Gravitational form factors

Feng-Kun Guo

Institute of Theoretical Physics, Chinese Academy of Sciences

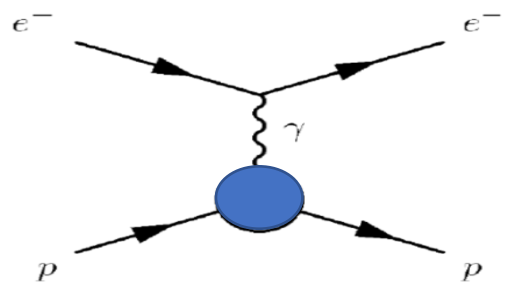
Xiong-Hui Cao, FKG, Q.-Z. Li, D.-L. Yao, Precise determination of nucleon gravitational form factors, Nature Commun. 16 (2025) 6979;
Xiong-Hui Cao, FKG, Q.-Z. Li, D.-L. Yao, Gravitational form factors of pions, kaons and nucleons from dispersion relations, Eur. Phys. J. ST [arXiv:2507.05375].

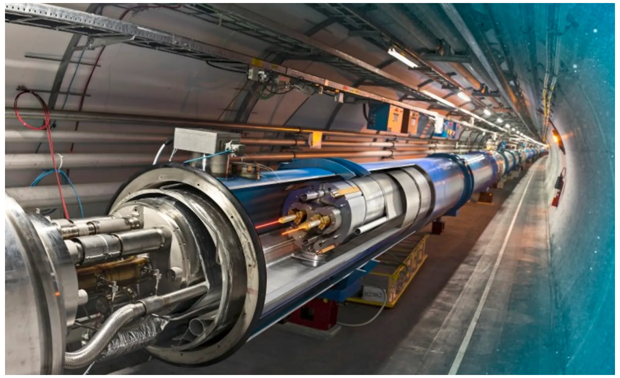
Proton has a size

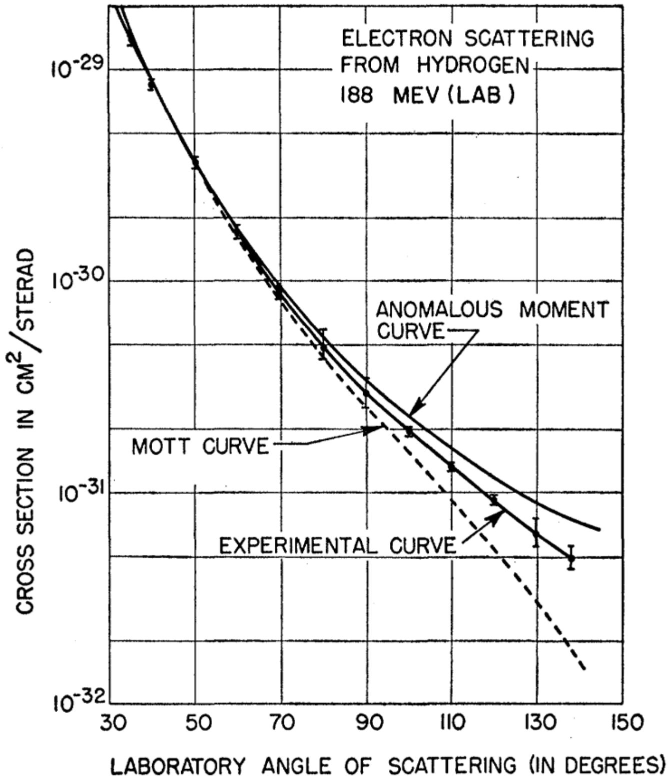
Electron Scattering from the Proton

Robert Hofstadter and Robert W. McAllister
Phys. Rev. **98**, 217 – Published 1 April 1955

well by the following choices of size. At 188 Mev, the data are fitted accurately by an rms radius of $(7.0\pm 2.4)\times 10^{-14}$ cm. At 236 Mev, the data are well fitted by an rms radius of $(7.8\pm 2.4)\times 10^{-14}$ cm. At 100 Mev the data are relatively insensitive to the radius but the experimental results are fitted by both choices given above. The 100-Mev data serve therefore as a valuable check of the apparatus. A compromise value fitting all the experimental results is $(7.4\pm 2.4)\times 10^{-14}$ cm. If the proton were a spherical ball of charge, this rms radius would indicate a true radius of 9.5×10^{-14} cm, or in round numbers 1.0×10^{-13} cm. It is to be noted that if our interpretation is correct the Coulomb law of force has not been violated at distances as small as 7×10^{-14} cm.







Hofstadter (1956)

Proton radius ~0.84 fm

Quantum world is different:
proton has more than one size, depending on the probe

Gravitational form factors

- Electro-magnetic form factors: matrix elements of em current
  - Probes distributions of electric charge, magnetic moment densities, etc.
    - proton is not pointlike, $r_E^p = 0.74(24)$ fm   R. Hofstadter, R. McAllister (1955); R. Hofstadter (1956)
- Gravitational form factors: matrix elements of energy-momentum tensor

$$\left\langle N(p') \left| \hat{T}^{\mu\nu}(0) \right| N(p) \right\rangle = \frac{1}{4m_N} \bar{u}(p') \left[A(t) P^\mu P^\nu + J(t) \left(i P^{\{\mu} \sigma^{\nu\}\rho} \Delta_\rho \right) + D(t) \left(\Delta^\mu \Delta^\nu - t g^{\mu\nu} \right) \right] u(p)$$

  - Trace anomaly: origin of mass
  - Fundamental properties of the nucleon
    - Energy/mass: $m_N = \int d^3 r T^{00}(r), \quad A(0) = 1$
    - Spin: $J^i = \epsilon^{ijk} \int d^3 r r^j T^{0k}(r), \quad J(0) = 1/2$
    - D-term: $D = -\frac{m_N}{2} \int d^3 r \left(r^i r^j - \frac{1}{3} \delta^{ij} \right) T^{ij}(r), \quad D(0):$ "last global unknown property"
  - Forces inside hadron, mechanical picture of quark confinement?

