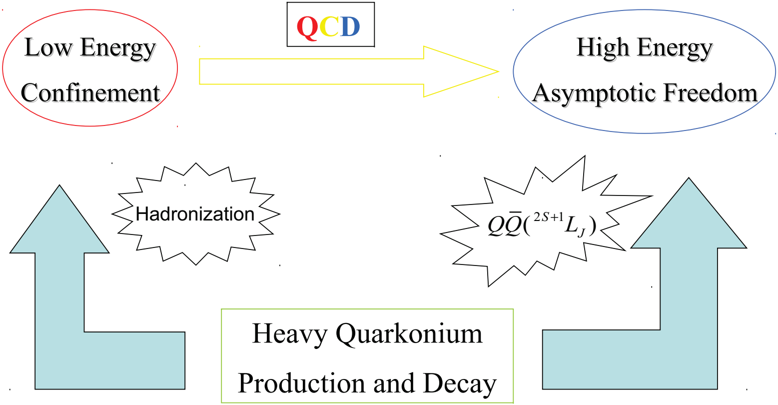
What is heavy quarkonium?

- Heavy quarkonium is one of the simplest QCD bound state constituted by heavy quark pair $Q\bar{Q}$ ($\Lambda_{QCD} \ll m_Q$).
- It is labeled by the spectroscopic notation $n^{2S+1}L_J$.
  - Its parity $P = (-1)^{L+1}$.
  - Its charge conjugation parity $C = (-1)^{L+S}$.
- There are 3 typical energy scales besides Λ_{QCD}:



- It is an approximately non-relativistic system:
  - Charmonium: $v_c^2 \approx 0.23 \ll 1$
  - Bottomonium: $v_b^2 \approx 0.08 \ll 1$

Quarkonium production–Ideal laboratory to probe QCD



J/ψ and Υ are ideal candidates for their large leptonic branching functions!

Factorization of heavy quarkonium production I

- The general factorization formalism:

$$\sigma_{(A+B\rightarrow quarkonium+X)} = \sum_n \int \sigma_{A+B\rightarrow(Q\bar{Q})_n+X} \times f[(Q\bar{Q})_n \rightarrow quarkonium]$$

Models in early stage:

1. Color-singlet model (CSM):
   - The $Q\bar{Q}$ carries the same quantum number as the heavy meson.
   - There are un-canceled infrared divergences in theoretical calculation.
   - Unable to explain J/ψ photoproduction and hadroproduction data.
2. color evaporation model (CEM):
   - All Fock states almost share the same hard part.
   - Unable to predict polarization.

Factorization of heavy quarkonium production II

NRQCD factorization formalism:

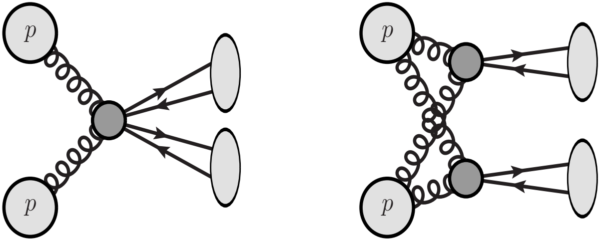
1. Double expansion in α_s for hard part and in v^2 for non-perturbative universal long distance matrix element (LDME).
2. The $Q\bar{Q}$ could be in color octet (CO), which explain the J/ψ surplus problem in hadron collider.
3. Infrared divergences are absorbed into CO LDMEs.

The challenge of NRQCD factorization:

1. The long-standing J/ψ and $\psi(2S)$ polarization puzzle.
2. The universality of the NRQCD LDMEs for J/ψ production up to QCD NLO.
3. The success of CSM to account for J/ψ production in e^+e^- annihilation, and η_c meson hadroproduction at the LHC.

Production mechanism of double quarkonium states

- Multi-final states can be produced through single parton scattering (SPS) or multi-parton scattering (MPS).

