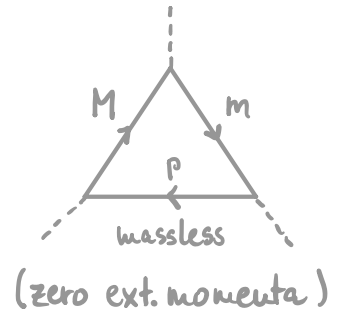


Problem Set 1 (16 August 2026)

Exercise 1:

Perform the region analysis for the integral:

$$I = \mu^{2\epsilon} \int \frac{d^D p}{(2\pi)^D} \frac{1}{p^2(p^2 - M^2)(p^2 - m^2)} ; \quad M^2 \gg m^2$$



Consider the five regions:

$$\text{i)} \quad |p^2| \gg M^2 \quad \text{ii)} \quad |p^2| \sim M^2 \quad \text{iii)} \quad M^2 \gg |p^2| \gg m^2$$

$$\text{iv)} \quad |p^2| \sim m^2 \quad \text{v)} \quad |p^2| \ll m^2$$

In each case, Taylor expand the integrand and keep the leading term only.

Solution:

We first derive the exact expression for the integral, using that:

$$\frac{1}{p^2(p^2-M^2)(p^2-m^2)} = \frac{1}{M^2 m^2 p^2} + \frac{1}{M^2-m^2} \left[\frac{1}{M^2} \frac{1}{p^2-M^2} - \frac{1}{m^2} \frac{1}{p^2-m^2} \right]$$

We now employ a standard formula for one-loop integrals.

One-loop master integral:

$$\int \frac{d^D p}{(2\pi)^D} \frac{1}{(p^2-\Delta)^n} = (-1)^n \frac{i}{(4\pi)^{D/2}} \frac{\Gamma(n-D/2)}{\Gamma(n)} \Delta^{D/2-n}$$

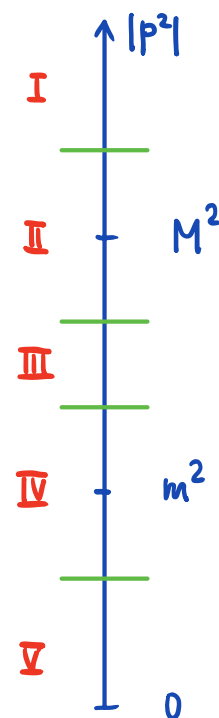
This leads to the exact answer:

$$\begin{aligned} I &= \frac{i}{(4\pi)^{2-\epsilon}} \Gamma(\epsilon) \frac{\mu^{2\epsilon}}{1-\epsilon} \frac{(M^2)^{-\epsilon} - (m^2)^{-\epsilon}}{M^2 - m^2} = -\frac{i}{16\pi^2} \frac{\ln M^2/m^2}{M^2 - m^2} + \mathcal{O}(\epsilon) \\ &= -\frac{i}{16\pi^2} \frac{1}{M^2} \left[1 + \frac{m^2}{M^2} + \frac{m^4}{M^4} + \dots \right] \ln \frac{M^2}{m^2} + \mathcal{O}(\epsilon) \end{aligned}$$

We now perform the region analysis in dimensional regularization. We need to consider 5 regions in total:

I. $|p^2| \gg M^2 \gg m^2$

$$\begin{aligned} I_I &= \mu^{2\epsilon} \int \frac{d^D p}{(2\pi)^D} \frac{1}{(p^2)^3} \left[1 + \frac{M^2}{p^2} + \dots \right] \left[1 + \frac{m^2}{p^2} + \dots \right] \\ &= 0 \quad (\text{scaleless integrals}) \end{aligned}$$



II. $|p^2| \sim M^2 \gg m^2$

$$I_{II} = \mu^{2\epsilon} \int \frac{d^D p}{(2\pi)^D} \frac{1}{(p^2)^2 (p^2 - M^2)} \left[1 + \frac{m^2}{p^2} + \frac{m^4}{(p^2)^2} + \dots \right]$$

The integrals can be evaluated using a master formula.