D-term: M. Polyakov, PLB 555 (2003) 57; M. Polyakov, P. Schweitzer, IJMPA 33 (2018) 1830025;

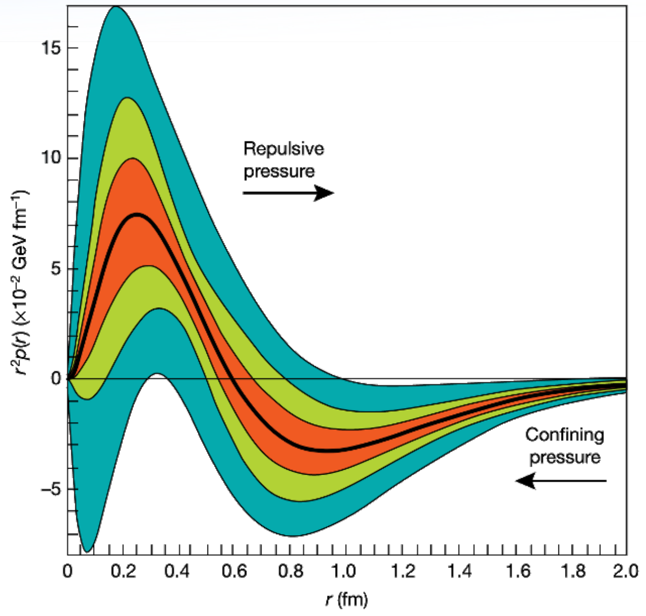
Trace anomaly, attractive force with a strength similar to the confinement string tension: X. Ji, C. Yang, Research 9(2026) 1155

Gravitational form factors

- Results from JLab
  - Pressure distribution inside proton using DVCS data

    V. D. Burkert, L. Elouadrhiri, F. X. Girod, Nature 557 (2018) 396

    "We find a strong repulsive pressure near the centre of the proton (up to 0.6 femtometres) and a binding pressure at greater distances. The average peak pressure near the centre is about 10^{35} pascals, which exceeds the pressure estimated for the most densely packed known objects in the Universe, neutron stars."

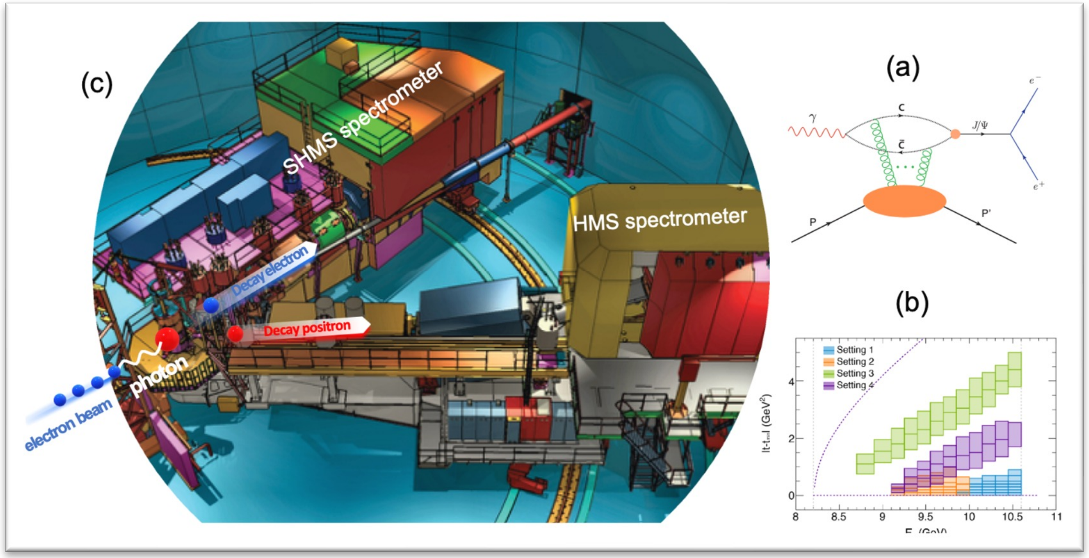


  - Nucleon GFFs from near-threshold J/ψ photoproduction with two models   B. Duran et al. [J/ψ-007], Nature 615 (2023) 7954
    - Holographic QCD
    - GPD + vector-meson dominance

Theoretical approach GFF functional form	$\sqrt{\langle r_m^2\rangle}$ (fm)	$\sqrt{\langle r_s^2\rangle}$ (fm)
Holographic QCD Tripole-tripole	0.755±0.035	1.069±0.056
GPD + VMD Tripole-tripole	0.472±0.042	0.695±0.071

Updated GPD analysis: 0.77 ± 0.07 1.20 ± 0.13

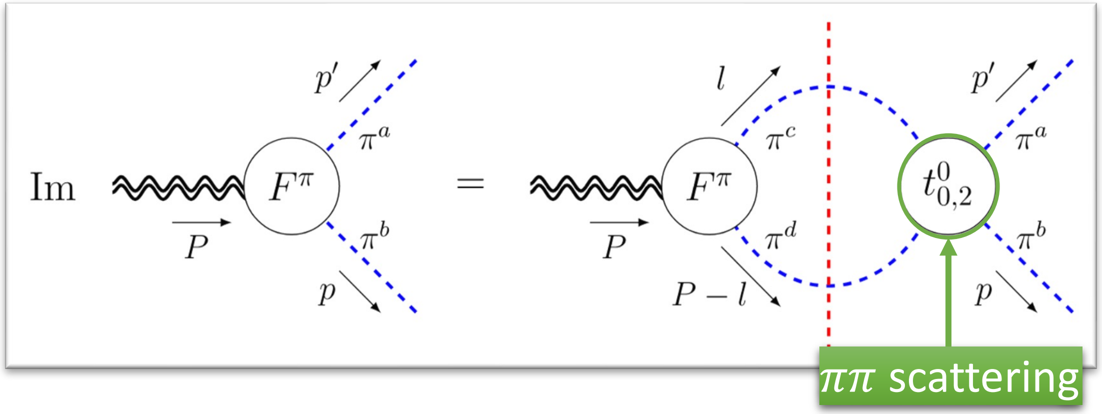
Y. Guo et al., PRD 108 (2023) 034003



But the main mechanism for this reaction could be different, so no reliable exp. results: M.-L. Du et al., EPJC 80 (2020) 1053