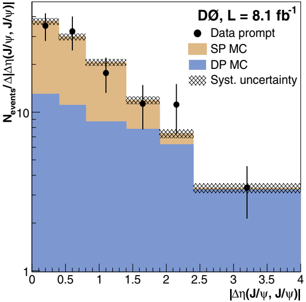


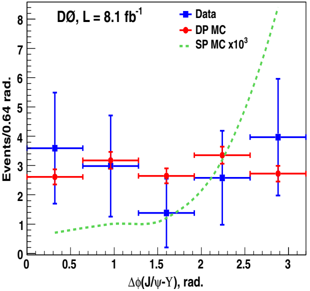
Schematical representation of SPS (left) and DPS (right) for a proton-proton collision. (Figure from Phys. Rev. D 95, 034029)

- The key parameter in MPS is σ_{eff}, which is supposed to be universal.

σ_{eff} from experimental fit I

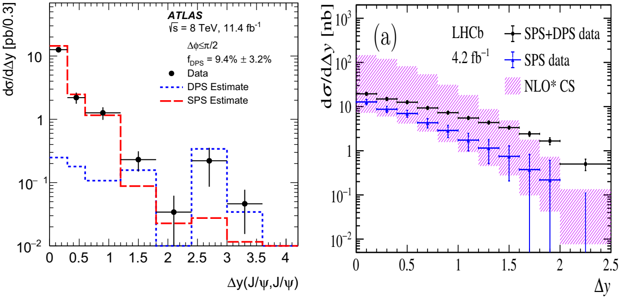
- In 2014, D0 Collaboration got $\sigma_{\text{eff}} = 4.8 \pm 0.5 \pm 2.5$ mb from double J/ψ hadroproduction. (left panel)
- Later in 2016 they got $\sigma_{\text{eff}} = 2.2 \pm 0.7 \pm 0.9$ mb from $J/\psi + \Upsilon$ hadroproduction. (right panel)





$\sigma_{\rm eff}$ from experimental fit II

- In 2017, ATLAS Collaboration got $\sigma_{\rm eff} = 6.3 \pm 1.6 \pm 1.0$ mb from double J/ψ hadroproduction. (left panel)
- Later in 2023, the LHCb Collaboration got $\sigma_{\rm eff} = 13.1 \pm 1.8 \pm 2.3$ mb from double J/ψ hadroproduction. (right panel)



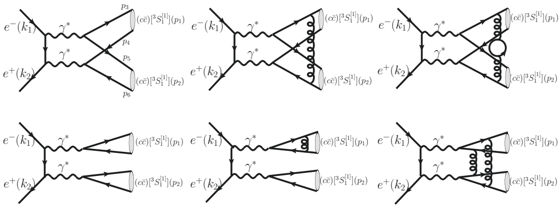
The experimental measurements are not consistent with each other too!

Why do we need higher order QCD and relativistic corrections?

- NRQCD is an effective field theory in double expansion of α_s and v^2.
- In heavy quarkonium production, higher order QCD corrections were found to be very large, and the K-factor could be orders of magnitude.
- The NLO v^2 corrections played an important role to resolve the large discrepancy of double charmonium (He et al. 2007) and single J/ψ (He et al. 2010, Jia 2010) production in e^+e^-.
- For prompt J/ψ photo- and hadroproduction, the relativistic corrections have considerable influence on both yield and polarization. (He and Kniehl 2014,2015)
- For double J/ψ hadroproduction, the NLO relativistic corrections are even more important than QCD corrections to near threshold region. (He, Jin and Kniehl 2014,2015)

Mode 1: $e^+e^- \to 2\gamma^* \to 2J/\psi$

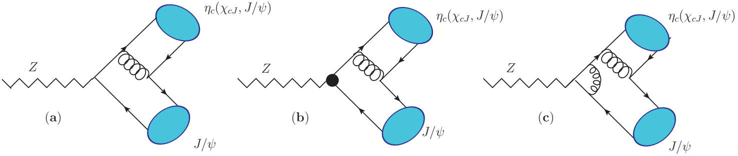
- The LO process was introduced in 2002 by Bodwin et al..
- The NLO QCD corrections were obtained by Gong and Wang in 2008.
- The remarkable non-trivial NNLO QCD corrections were calculated very recently by 2 groups (Jia and Wang) in 2023.
- Representative Feynman diagrams up to QCD NNLO. (Huang et al. JHEP02 (2024) 055)



The total cross section drops down fast as $\sqrt{s}$ increase.