Useful integral:

$$\int \frac{d^D p}{(2\pi)^D} \frac{1}{p^2 - M^2} \frac{1}{(p^2)^a} = - \frac{i}{(4\pi)^{D/2}} \Gamma(1 - \frac{D}{2}) (M^2)^{\frac{D}{2} - 1 - a}$$

This leads to:

$$I_{II} = \frac{i}{(4\pi)^{2-\epsilon}} \frac{\Gamma(\epsilon)}{1-\epsilon} \left(\frac{\mu^2}{M^2} \right)^\epsilon \frac{1}{M^2} \left[1 + \frac{m^2}{M^2} + \frac{m^4}{M^4} + \dots \right]$$

We can not only evaluate the leading term, but also power corrections in $m^2/M^2 \ll 1$.

III. $M^2 \gg |p^2| \gg m^2$

$$\begin{aligned} I_{III} &= \mu^{2\epsilon} \int \frac{d^D p}{(2\pi)^D} \frac{1}{(p^2)^2} \frac{1}{(-M^2)} \left[1 + \frac{p^2}{M^2} + \dots \right] \left[1 + \frac{m^2}{p^2} + \dots \right] \\ &= 0 \quad (\text{scaleless integrals}) \end{aligned}$$

IV. $|p^2| \sim m^2 \ll M^2$

$$\begin{aligned} I_{\text{IV}} &= \mu^{2\epsilon} \int \frac{d^D p}{(2\pi)^D} \frac{1}{p^2(p^2 - m^2)} \frac{1}{(-M^2)} \left[1 + \frac{p^2}{M^2} + \frac{(p^2)^2}{M^4} + \dots \right] \\ &= - \frac{i}{(4\pi)^{2-\epsilon}} \frac{\Gamma(\epsilon)}{1-\epsilon} \left(\frac{\mu^2}{m^2} \right)^\epsilon \frac{1}{M^2} \left[1 + \frac{m^2}{M^2} + \frac{m^4}{M^4} + \dots \right] \end{aligned}$$

V. $|p^2| \ll m^2 \ll M^2$

$$\begin{aligned} I_{\text{V}} &= \mu^{2\epsilon} \int \frac{d^D p}{(2\pi)^D} \frac{1}{p^2} \frac{1}{(-m^2)(-M^2)} \left[1 + \frac{p^2}{M^2} + \dots \right] \left[1 + \frac{p^2}{m^2} + \dots \right] \\ &= 0 \quad (\text{scaleless integrals}) \end{aligned}$$

Notice that non-zero contributions arise only if the loop momentum is assumed to be commensurate with one of the external scales, $|p^2| \sim M^2$ or $|p^2| \sim m^2$.

Note also that in all cases we integrate over all loop momenta after the expansions of the integrand have been performed. The sum $I_{\text{II}} + I_{\text{IV}}$ precisely reproduces the exact result on page A.1, including the power-suppressed corrections.

It is instructive to take a closer look at these two integrals. Notice that

$$I_{II} = \mu^{2\epsilon} \int \frac{d^D p}{(2\pi)^D} \frac{1}{(p^2)^2 (p^2 - M^2)} \left[1 + \frac{m^2}{p^2} + \frac{m^4}{(p^2)^2} + \dots \right]$$

is UV-finite but receives IR-singular contributions from the region where $p^2 \rightarrow 0$. Expanding the result in ϵ , we obtain:

$$\begin{aligned} I_{II} &= \frac{i}{(4\pi)^{2-\epsilon}} \frac{\Gamma(\epsilon)}{1-\epsilon} \left(\frac{\mu^2}{M^2} \right)^\epsilon \frac{1}{M^2} \left[1 + \frac{m^2}{M^2} + \frac{m^4}{M^4} + \dots \right] \\ &= - \frac{i}{16\pi^2 M^2} \left[- \left(\frac{1}{\epsilon_{IR}} - \gamma_\epsilon + \ln 4\pi \right) + \ln \frac{M^2}{\mu^2} - 1 + \mathcal{O}(\epsilon) \right] + \mathcal{O}\left(\frac{m^2}{M^4}\right) \\ &\equiv I_{\text{hard}} \end{aligned}$$