Unitarity relation for the pion GFFs

- Probability conservation: S-matrix unitarity



- Discontinuity of A^π and D^π

$$\text{Disc}\left\langle \pi^a(p')\pi^b(p)\left|\hat{T}^{\mu\nu}(0)\right|0\right\rangle = \frac{\delta^{ab}}{2}\left[\text{Disc}\,A^\pi(t)\Delta^\mu\Delta^\nu + \text{Disc}\,D^\pi(t)\left(P^\mu P^\nu - tg^{\mu\nu}\right)\right]$$

$$= 2i\frac{2p_\pi}{\sqrt{t}}\frac{\delta^{ab}}{2}\left[\left(A^\pi(t)\right)^*\left(\frac{4}{3t}p_\pi^2\left(t_0^0(t)-t_2^0(t)\right)\left(P^\mu P^\nu - tg^{\mu\nu}\right) + t_2^0(t)\Delta^\mu\Delta^\nu\right) + \left(D^\pi(t)\right)^* t_0^0(t)\left(P^\mu P^\nu - tg^{\mu\nu}\right)\right]$$

$$\text{Im}\,A^\pi(t) = \frac{2p_\pi}{\sqrt{t}}\left(t_2^0(t)\right)^* A^\pi(t)\,,$$

$$\text{Im}\,D^\pi(t) = \frac{2p_\pi}{\sqrt{t}}\left[\frac{4}{3}\frac{p_\pi^2}{t}\left(t_0^0(t)-t_2^0(t)\right)^* A^\pi(t) + \left(t_0^0(t)\right)^* D^\pi(t)\right] \Rightarrow \text{Im}\,\Theta^\pi(t) = \frac{2p_\pi}{\sqrt{t}}\left(t_0^0(t)\right)^*\Theta^\pi(t)$$

- Decomposition into $J^{PC} = 0^{++}, 2^{++}$ matrix elements (conserved separately):

K. Raman (1971)

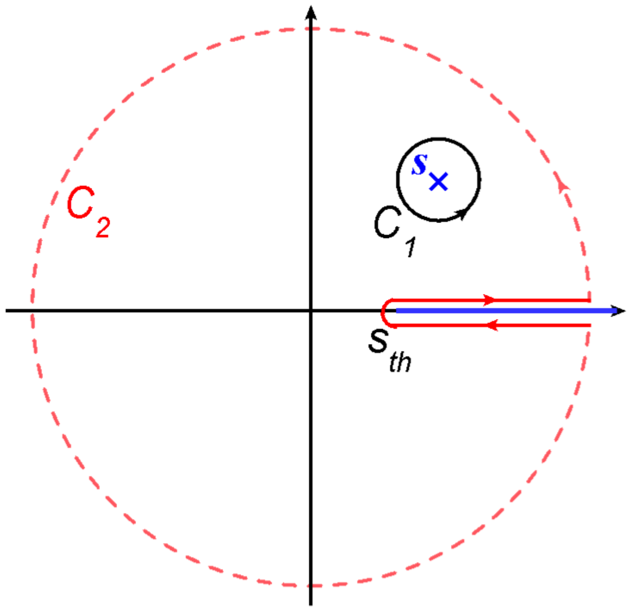
$$\left\langle \pi^a(p')\pi^b(p)\left|\hat{T}^{\mu\nu}(0)\right|0\right\rangle = \delta^{ab}\left\{\frac{1}{3}\left(g^{\mu\nu} - \frac{P^\mu P^\nu}{P^2}\right)\Theta^\pi(t) + \left[\Delta^\mu\Delta^\nu + \frac{\Delta^2}{3t}\left(P^\mu P^\nu - tg^{\mu\nu}\right)\right]A^\pi(t)\right\}$$

trace part: $\left\langle \pi^a(p')\pi^b(p)\left|\hat{T}^\mu_{\ \mu}(0)\right|0\right\rangle = \delta^{ab}\Theta^\pi(t)$, $\quad \Theta^\pi(t) = -\frac{1}{2}\left(4p_\pi^2 A^\pi(t) + 3tD^\pi(t)\right)$

Dispersion relation

- Causality ⇒ analyticity, Cauchy's theorem ⇒ dispersion relation:

$$f(s) = \frac{1}{2\pi i}\oint_{C_1} dz\,\frac{f(z)}{z-s} = \frac{1}{2\pi i}\int_{s_{\rm th}}^{\infty} dz\,\frac{{\rm disc}f(z)}{z-s}$$



- Schwarz reflection principle: discontinuity along the cut

$${\rm disc}f(z) = f(z+i\epsilon) - f(z-i\epsilon) = 2i\,{\rm Im}f(z+i\epsilon)$$

- One or more subtractions might be necessary if $\lim_{z\to\infty} f(z) \neq 0$
  - E.g., once-subtracted dispersion relation

$$f(s) - f(s_0) = \frac{s-s_0}{2\pi i}\int_{s_{\rm th}}^{\infty} dz\,\frac{{\rm disc}f(z)}{(z-s)(z-s_0)}$$

$\pi\pi$-$K\bar{K}$ coupled channels

- Data for $\pi\pi$ phase shifts known precisely from Roy(-like) equation analyses — Bern group; Madrid-Krakow group
- Generalization to coupled channels: isoscalar, scalar $\pi\pi$-$K\bar{K}$; $f_0(500)$, $f_0(980)$ mesons
  - ☐ Unitarity relation for $\Theta^\pi(t)$ ⇒ matrix relation for coupled channels (both pion and kaon trace GFFs):

$$\mathrm{Im}\,\Theta^\pi(t) = \frac{2p_\pi}{\sqrt{t}}\left(t_0^0(t)\right)^*\Theta^\pi(t)$$

$$\mathrm{Im}\boldsymbol{\Theta}(t) = [\mathbf{T}_0^0(t)]^*\,\boldsymbol{\Sigma}_0^0(t)\,\boldsymbol{\Theta}(t), \quad \boldsymbol{\Theta}(t) = \begin{pmatrix}\Theta^\pi(t)\\ \frac{2}{\sqrt{3}}\Theta^K(t)\end{pmatrix}$$

phase-space factor

$$\boldsymbol{\Sigma}_0^0(t) \equiv \mathrm{diag}(\sigma_\pi\theta(t-t_\pi), \sigma_K\theta(t-t_K))$$

$$\text{with } \sigma_i(t) \equiv \sqrt{1-4m_i^2/t}\ (i=\pi,K)$$

$\pi\pi$-$K\bar{K}$ T-matrix

$$\mathbf{T}_0^0(t) = \begin{pmatrix}\frac{\eta_0^0(t)e^{2i\delta_0^0(t)}-1}{2i\sigma_\pi} & |g_0^0(t)|e^{i\Psi_0^0(t)}\\ |g_0^0(t)|e^{i\Psi_0^0(t)} & \frac{\eta_0^0(t)e^{2i(\Psi_0^0(t)-\delta_0^0(t))}-1}{2i\sigma_K}\end{pmatrix}$$

$$\eta_0^0(t) = \sqrt{1-4\sigma_\pi\sigma_K|g_0^0(t)|^2\theta\left(t-t_K\right)}$$