Mode 2: $e^+e^- \to Z \to 2J/\psi$

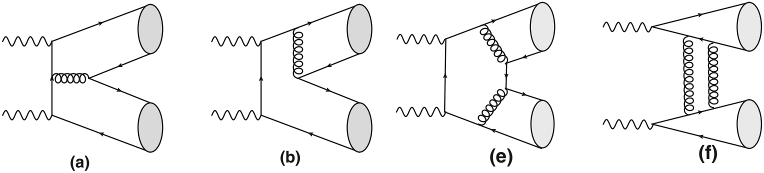
- The idea to study Z-boson contribution was raised by Chen et al. in 2013, and LO predictions were given.
- The NLO QCD corrections were considered by 2 groups (Berezhnoy et al. in 2021 and Luo et al. in 2022).
- Representative Feynman diagrams up to QCD NLO. (Luo et al. arXiv: 2209.08802)



The cross section is tiny.

Mode 3: $\gamma\gamma \to 2J/\psi$

- The 2 J/ψ photoproduction was first considered by Qiao in 2002 as nonignorable contribution to single J/ψ photoproduction.
- The NLO QCD corrections were computed in 2020 by Yang et al..
- Representative Feynman diagrams up to QCD NLO. (Yang et al. Eur. Phys. J. C80 (2020), 806)



1. The production mechanism is relative simpler.
2. The NLO QCD corrections were large and negative.
3. The cross section is not small and increases as $\sqrt{s}$ becomes larger.

Prompt $2J/\psi$ photoproduction

- At high energy the quark or gluon content of photon can also participate in the hard collisions leading to 3 channels: Direct, Single-resolved, Double-resolved.
- The single-resolved channel played an important role to understand single J/ψ photoproduction at LEP. (Klasen 2002)
- The general formalism

$$\mathrm{d}\sigma(e^+e^- \to e^+e^- + 2J/\psi + X) = \sum_{i,j,H_1,H_2} \int \mathrm{d}x_1 \mathrm{d}x_2 f_\gamma(x_1) f_\gamma(x_2) \int dx_i dx_j f_{i/\gamma}(x_i) f_{j/\gamma}(x_j) \times \mathrm{d}\hat{\sigma}(i + j \to 2J/\psi + X),$$

where f_γ is the photon spectrum of $e^+(e^-)$, and $f_{i/\gamma}(x_i)$ is the PDF of parton i in the resolved photon.

Prompt $2J/\psi$ photoproduction

- We investigate 3 experiments: CEPC and FCC-ee at $\sqrt{s} = 92$ GeV, and CLIC at $\sqrt{s} = 3$TeV.
- We verified that both single and double resolved contributions are tiny up to $\sqrt{s} = 3$ TeV.
- In direct case, the CO channels are much suppressed by the NRQCD LDMEs.
- The CS channel includes:

$$\begin{aligned} \gamma + \gamma &\to (c\bar{c})_1({}^3S_1^{[1]}) + (c\bar{c})_2({}^3S_1^{[1]}), \\ \gamma + \gamma &\to (c\bar{c})_1({}^3P_{J_1}^{[1]}) + (c\bar{c})_2({}^3P_{J_2}^{[1]}). \end{aligned}$$

- It turns out that the SDCs of double P-wave cases are about one order of magnitude smaller.
- Further suppression factor of the branching functions.

NRQCD factorization formalism up to v^2 NLO

- The partonic cross section in NRQCD factorization up to v^2 order:

$$\hat{\sigma}(i+j \to 2J/\psi + X) = \sum_{m,n,H_1,H_2} \left(\frac{F^{ij}(m,n)}{m_c^{d_{\mathcal{O}(m)}-4} m_c^{d_{\mathcal{O}(n)}-4}} \times \langle \mathcal{O}^{H_1}(m) \rangle \langle \mathcal{O}^{H_2}(n) \rangle + \frac{G_1^{ij}(m,n)}{m_c^{d_{\mathcal{P}(m)}-4} m_c^{d_{\mathcal{O}(n)}-4}} \times \langle \mathcal{P}^{H_1}(m) \rangle \langle \mathcal{O}^{H_2}(n) \rangle + \frac{G_2^{ij}(m,n)}{m_c^{d_{\mathcal{O}(m)}-4} m_c^{d_{\mathcal{P}(n)}-4}} \times \langle \mathcal{O}^{H_1}(m) \rangle \langle \mathcal{P}^{H_2}(n) \rangle \right) \times \mathrm{Br}(H_1 \to J/\psi + X) \mathrm{Br}(H_2 \to J/\psi + X)$$

