Collinear gluon field:

The relevant two-point function to consider is:

general cov. gauge

$$\langle 0| T\{A_c^{\mu,a}(x)\, A_c^{\nu,b}(0)\}|0\rangle = \int \underbrace{\frac{d^4p}{(2\pi)^4}}_{\lambda^4} e^{-ip\cdot x} \underbrace{\frac{i\delta_{ab}}{p^2+i\epsilon}}_{\lambda^{-2}} \Big[\underbrace{-g^{\mu\nu}}_{1} + (1-\xi)\frac{p^\mu p^\nu}{p^2}\Big]$$

various scalings

It would be incorrect to work in Feynman gauge ($\xi=1$) and conclude that $A_a^\mu \sim \lambda$. Rather, we need to decompose:

$$A_c^\mu = n\cdot A_c \frac{\bar{n}^\mu}{2} + \bar{n}\cdot A_c \frac{n^\mu}{2} + A_{c,\perp}^\mu$$

It follows from above that:

$$\underbrace{\langle 0| T\{n\cdot A_c(x)\; n\cdot A_c(0)\}|0\rangle}_{\boxed{\sim\lambda^4}} = \int \underbrace{\frac{d^4p}{(2\pi)^4}}_{\lambda^4} e^{-ip\cdot x} \underbrace{\frac{i\delta_{ab}}{p^2+i\epsilon}}_{\lambda^{-2}} \Big[0 + (1-\xi)\underbrace{\frac{(n\cdot p)^2}{p^2}}_{\frac{\lambda^4}{\lambda^2}}\Big]$$

$$\underbrace{\langle 0| T\{\bar{n}\cdot A_c(x)\; \bar{n}\cdot A_c(0)\}|0\rangle}_{\boxed{\sim\lambda^0}} = \int \underbrace{\frac{d^4p}{(2\pi)^4}}_{\lambda^4} e^{-ip\cdot x} \underbrace{\frac{i\delta_{ab}}{p^2+i\epsilon}}_{\lambda^{-2}} \Big[0 + (1-\xi)\underbrace{\frac{(\bar{n}\cdot p)^2}{p^2}}_{\frac{\lambda^0}{\lambda^2}}\Big]$$

$$\underbrace{\langle 0| T\{A_{c,\perp}^{\mu,a}(x)\; A_{c,\perp}^{\nu,b}(0)\}|0\rangle}_{\boxed{\sim\lambda^2}} = \int \underbrace{\frac{d^4p}{(2\pi)^4}}_{\lambda^4} e^{-ip\cdot x} \underbrace{\frac{i\delta_{ab}}{p^2+i\epsilon}}_{\lambda^{-2}} \Big[\underbrace{-g_\perp^{\mu\nu}}_{1} + (1-\xi)\underbrace{\frac{p_\perp^\mu p_\perp^\nu}{p^2}}_{\frac{\lambda^2}{\lambda^2}}\Big]$$

same

$$\Rightarrow \boxed{n\cdot A_c \sim \lambda^2\,, \quad \bar{n}\cdot A_c \sim 1\,, \quad A_{c,\perp}^\mu \sim \lambda}$$

This also works for the mixed terms, e.g.:

$$\underbrace{\langle 0 | T\{ n\cdot A_c(x)\, \bar{n}\cdot A_c(0)\} | 0\rangle}_{\sim \lambda^2} = \int \underbrace{\frac{d^4p}{(2\pi)^4}}_{\lambda^4} e^{-ip\cdot x} \underbrace{\frac{i\delta_{ab}}{p^2+i\epsilon}}_{\lambda^{-2}}$$

$$\times \Big[\underbrace{-n\cdot\bar{n}}_{1} + (1-\xi)\underbrace{\frac{n\cdot p\ \bar{n}\cdot p}{p^2}}_{\frac{\lambda^2\cdot 1}{\lambda^2}} \Big]$$

etc.

We observe that the different components of the collinear gluon field scale like the components of a collinear momentum, such that the covariant collinear derivative

$$\boxed{iD_c^\mu = i\partial^\mu + g_s A_c^{\mu,a} t^a \sim (\lambda^2, 1, \lambda)}$$

has homogeneous power counting in λ !