J. Donoghue, J. Gasser, H. Leutwyler, NPB 343 (1991) 341

Muskhelishvili-Omnès representation

- Coupled-channel: solution known as the Muskhelishvili-Omnès (MO) representation
  - ☐ The above can be generalized to $\pi\pi$-$K\bar{K}$ coupled channels (matching point: $\sim$ 1.3 GeV)
  - ☐ Take isoscalar scalar $\pi\pi$-$K\bar{K}$ as example

$$\boldsymbol{\Omega}_0^0(t) = \frac{1}{\pi}\int_{t_\pi}^{\infty} \frac{\mathrm{d}t'}{t'-t}[\mathbf{T}_0^0(t)]^* \,\boldsymbol{\Sigma}_0^0(t)\,\boldsymbol{\Omega}_0^0(t')$$

Precisely known scattering phases

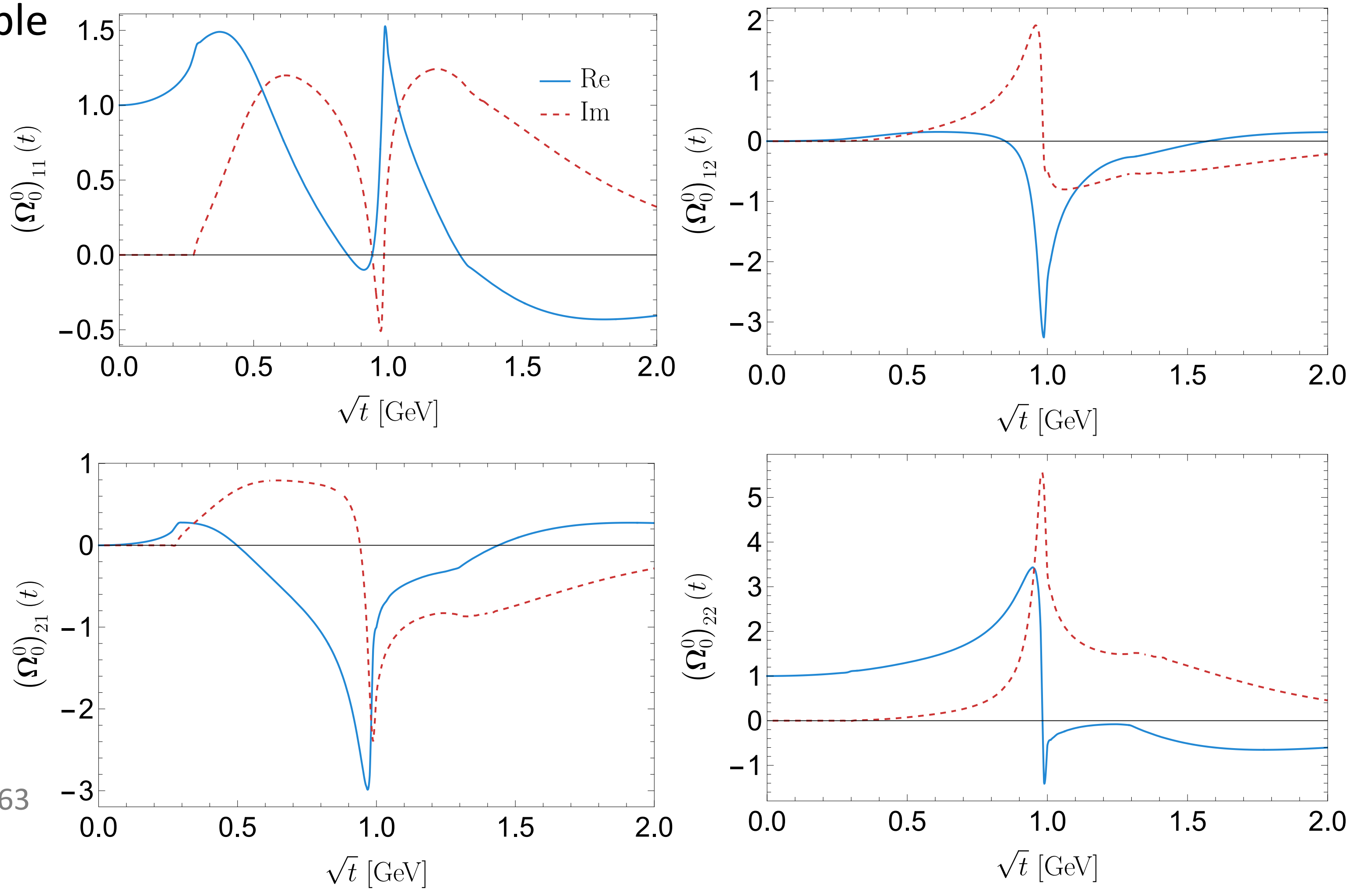
$\pi\pi$ phase shifts: Roy equation

I. Caprini et al. (2012);

$\pi\pi \to K\bar{K}$: Roy-Steiner equation

P. Büttiker et al. (2004);

M. Hoferichter et al., JHEP 06 (2012) 063



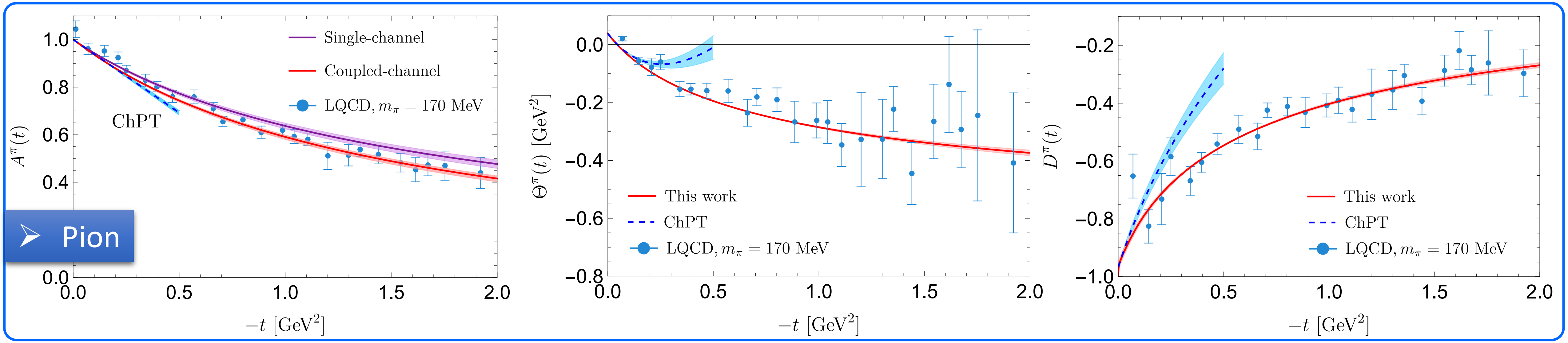
Pion and kaon GFFs **Prediction, NOT fit**

At low-energy, matching to chiral perturbation theory;
then no free parameters!