- The definitions of S-wave 4-fermion operators in NRQCD:

$$\mathcal{O}^H(^3S_1^{[1]}) = \chi^\dagger \sigma^i \psi (a_H^\dagger a_H) \psi^\dagger \sigma^i \chi,$$

$$\mathcal{P}^H(^3S_1^{[1]}) = \chi^\dagger \sigma^i \psi (a_H^\dagger a_H) \psi^\dagger \sigma^i \left(-\frac{i}{2}\overleftrightarrow{\mathbf{D}}\right)^2 \chi + \mathrm{H.c.},$$

NLO QCD corrections

- QGRAF generate Feynman diagrams and Feynman amplitude.
- Self-written Form code to handle Dirac and Color algebra.
- Self-developed Mathematica code to perform the partial fraction decomposition.
- Reduze 2 and FIRE 6 to reduce the one-loop scalar integrals to master integrals.
- Package-X and QCD-Loop to evaluate the master integrals analytically.

Our results agree with previous calculation when choosing the same inputs.

Expansion of squared amplitude – Explicit case

- Expansion the amplitude in series of q_1 and q_2:

$$A(q_1, q_2) = A(0,0) + q_1^{\alpha_1} A_{\alpha_1,0} + q_2^{\beta_1} A_{0,\beta_1}(0,0) + \frac{1}{2} q_1^{\alpha_1} q_1^{\alpha_2} A_{\alpha_1\alpha_2,0}(0,0) + \frac{1}{2} q_2^{\beta_2} q_2^{\beta_2} A_{0,\beta_1\beta_2}(0,0) + \cdots .$$

where

$$A_{\alpha_1\cdots\alpha_m,\beta_1\cdots\beta_n}(0,0) \equiv \left.\frac{\partial^{m+n} A(q_1, q_2)}{\partial q_1^{\alpha_1}\cdots\partial q_1^{\alpha_m}\partial q_2^{\beta_1}\cdots\partial q_2^{\beta_n}}\right|_{q_1=0,q_2=0},$$

- Decompose the higher rank tensor into S-wave:

$$q^\mu q^\nu = \frac{|\mathbf{q}|^2}{3}\left(-g^{\mu\nu} + \frac{P\mu P^\nu}{4E_q^2}\right) = \frac{|\mathbf{q}|^2}{3}\Pi^{\mu\nu}$$

Expansion of squared amplitude – Implicit case

- The $A(0,0)$ still depends on $\mathbf{q}_i^2$ implicitly through $P_i = 4E_{q_i}^2$ and the Mandelstam variables

$$t = (k_1 - P_1)^2 = (k_2 - P_2)^2, \quad u = (k_1 - P_2)^2 = (k_2 - P_1)^2,$$

- We introduce 2 new variables $\hat{t}$ and $\hat{u}$, and their non-relativistic limit:

$$\hat{t} \equiv t + \frac{s - 4E_{q_1}^2 - 4E_{q_2}^2}{2}, \quad \hat{u} \equiv u + \frac{s - 4E_{q_1}^2 - 4E_{q_2}^2}{2}$$

$$\hat{t}_0 \equiv t_0 + \frac{s - 8m_c^2}{2}, \quad \hat{u}_0 \equiv u_0 + \frac{s - 8m_c^2}{2}$$