Similarly, the integral

$$I_{IV} = \mu^{2\epsilon} \int \frac{d^D p}{(2\pi)^D} \frac{1}{p^2 (p^2 - m^2)} \frac{1}{(-M^2)} \left[1 + \frac{p^2}{M^2} + \frac{(p^2)^2}{M^4} + \dots \right]$$

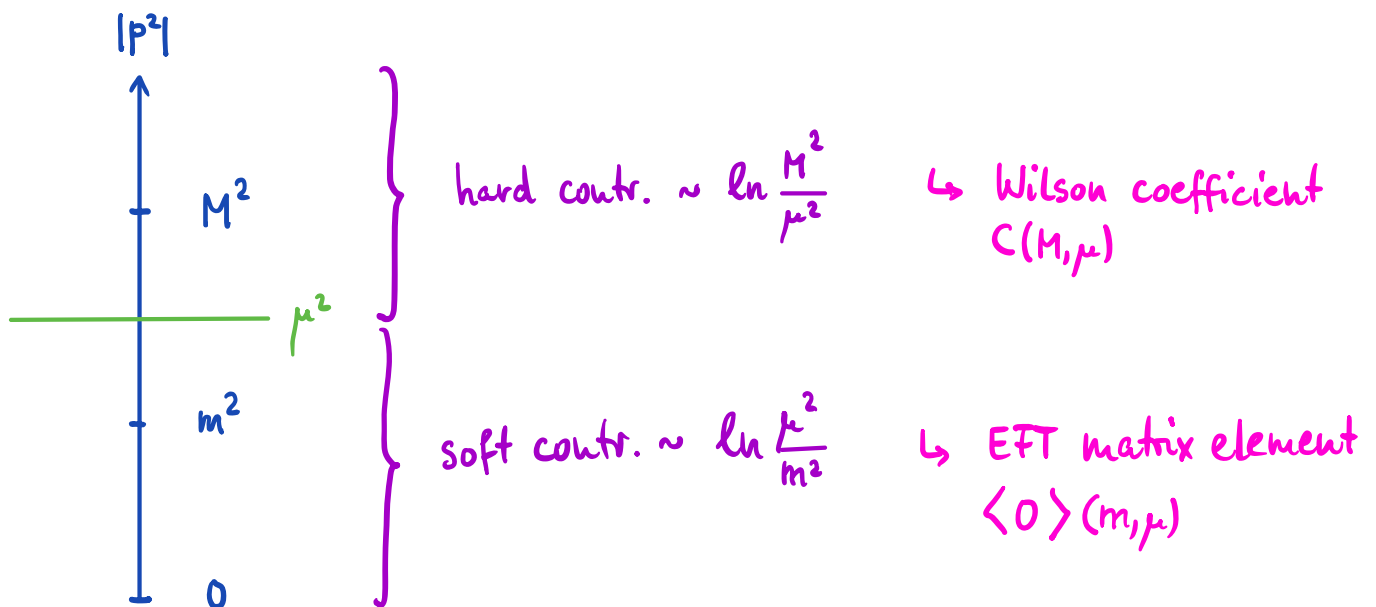
is IR-finite but UV-divergent for $|p^2| \rightarrow \infty$. Expanding in ϵ we find:

$$\begin{aligned} I_{IV} &= - \frac{i}{16\pi^2 M^2} \left[+ \left(\frac{1}{\epsilon_{UV}} - \gamma_\epsilon + \ln 4\pi \right) - \ln \frac{m^2}{\mu^2} + 1 + \mathcal{O}(\epsilon) \right] + \mathcal{O}\left(\frac{m^2}{M^4}\right) \\ &\equiv I_{\text{soft}} \end{aligned}$$