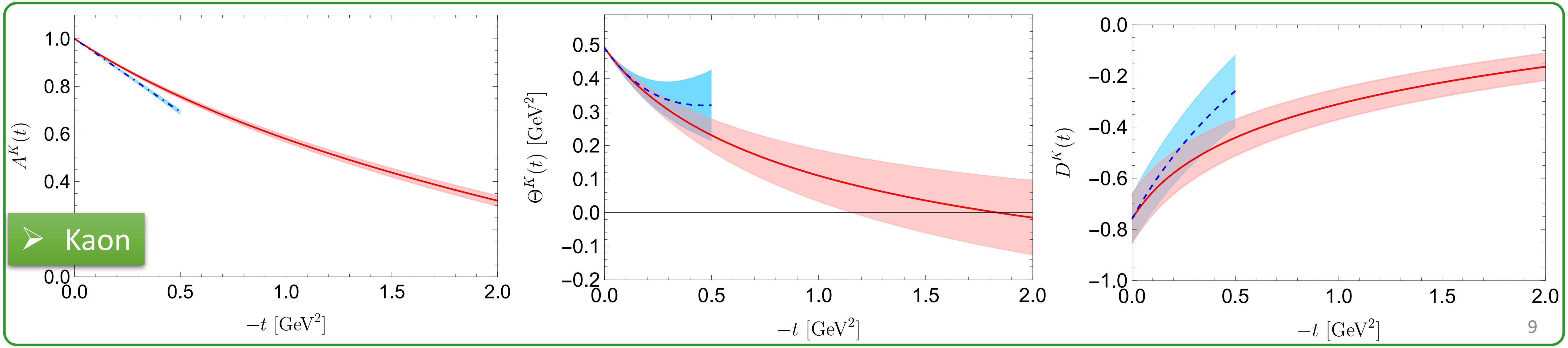
J. Donoghue, H. Leutwyler, ZPC 52 (1991) 343

LQCD ($m_\pi = 170$ MeV): D.C. Hackett et al., PRL 132 (2024) 251904

➢ Pion



➢ Kaon

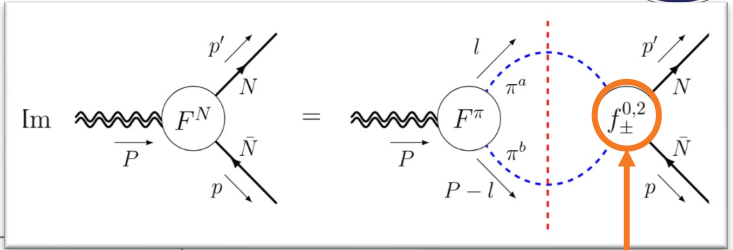


Unitarity relation for nucleon GFFs

- Partial-wave amplitudes for $\pi\pi \to N\bar{N}$ W. Frazer, J. Fulco (1960); G. Höhler (1983)

$$A^I(t,s) = -\frac{8\pi}{p_N^2}\sum_{J=0}^{\infty}\left(J+\frac{1}{2}\right)(p_\pi p_N)^J\left\{P_J(\cos\theta)f_+^J(t) - \frac{m_N\cos\theta}{\sqrt{J(J+1)}}P'_J(\cos\theta)f_-^J(t)\right\},$$

$$B^I(t,s) = 8\pi\sum_J \frac{J+\frac{1}{2}}{\sqrt{J(J+1)}}(p_\pi p_N)^{J-1}P'_J(\cos\theta)\,f_-^J(t)$$



$\pi\pi/K\bar{K} \to N\bar{N}$ amp.

$I = +/-$ for even/odd J;
$f_\pm^J$: $\pi\pi \to N\bar{N}$ partial-wave amp. with
$+/-$ for parallel/antiparallel $N\bar{N}$ helicities

- Discontinuity of the nucleon GFFs

$$\mathrm{Im}\,A^s(t) = \frac{3p_\pi^5}{\sqrt{6t}}\left[f_-^2(t) + \sqrt{\frac{3}{2}}\frac{m_N}{p_N^2}\left(m_N\sqrt{\frac{2}{3}}f_-^2(t) - f_+^2(t)\right)\right]^* A^\pi(t)$$

$$\mathrm{Im}\,J^s(t) = \frac{3p_\pi^5}{2\sqrt{6t}}\left(f_-^2(t)\right)^* A^\pi(t),$$

$$\Rightarrow \mathrm{Im}\,\Theta^s(t) = -\frac{3p_\pi}{4p_N^2\sqrt{t}}\left(f_+^0(t)\right)^*\Theta^\pi(t)$$

$$\mathrm{Im}\,D^s(t) = -\frac{3m_N p_\pi}{2p_N^2\sqrt{t}}\left[\frac{4p_\pi^2}{3t}\left(\left(f_+^0(t)\right)^* - (p_\pi p_N)^2\left(f_+^2(t)\right)^*\right)A^\pi(t) + \left(f_+^0(t)\right)^* D^\pi(t)\right]$$

coupled-channel

$$\mathrm{Im}\,\Theta^s(t) = -\frac{3}{4p_N^2\sqrt{t}}\left[p_\pi\left(f_+^0(t)\right)^*\Theta^\pi(t)\theta(t-t_\pi) + \frac{4}{3}p_K\left(h_+^0(t)\right)^*\Theta^K(t)\theta(t-t_K)\right]$$

$K\bar{K} \to N\bar{N}$ amplitude

- Decomposition into $J^{PC} = 0^{++}, 2^{++}$ matrix elements

$$\left\langle N(p')\bar{N}(p)\left|\hat{T}^{\mu\nu}(0)\right|0\right\rangle = \bar{u}(p')\left(T_S^{\mu\nu} + T_T^{\mu\nu}\right)v(p)$$

$$T_S^{\mu\nu} = \frac{1}{3}\left(g^{\mu\nu} - \frac{P^\mu P^\nu}{P^2}\right)\Theta^s(t),\qquad \Theta^s(t) = \frac{1}{4m_N}\left[-4p_N^2A^s(t) + 2tJ^s(t) - 3tD^s(t)\right]$$