- In the parton center of rest frame, t,u and t_0, u_0 are related through

$$\frac{\hat{t}}{\hat{t}_0} = \frac{\hat{u}}{\hat{u}_0} = \sqrt{\frac{\lambda(s, 4E_{q_1}^2, 4E_{q_2}^2)}{\lambda(s, 4m_c^2, 4m_c^2)}} \equiv k,$$

where

$$\lambda(a,b,c) = a^2 + b^2 + c^2 - 2ab - 2bc - 2ac.$$

Expansion of phase space

- The two-body phase space can be written in a simple form with $\hat{t}$ and $\hat{u}$, or with $\hat{t}_0$ and $\hat{u}_0$ as

$$\mathrm{d}\Phi_2 = \frac{\mathrm{d}\hat{t}\mathrm{d}\hat{u}}{8\pi s}\,\delta(\hat{t}+\hat{u}) = k\frac{\mathrm{d}\hat{t}_0\mathrm{d}\hat{u}_0}{8\pi s}\,\delta(\hat{t}_0+\hat{u}_0) = k\;\mathrm{d}\Phi_{20}.$$

  where

$$k = \sqrt{1 - \frac{8\mathbf{q}_1^2 + 8\mathbf{q}_2^2}{s-16m_c^2} + \frac{16(\mathbf{q}_1^2-\mathbf{q}_2^2)^2}{s(s-16m_c^2)}},$$

- All dependence on $\mathbf{q}_i^2$ are factorized into k, which makes the expansion of phase space become trivial.

Near threshold region, the expansion will be spoiled, for the actual expansion parameter is not $v^2 = \mathbf{q}^2/m_c^2$ anymore, but $8m_c^2v^2/(s-16m_c^2)$.

"Rigorous" NRQCD results

- The formal results of matching:

$$\frac{F({}^3S_1^{[1]},{}^3S_1^{[1]})}{m_c^4} = \frac{1}{2s}\int d\Phi_{20}|M|^2,$$
$$\frac{G^1({}^3S_1^{[1]},{}^3S_1^{[1]})}{m_c^6} = \frac{G^2({}^3S_1^{[1]},{}^3S_1^{[1]})}{m_c^6} = \frac{1}{2s}\int d\Phi_{20}(K|M|^2 + |N|^2).$$

where

$$|M|^2 = \overline{\sum}|A(0,0)|^2\Big|_{\mathbf{q}_{1,2}^2=0},$$
$$|N|^2 = \left\{\frac{\partial}{\partial \mathbf{q}_1^2}\left[\frac{m_c}{E_{q_1}}\overline{\sum}|A(0,0)|^2\right] + \frac{1}{3}\Pi_1^{\alpha_1\alpha_2}\mathrm{Re}\left[A^*(0,0)A_{\alpha_1\alpha_2,0}\right]\right\}\Big|_{\mathbf{q}_{1,2}^2=0}.$$

Modified NRQCD calculation

- The physical mass of J/ψ and $\psi(2S)$ are used in phase space integral.
- m_c is preserved in matrix element calculation.
- $\hat{t}_0$ and $\hat{u}_0$ can be related to $\hat{t}_p$ and $\hat{u}_p$ in a similar way as:

$$\frac{\hat{t}_p}{\hat{t}_0} = \frac{\hat{u}_p}{\hat{u}_0} = \sqrt{\frac{\lambda(s, M_{H_1}^2, M_{H_2}^2)}{\lambda(s, 4m_c^2, 4m_c^2)}}$$

- In such a way, the theoretical uncertainties due to choice of m_c are largely reduced.

Numerical inputs

- The LO LDMEs can be related to wave function in potential model calculation:

$$\mathcal{O}^{J/\psi}({}^3S_1^{[1]}) = 1.16\ \mathrm{GeV}^3, \mathcal{O}^{\psi(2S)}({}^3S_1^{[1]}) = 0.758\ \mathrm{GeV}^3.$$

- The ratios between LO and v^2 NLO LDMEs were calculated by Bodwin et al. in 2006,

$$\frac{\langle \mathcal{P}^{J/\psi}({}^3S_1^{[1]})\rangle}{\langle \mathcal{O}^{J/\psi}({}^3S_1^{[1]}\rangle} = 0.5\,\mathrm{GeV}^2 \simeq \frac{\langle \mathcal{P}^{\psi(2S)}({}^3S_1^{[1]})\rangle}{\langle \mathcal{O}^{\psi(2S)}({}^3S_1^{[1]})\rangle}.$$

- Λ^4_{QCD}=215 (326) MeV for tree (one-loop) calculation.
- m_c =1.5GeV, $M_{J/\psi} = 3.097$GeV, $M_{\psi(2S)} = 3.686$GeV, and $\mathrm{Br}(\psi(2S) \to J/\psi + X) = 61.4\%$ are taken from PDG 2023.
- We choose $\mu_r = \sqrt{s}$, and require $p_T^{J/\psi} > 2$GeV.