The divergences arise because after the Taylor expansion I_{II} no longer contains the IR regulator m^2 , and I_{IV} no longer contains the UV regulator M^2 . In the sum

$$I_{\text{hard}} + I_{\text{soft}} = -\frac{i}{16\pi^2 M^2} \left[\frac{1}{\epsilon_{\text{UV}}} - \frac{1}{\epsilon_{\text{IR}}} + \ln \frac{M^2}{m^2} + \mathcal{O}(\epsilon) \right] + \mathcal{O}\left(\frac{m^2}{M^4}\right)$$

the divergent terms cancel out. In EFT jargon, the hard contribution is absorbed into a Wilson coefficient, while the soft contribution corresponds to the matrix element of an EFT operator:



By construction, the dependence on the factorization scale μ cancels between the two contributions!

Exercise 2 :

Evaluate the collinear contribution

$$I_c = i \pi^{-D/2} \mu^{2\epsilon} \int d^D k \frac{1}{(k^2 + i0) [(k+p_1)^2 + i0] [2k \cdot p_2 + i0]}$$

to the Sudakov form factor.

Solution:

We combine the first two factors in the denominator using a standard Feynman parameter:

$$\frac{1}{(k^2 + i0) [(k+p_1)^2 + i0]} = \int_0^1 dx \frac{1}{(k^2 + 2x k \cdot p_1 + x p_1^2 + i0)^2}$$

For propagators that are linear in the loop momentum, it is convenient to use Feynman parameters $\lambda \in [0, \infty[$ and the following master formula.

Feynman parameters for linear propagators:

$$\frac{1}{A^a B^b} = \frac{\Gamma(a+b)}{\Gamma(a) \Gamma(b)} \int_0^\infty d\lambda \frac{\lambda^{b-1}}{(A + \lambda B)^{a+b}}$$

linear propagators

This gives:

$$\begin{aligned}
 & \frac{1}{(k^2 + i0) [(k+p_1)^2 + i0] [2k \cdot p_{2+} + i0]} \\
 &= 2 \int_0^1 dx \int_0^\infty d\lambda \frac{1}{\underbrace{(k^2 + 2xk \cdot p_1 + x p_1^2 + 2\lambda k \cdot p_{2+} + i0)^3}_{(k+xp_1+\lambda p_{2+})^2 - \Delta}} \\
 & \quad (k+xp_1+\lambda p_{2+})^2 - \Delta; \quad \Delta = (xp_1 + \lambda p_{2+})^2 - x p_1^2 - i0 \\
 & \quad = 2x\lambda p_1 \cdot p_{2+} - x(1-x) p_1^2 - i0
 \end{aligned}$$

We now use that:

$$2 p_1 \cdot p_{2+} \approx -q^2 = Q^2 \quad \text{up to } O(\lambda^2) \text{ corrections}$$

$$-p_1^2 - i0 = p_1^2$$

$$\hookrightarrow \Delta = \lambda x Q^2 + x(1-x) p_1^2$$

Changing variables from k to $\ell = k + xp_1 + \lambda p_{2+}$ yields:

$$\begin{aligned}
 I_c &= i \pi^{-D/2} \mu^{2\epsilon} \cdot (2\pi)^D \cdot 2 \int_0^1 dx \int_0^\infty d\lambda \underbrace{\int \frac{d^D \ell}{(2\pi)^D} \frac{1}{(\ell^2 - \Delta)^3}}_{\frac{-i}{(4\pi)^{D/2}} \frac{1}{2} \Gamma(1+\epsilon) \Delta^{-1-\epsilon}} \\
 &= \Gamma(1+\epsilon) \mu^{2\epsilon} \int_0^1 dx \int_0^\infty d\lambda \underbrace{(\lambda x Q^2 + x(1-x) p_1^2)^{-1-\epsilon}}_{-\frac{1}{\epsilon} \frac{1}{x Q^2} (\lambda x Q^2 + x(1-x) p_1^2)^{-\epsilon} \Big|_{\lambda=0}^\infty} \\
 &= \frac{\Gamma(1+\epsilon)}{Q^2} \frac{1}{\epsilon} \left(\frac{\mu^2}{p_1^2} \right)^\epsilon \int_0^1 dx x^{-1-\epsilon} (1-x)^{-\epsilon}
 \end{aligned}$$