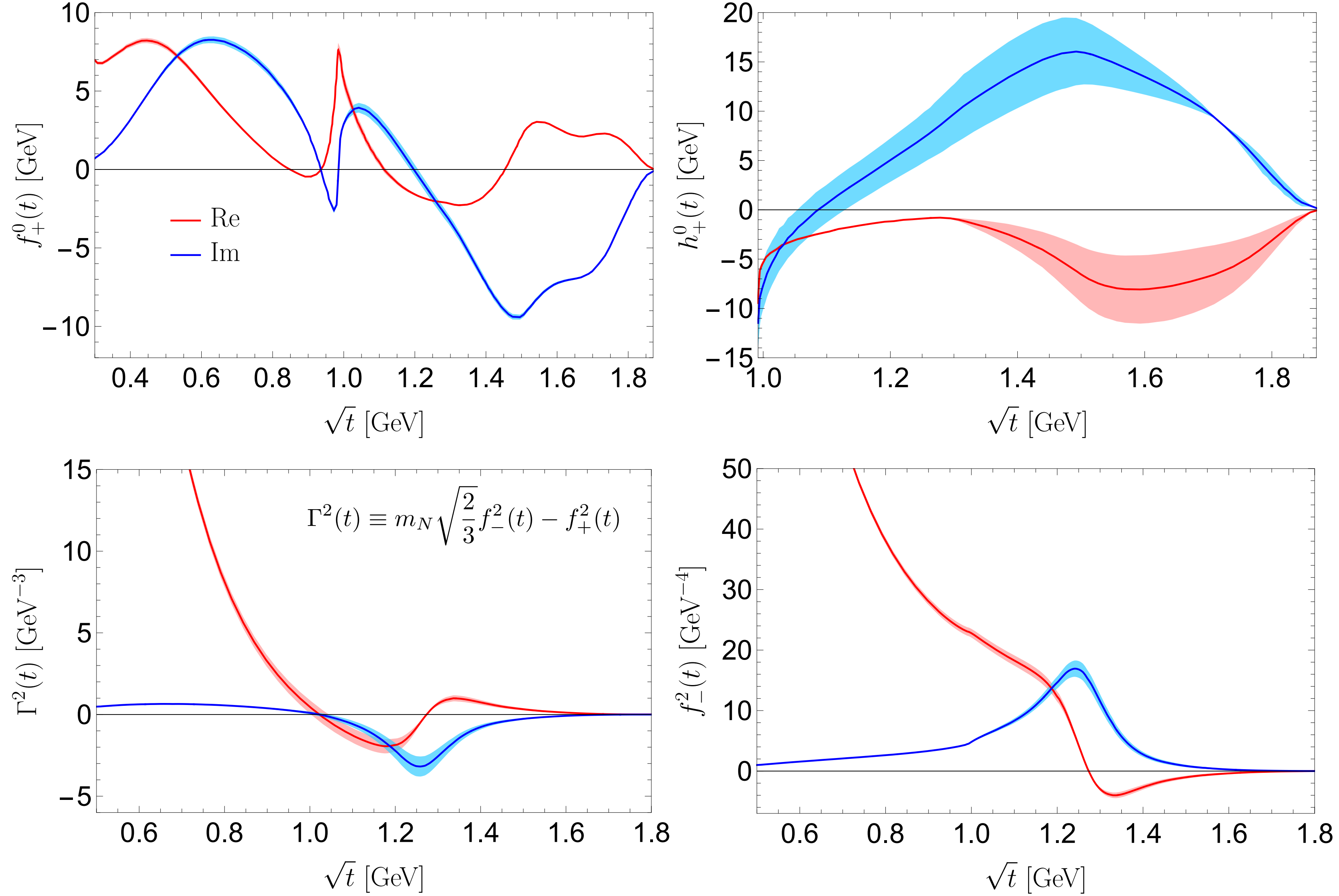
$$T_T^{\mu\nu} = \frac{1}{4m_N}\left[\Delta^\mu\Delta^\nu + \frac{\Delta^2}{3t}\left(P^\mu P^\nu - tg^{\mu\nu}\right)\right]A^s(t) + \left[i\Delta^{\{\mu}\sigma^{\nu\}\rho}P_\rho + \frac{2i\sigma^{\rho\kappa}\Delta_\rho P_\kappa}{3t}\left(P^\mu P^\nu - tg^{\mu\nu}\right)\right]J^s(t)$$

$\pi\pi/K\bar{K} \to N\bar{N}$ S-wave amplitudes

G.E. Hite, F. Steiner (1973)

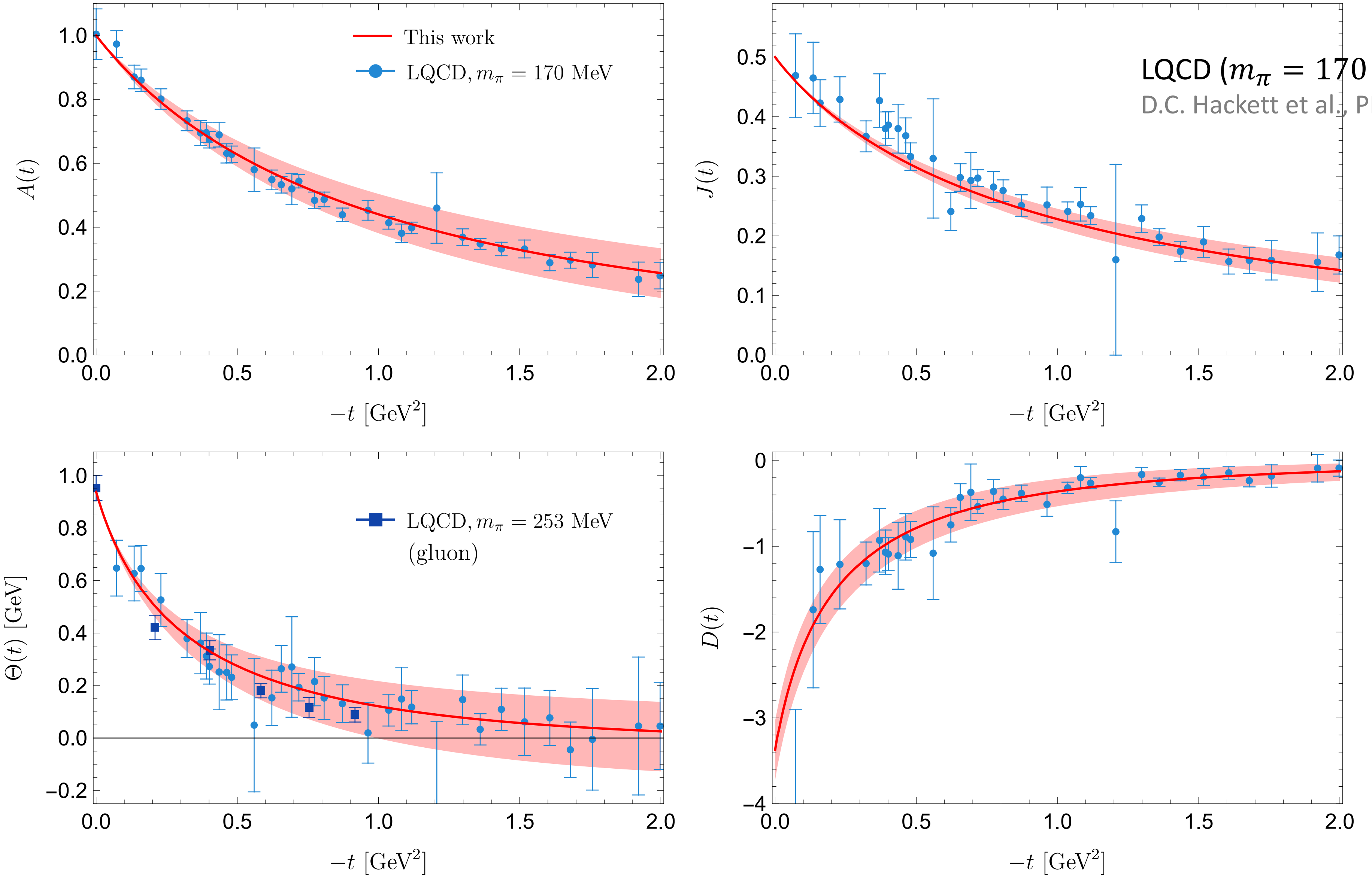
- **Input data:** $\pi\pi/K\bar{K} \to N\bar{N}$ S-wave amplitudes $f_\pm^{0,2}, h_+^0$ from Roy-Steiner equation analyses