The sources of photon and their spectra I

- At e^+e^- colliders, the photon can come from: (1)Bremsstrahlung, (2) Beamstrahlung, and (3) Laser back scattering.
- The Bremsstrahlung photon is described in WWA approximation:

$$f_\gamma^{WWA}(x) = \frac{\alpha}{2\pi}\left(\frac{1+(1-x)^2}{x}\log\left(\frac{Q^2_{\rm max}}{Q^2_{\rm min}}\right) + 2m_e^2 x\left(\frac{1}{Q^2_{\rm max}} - \frac{1}{Q^2_{\rm min}}\right)\right)$$

where $Q^2_{\rm min} = \frac{m_e^2 x^2}{1-x}$, $Q^2_{\rm max} = \left(\frac{\theta_c\sqrt{S}}{2}\right)^2(1-x) + Q^2_{\rm min}$.

- The spectrum of Beamstrahlung photon can be formulated as:

$$f_\gamma^{\rm beam}(x) = \frac{1}{\Gamma(\frac{1}{3})}\left(\frac{2}{3\Upsilon}\right)^{1/3} x^{-2/3}(1-x)^{-1/3}e^{-2x/(3\Upsilon(1-x))} \times$$
$$\left\{\frac{1-\sqrt{\Upsilon/24}}{g(x)}\left[1-\frac{1}{g(x)N_\gamma}\right](1-e^{-g(x)N_\gamma}) + \right.$$
$$\left.\sqrt{\frac{\Upsilon}{24}}\left[1-\frac{1}{N_\gamma}(1-e^{-N_\gamma})\right]\right\}$$

The sources of photon and their spectra II

- Υ is called effective beamstrahlung parameter, $N_\gamma = \frac{5\alpha\sigma_z m_e^2 \Upsilon}{2E_e\sqrt{1+\Upsilon^{2/3}}}$ is the averaged photon number emitted by e, and

$$g(x) = 1 - \frac{1}{2}\left((1+x)\sqrt{1+\Upsilon^{2/3}} + 1 - x\right)(1-x)^{2/3},$$

- The relevant parameters of CEPC,FCC-ee, and CLIC are:

Facility	collision energy	θ_c (mrad)	average Υ	bunch length σ_z(mm)	luminosity (ab^{-1}year^{-1})
FCC-ee	$\sqrt{s}$=92GeV	30	10^{-4}	15.5	17
CEPC	$\sqrt{s}$=92GeV	33	2×10^{-4}	8.7	15
CLIC	$\sqrt{s}$=3000GeV	20	5	0.044	0.6

NRQCD predictions at α_s and v^2 NLO – Cross section

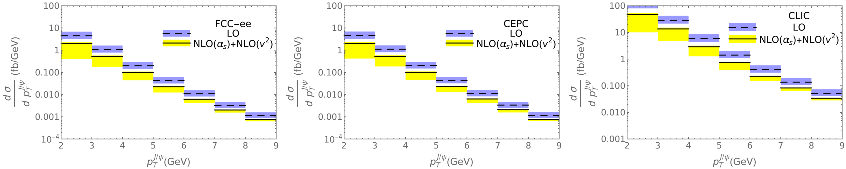
- The total cross sections (fb) and number of lepton pair event produced per year:

	$\sigma_{\rm LO}$	$\sigma_{\rm NLO}^{\alpha_s}$	$\sigma_{\rm NLO}^{v^2}$	$\sigma_{\rm NLO}^{\alpha_s,v^2}$	# of l^+l^-
FCC-ee	$5.88^{+2.99}_{-1.70}$	$3.21^{+0.18}_{-1.71}$	$5.17^{+2.64}_{-1.49}$	$2.65^{+0.34}_{-1.99}$	649^{+82}_{-488}
CEPC	$6.00^{+3.06}_{-1.73}$	$3.28^{+0.18}_{-1.75}$	$5.28^{+2.69}_{-1.53}$	$2.71^{+0.34}_{-2.03}$	584^{+75}_{-439}
CLIC	144^{+73}_{-36}	$78.9^{+3.9}_{-40.9}$	126^{+64}_{-32}	$64.6^{+7.5}_{-47.9}$	558^{+64}_{-414}

1. The Beamstrahlung contribution can be ignored in FCC-ee and CEPC cases, but enhance Bremsstrahlung predictions by about a factor of 2.6 at CLIC.
2. The v^2 corrections lead 12% reduction to LO predictions, and about 17^{+39}_{-6}% on top of NLO QCD corrections.

NRQCD predictions at α_s and v^2 NLO – p_T distribution

- The $p_T^{J/\psi}$ distribution and number of lepton pair event produced per year in each bin:

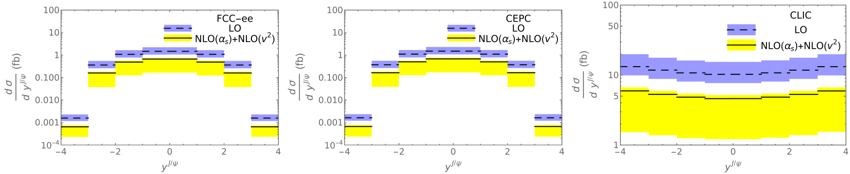


	1	2	3	4	5	6
FCC-ee	486^{+66}_{-386}	128^{+14}_{-83}	$24.4^{+2.3}_{-13.7}$	$5.6^{+0.4}_{-2.5}$	$1.5^{+0.1}_{-0.5}$	$0.5^{+0.0}_{-0.1}$
CEPC	438^{+59}_{-347}	115^{+13}_{-75}	$22.0^{+2.1}_{-12.3}$	$5.0^{+0.4}_{-2.3}$	$1.4^{+0.1}_{-0.5}$	$0.5^{+0.0}_{-0.1}$
CLIC	407^{+50}_{-319}	118^{+13}_{-77}	$25.2^{+2.4}_{-14.2}$	$6.4^{+0.5}_{-3.0}$	$2.0^{+0.1}_{-0.7}$	$0.7^{+0.0}_{-0.2}$

The $\alpha_s + v^2$ corrections lead to a reduction of $56^{+38}_{-27}\%$ in the fist bin down to $34^{+27}_{-21}\%$ in the last bin.

NRQCD predictions at α_s and v^2 NLO – y distribution

- The $y^{J/\psi}$ distribution and number of lepton pair event produced per year in each bin:

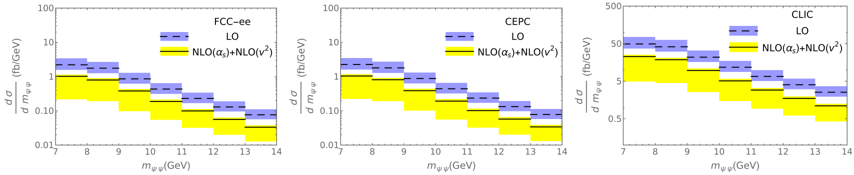


	1	2	3	4
FCC-ee	$0.16^{+0.0}_{-0.1}$	$39.4^{+5.3}_{-30.2}$	121^{+15}_{-90}	164^{+21}_{-123}
CEPC	$0.144^{+0.021}_{-0.093}$	$35.5^{+4.8}_{-27.2}$	109^{+14}_{-81}	147^{+19}_{-110}
CLIC	$51.5^{+6.2}_{-38.3}$	$45.9^{+5.5}_{-34.0}$	$41.9^{+5.0}_{-31.0}$	$39.9^{+4.8}_{-29.4}$

The $\alpha_s + v^2$ corrections lead to an almost uniformly reduction about $55^{+37}_{-26}\%$ except for the outermost bin.