Using the identity

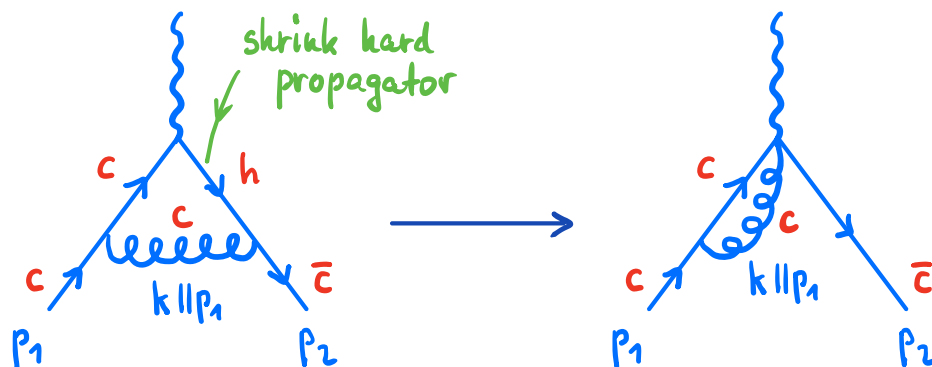
$$\int_0^1 dx x^a (1-x)^b = \frac{\Gamma(a+1) \Gamma(b+1)}{\Gamma(a+b+2)}$$

we finally obtain:

$$\begin{aligned} I_c &= \frac{\Gamma(1+\epsilon)}{Q^2} \frac{\Gamma(-\epsilon) \Gamma(1-\epsilon)}{\epsilon \Gamma(1-2\epsilon)} \left(\frac{\mu^2}{P_1^2} \right)^\epsilon \\ &= \frac{\Gamma(1+\epsilon)}{Q^2} \left[-\frac{1}{\epsilon^2} - \frac{1}{\epsilon} \ln \frac{\mu^2}{P_1^2} - \frac{1}{2} \ln^2 \frac{\mu^2}{P_1^2} + \frac{\pi^2}{6} + \mathcal{O}(\epsilon) \right] \end{aligned}$$

This agrees with the expression given in the lectures.

It is interesting to give a "diagrammatic EFT interpretation" of this integral obtained by shrinking the hard propagator to a point:



The SCET Feynman rules must reproduce the collinear loop graph shown on the right. To this end, SCET must contain an unusual $q\bar{q}g\gamma$ -vertex not present in the full theory.

Exercise 3 :

Evaluate the integral

$$I = \int_1^{\infty} dx \frac{x^{-\epsilon}}{x + \mu x^3} \quad ; \quad 0 < \mu \ll 1 \quad (\epsilon > -2)$$

using the method of regions, dropping terms of $O(\mu^2)$.

Solution:

The integral depends on two "scales": the lower integration limit $x_{\min} = 1$ and the parameter μ . We can rewrite:

$$I = \int_0^{\infty} dx \frac{x^{-\epsilon}}{x + \mu x^3} \Theta(x-1)$$

The exact expression for the integral is rather complicated.