M. Hoferichter et al., Phys. Rept. 625 (2016) 1; PLB 853 (2024) 138698; X.-H. Cao et al., JHEP 12 (2022) 073



Nucleon GFFs: results **Prediction, NOT fit**

LQCD ($m_\pi = 170$ MeV):
D.C. Hackett et al., PRL 132 (2024) 251904



LQCD ($m_\pi = 253$ MeV), gluon part only:
B. Wang et al. [χQCD], PRD 109 (2024) 094504

Nucleon GFFs: results

- D-term: $D \equiv D(0)$
- Various radii in the Breit frame
  - From the trace FF (scalar radius):

$$\langle r_\Theta^2 \rangle = \frac{6\dot{\Theta}(0)}{m_N} = 6\dot{A}(0) - \frac{9D}{2m_N^2}$$

  - Radius of the energy density (mass radius):

$$\langle r_{\rm Mass}^2 \rangle = 6\dot{A}(0) - \frac{3D}{2m_N^2}$$

  - Mechanical radius: M. Polyakov, PLB 555 (2003) 57; M. Polyakov, P. Schweitzer, IJMPA 33 (2018) 183005; C. Lorcé et al., EPJC 79 (2019) 89

$$\langle r_{\rm Mech}^2 \rangle = \frac{6D}{\int_{-\infty}^0 {\rm d}t\ D(t)}$$

  - Radius of the density $J(t) + \frac{2}{3} t \frac{{\rm d}J(t)}{{\rm d}t}$:

$$\langle r_J^2 \rangle = 20J'(0)$$

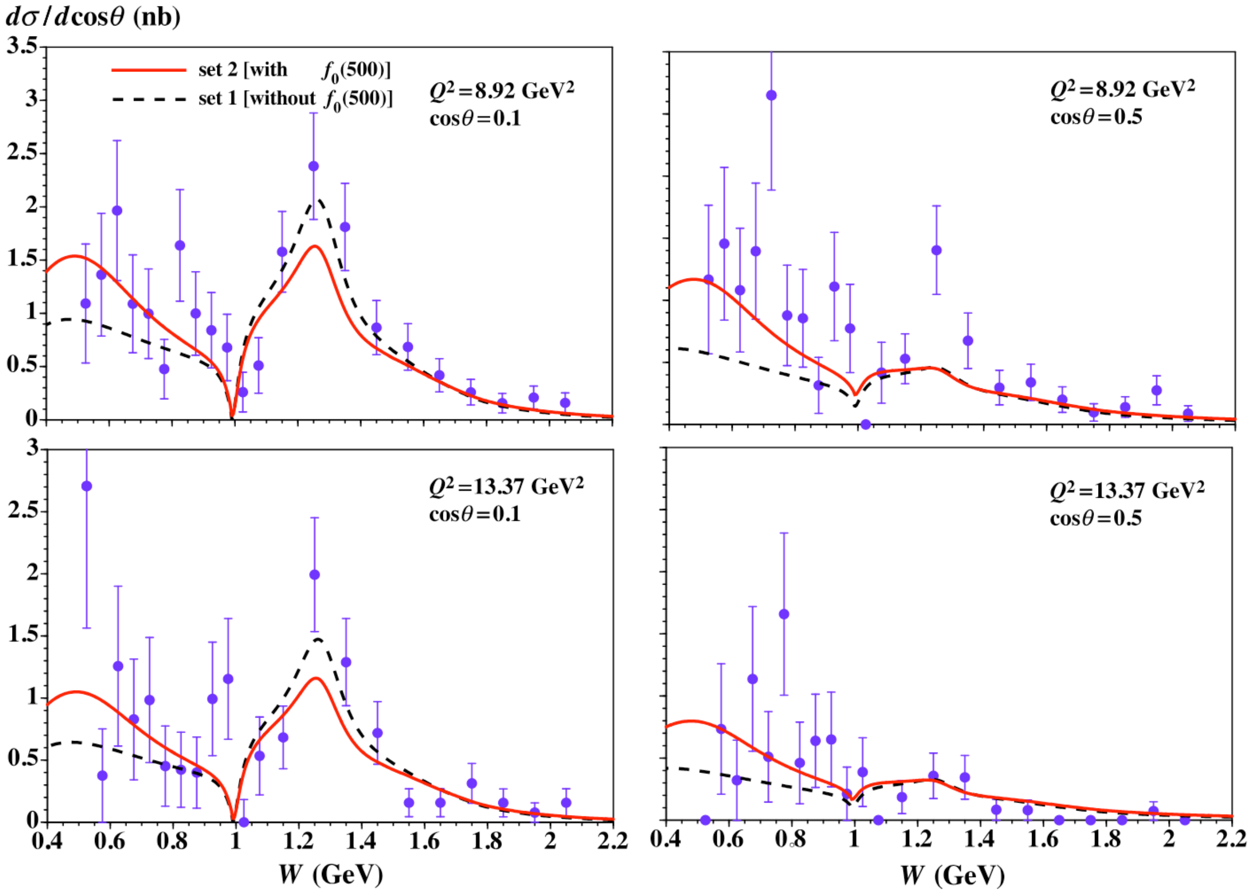
M. Polyakov, PLB 555 (2003) 57; C. Lorcé et al., PLB 776 (2018) 38