NRQCD predictions at α_s and v^2 NLO – m_{jj} distribution

- The invariant mass distribution and number of lepton pair event produced per year in each bin:

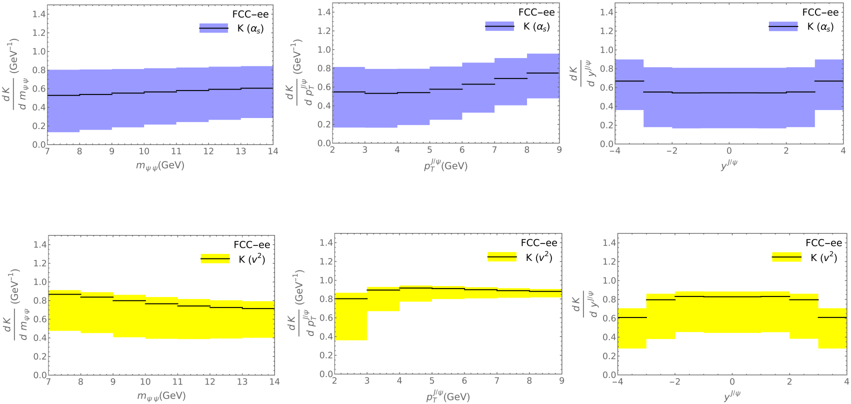


	1	2	3	4	5	6
FCC-ee	250^{+30}_{-197}	195^{+25}_{-148}	$93.5^{+12.4}_{-70.0}$	$46.1^{+6.5}_{-32.9}$	$24.4^{+3.4}_{-16.7}$	$13.8^{+1.9}_{-8.9}$
CEPC	225^{+27}_{-177}	176^{+23}_{-134}	$84.2^{+11.0}_{-62.3}$	$41.6^{+5.8}_{-29.6}$	$22.0^{+3.1}_{-15.0}$	$12.4^{+1.7}_{-8.1}$
CLIC	197^{+22}_{-154}	162^{+20}_{-123}	$83.8^{+10.7}_{-61.6}$	$44.4^{+5.8}_{-31.5}$	$25.2^{+3.3}_{-17.2}$	$15.1^{+2.0}_{-9.9}$

The reduction factor is the same as that in $y^{J/\psi}$ distribution.

Effect of NLO QCD or relativistic corrections

- The NLO QCD or relativistic corrections can be characterized by $k(\alpha_s) = d\sigma^{\mathrm{NLO}(\alpha_s)}/d\sigma^{\mathrm{LO}}$ and $k(v^2) = d\sigma^{\mathrm{NLO}(v^2,\alpha_s)}/d\sigma^{\mathrm{NLO}(\alpha_s)}$.
- For 3 experiment, the $k(\alpha_s)$, $k(v^2)$ are similar, we illustrate them in FCC-ee cases.



A few words about NLO QCD corrections

- Although the uncertainties due to μ variation still large, the band become narrow at QCD NLO.
- However, the predictions strongly depend on choice of the default μ.
- Differential cross sections of prompt double J/ψ photoproduction at FCC-ee with three different $\mu_r = \xi m_T$.

p_T (GeV)	$2-3$	$3-4$	$4-5$	$5-6$	$6-7$	$7-8$	$8-9$
$\mu_r = \sqrt{p_T^2 + 4m_c^2}/2$	-9.42	-1.90	-0.272	-0.0430	-0.00726	-0.00120	-0.000124
$\mu_r = \sqrt{p_T^2 + 4m_c^2}$	-0.162	0.0673	0.0270	0.00974	0.00370	0.00148	0.000620
$\mu_r = 2\sqrt{p_T^2 + 4m_c^2}$	1.81	0.519	0.0100	0.023	0.00647	0.00213	0.000790

The NLO QCD correction is negative and its absolute value increase as μ decrease that can even lead to un-physical prediction.

Conclusion and summary

1. Quarkonium pair hadroproduction can provide very rich probes to the perturbative and nonperturbative aspects of QCD as well as information about the partons inside proton.
2. In prompt double J/ψ photoproduction, both the NLO QCD and relativistic corrections are negative.
3. There may be enough numbers of J/ψ that can be observed at FCC-ee, CEPC and CLIC.

Thank you!