One finds:

$$I = \frac{1}{\epsilon} \left[1 - {}_2F_1 \left(1, \frac{\epsilon}{2}; 1 + \frac{\epsilon}{2}; -\frac{1}{\mu} \right) \right]$$

hypergeometric function
↙

HypExp
package
↘

$$= \frac{1}{\epsilon} \left[1 - \Gamma\left(1 + \frac{\epsilon}{2}\right) \Gamma\left(1 - \frac{\epsilon}{2}\right) \mu^{\epsilon/2} \right] + \frac{\mu}{2-\epsilon} + O(\mu^2)$$

Let us reproduce this result, including the first-order correction in μ , using the method of regions. The two relevant regions are $x \sim 1$ and $x \sim \mu^{-1/2}$.

Region 1: $x \sim 1$

$$\begin{aligned}
 I_1 &= \int_0^{\infty} dx \frac{x^{-\epsilon}}{x} \left[1 - \mu x^2 + \mathcal{O}(\mu^2) \right] \Theta(x-1) \\
 &= \int_1^{\infty} dx x^{-1-\epsilon} - \mu \int_1^{\infty} dx x^{1-\epsilon} + \mathcal{O}(\mu^2) \\
 &= -\frac{1}{\epsilon} x^{-\epsilon} \Big|_{x=1}^{\infty} - \frac{\mu}{2-\epsilon} x^{2-\epsilon} \Big|_{x=1}^{\infty} + \mathcal{O}(\mu^2) \\
 &= \frac{1}{\epsilon} + \frac{\mu}{2-\epsilon} + \mathcal{O}(\mu^2)
 \end{aligned}$$

Note that the first integral converges for $\epsilon > 0$, while for the second one we need to assume that $\epsilon > 2$. But this is o.k., since at the final result is defined by analytic continuation in ϵ to the region $\epsilon > -2$.

Region 2: $x \sim \mu^{-1/2}$

This region is chosen such that the two terms in the denominator are of the same order. Setting $x = \frac{y}{\sqrt{\mu}}$, we obtain:

$$\begin{aligned}
 I_2 &= \int_0^{\infty} dy \frac{y^{-\epsilon}}{y(1+y^2)} \mu^{\epsilon/2} \Theta\left(y - \frac{1}{\sqrt{\mu}}\right) \\
 &\quad \quad \quad \begin{array}{l} \text{must expand, since } y = \mathcal{O}(1) \\ \text{in this region} \end{array} \\
 &\quad \quad \quad \Theta(y^2 - \mu) \\
 &= \mu^{\epsilon/2} \int_0^{\infty} dy \frac{y^{-\epsilon}}{y(1+y^2)} \left[\Theta(y^2) - \mu \delta(y^2) + \mathcal{O}(\mu^2) \right]
 \end{aligned}$$

Changing variables from y to $z=y^2$, we find:

$$I_2 = \mu^{\epsilon/2} \frac{1}{2} \int_0^\infty dz \frac{z^{-\epsilon/2}}{z(1+z)} \left[1 - \mu \delta(z) + O(\mu^2) \right]$$

The contribution from the Dirac δ -function converges if we assume that $\epsilon < -2$, and if that condition is satisfied it evaluates to zero. The integral for the first term converges for $-2 < \epsilon < 0$. Setting $w = \frac{1}{1+z}$, we find:

$$\int_0^\infty dz \frac{z^{-\epsilon/2}}{z(1+z)} = \int_0^1 dw w^{\epsilon/2} (1-w)^{-1-\epsilon/2} = \Gamma(1+\frac{\epsilon}{2}) \Gamma(-\frac{\epsilon}{2})$$

This gives:

$$\begin{aligned} I_2 &= \frac{\mu^{\epsilon/2}}{2} \Gamma(1+\frac{\epsilon}{2}) \Gamma(-\frac{\epsilon}{2}) + O(\mu^2) \\ &= -\frac{1}{\epsilon} \frac{\mu^{\epsilon/2}}{2} \Gamma(1+\frac{\epsilon}{2}) \Gamma(1-\frac{\epsilon}{2}) + O(\mu^2) \end{aligned}$$

The sum $I_1 + I_2$ correctly reproduces the exact result for I given on page A.9, including the power correction of $O(\mu)$. Any other region of x values leads to scaleless integrals.