Quantity	Result	Error budget
D-term	$-3.38^{+0.34}_{-0.35}$	$+(0.18)_{\rm ChPT}(0.12)_{\rm pwa}(0.26)_{\rm eff}$
		$-(0.16)_{\rm ChPT}(0.12)_{\rm pwa}(0.29)_{\rm eff}$
$\sqrt{\langle r_\Theta^2 \rangle}$ [fm]	$0.97^{+0.03}_{-0.03}$	$+(0.01)_{\rm ChPT}(0.01)_{\rm pwa}(0.03)_{\rm eff}$
		$-(0.02)_{\rm ChPT}(0.01)_{\rm pwa}(0.02)_{\rm eff}$
$\sqrt{\langle r_{\rm Mass}^2 \rangle}$ [fm]	$0.70^{+0.03}_{-0.04}$	$+(0.02)_{\rm ChPT}(0.01)_{\rm pwa}(0.02)_{\rm eff}$
		$-(0.02)_{\rm ChPT}(0.01)_{\rm pwa}(0.03)_{\rm eff}$
$\sqrt{\langle r_{\rm Mech}^2 \rangle}$ [fm]	$0.72^{+0.09}_{-0.08}$	$+(0.02)_{\rm ChPT}(0.00)_{\rm pwa}(0.09)_{\rm eff}$
		$-(0.03)_{\rm ChPT}(0.01)_{\rm pwa}(0.07)_{\rm eff}$
$\sqrt{\langle r_J^2 \rangle}$ [fm]	$0.70^{+0.02}_{-0.02}$	$+(0.01)_{\rm ChPT}(0.01)_{\rm pwa}(0.01)_{\rm eff}$
		$-(0.01)_{\rm ChPT}(0.00)_{\rm pwa}(0.02)_{\rm eff}$

- ChPT: NLO ChPT inputs
- pwa: $\pi\pi/K\bar{K} \to N\bar{N}$
- eff: effective poles m_S, m_D

Two-photon processes

- $\gamma\gamma^{(*)} \to \pi\pi/K\bar{K}/\bar{N}N$: the same partial waves enter $[0^{++}, 2^{++}]$, check dispersive inputs
  - ☐ Dispersive analyses of $\gamma\gamma^* \to \pi\pi$ M. Hoferichter, D.R. Phillips, C. Schat EPJC 71 (2011) 1743; I. Danilkin, M. Vanderhaeghen, PLB 789 (2019) 366
- Analysis with generalized distribution amplitude (GDA) parametrization S. Kumano, Q.-T. Song, O. V. Teryaev, PRD 97 (2018) 014020



$$\frac{d\sigma}{d(\cos\theta)} = \frac{\pi\alpha^2}{4(Q^2+s)}\sqrt{1-\frac{4m_\pi^2}{s}}|A_{++}|^2,$$

$$A_{++} = \sum_a \frac{e_q^2}{2}\int_0^1 dz\,\frac{2z-1}{z(1-z)}\Phi_q^{\pi^0\pi^0}(z,\xi,W^2),$$

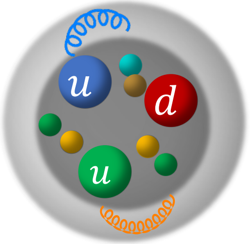
$$\int_0^1 d\,z(2z-1)\Phi_q^{\pi^0\pi^0}(z,\zeta,W^2) = \frac{2}{(P^+)^2}\langle\pi^0\pi^0|T_q^{++}(0)|0\rangle$$

HAPOF-122: 宋勤涛 Generalized distribution amplitudes and gravitational form factors of hadrons

Summary

- The pion, kaon and nucleon GFFs are precisely determined using dispersive method with inputs:

- $D^N = -3.38^{+0.34}_{-0.35}$, $\sqrt{\langle r_\Theta^2 \rangle} = 0.97^{+0.03}_{-0.03}$ fm > $\sqrt{\langle r_{E,p}^2 \rangle} \simeq 0.84$ fm > $\sqrt{\langle r_{\text{Mass}}^2 \rangle} = 0.70^{+0.03}_{-0.04}$ fm



May be regarded as a confinement radius

proton charge radius

Bag radius in MIT bag model X. Ji, Front. Phys. (Beijing) 16 (2021) 64601

  - Gluons distributed over a larger region than (anti-)quarks
  - Foundation for further analyses, interpretations, etc.

- Question: How the GFFs can be reliably measured experimentally?

Thank you for your attention!

Pion and nucleon GFFs

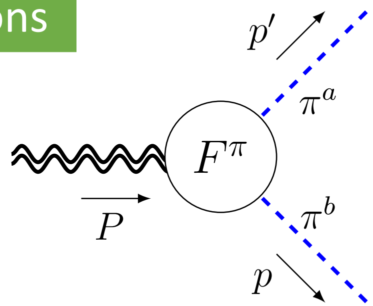
- Definitions
  - Gravitational form factors (GFFs) for spin-0 particles, e.g., for pion:

$$\left\langle \pi^a(p') \left| \hat{T}^{\mu\nu}(0) \right| \pi^b(p) \right\rangle = \frac{\delta^{ab}}{2} \left[A^\pi(t) P^\mu P^\nu + D^\pi(t) \left(\Delta^\mu \Delta^\nu - t g^{\mu\nu} \right) \right]$$

$P^\mu = p'^\mu + p^\mu, \Delta^\mu = p'^\mu - p^\mu$

Crossing symmetry, for constructing dispersion relations

$$\left\langle \pi^a(p') \pi^b(p) \left| \hat{T}^{\mu\nu}(0) \right| 0 \right\rangle = \frac{\delta^{ab}}{2} \left[A^\pi(t) \Delta^\mu \Delta^\nu + D^\pi(t) \left(P^\mu P^\nu - t g^{\mu\nu} \right) \right]$$



  - Nucleon GFFs

$$\left\langle N(p') \bar{N}(p) \left| \hat{T}^{\mu\nu}(0) \right| 0 \right\rangle = \frac{1}{4m_N} \bar{u}(p') \left[\hat{A}(t) \Delta^\mu \Delta^\nu + \hat{J}(t) \left(i \Delta^{\{\mu} \sigma^{\nu\}\rho} P_\rho \right) + \hat{D}(t) \left(P^\mu P^\nu - t g^{\mu\nu} \right) \right] u(p)$$