Important note:

The presence of a field with unsuppressed power counting, $\bar{n}\cdot A_c \sim 1$, is worrisome, since this will lead to an infinite number of operators with the same power counting in the EFT. We will see the implications of this observation later.

Ultra-soft quark field:

We have:

$$\langle 0 | T\{ q_{us}(x)\, \bar{q}_{us}(0) \} | 0 \rangle = \underset{\lambda^8}{\int \frac{d^4k}{(2\pi)^4}}\ \underset{1}{e^{-ik\cdot x}}\ \underset{\frac{\lambda^2}{\lambda^4}}{\frac{i \not{k}}{k^2 + i\epsilon}} \sim \lambda^6$$

This gives:

$$\boxed{q_{us} \sim \lambda^3}$$

There are no large or small components in this case.

Ultra-soft gluon fields:

We find:

$$\langle 0 | T\{ A^{\mu,a}_{us}(x), A^{\nu,b}_{us}(0) \} | 0 \rangle = \underset{\lambda^8}{\int \frac{d^4k}{(2\pi)^4}}\ e^{-ik\cdot x}\ \underset{\lambda^{-4}}{\frac{i\delta_{ab}}{k^2 + i\epsilon}} \left[\underset{1}{-g^{\mu\nu}} + (1-\xi) \underset{\lambda^0}{\frac{k^\mu k^\nu}{k^2}} \right]$$

$$\sim \lambda^4$$

This gives:

$$\boxed{A^{\mu,a}_{us} \sim \lambda^2}$$

In the following we will derive the effective Lagrangian of collinear and ultra-soft fields, focussing on the Dirac Lagrangian. The Yang-Mills Lagrangian can be discussed in an analogous way.

Introducing the decompositions

$$\psi \longrightarrow \psi_c + q_{us} = \xi_n + \eta_n + q_{us}$$

$$A^\mu \longrightarrow A^\mu_c + A^\mu_{us}$$

in the Dirac Lagrangian, we obtain:

$$\bar{\psi}\, i\not{D}\, \psi \longrightarrow (\overset{\lambda}{\bar{\xi}_n} + \overset{\lambda^2}{\bar{\eta}_n} + \overset{\lambda^3}{\bar{q}_{us}})\, i\not{D}_{c+us}\, (\overset{\lambda}{\xi_n} + \overset{\lambda^2}{\eta_n} + \overset{\lambda^3}{q_{us}})$$

where:

$$i\not{D}_{c+us} = (\overset{\lambda^2}{in\cdot D_c} + \overset{\lambda^2}{g_s\, n\cdot A_{us}})\frac{\not{\bar{n}}}{2} + (\overset{1}{i\bar{n}\cdot D_c} + \overset{\lambda^2}{g_s\, \bar{n}\cdot A_{us}})\frac{\not{n}}{2}$$
$$+ \overset{\lambda}{i\not{D}^\perp_c} + \overset{\lambda^2}{g_s \not{A}^\perp_{us}}$$

$\mathcal{O}(\lambda^4)$:

$$\bar{\xi}_n \frac{\not{\bar{n}}}{2}(in\cdot D_c + g_s\, n\cdot A_{us})\,\xi_n + \bar{\eta}_n \frac{\not{n}}{2}\, i\bar{n}\cdot D_c\, \eta_n$$
$$+ \bar{\xi}_n\, i\not{D}^\perp_c\, \eta_n + \bar{\eta}_n\, i\not{D}^\perp_c\, \xi_n$$

$\mathcal{O}(\lambda^5)$:

$$\bar{\xi}_n\, g_s \not{A}^\perp_{us}\, \eta_n + \bar{q}_{us}\, g_s \not{A}^\perp_c\, \xi_n + \bar{q}_{us}\frac{\not{n}}{2} g_s\, \bar{n}\cdot A_c\, \eta_n + \text{h.c.}$$

In the $\mathcal{O}(\lambda^5)$ Lagrangian we have used that terms involving only a single collinear field are not allowed by momentum conservation. Note that <u>all</u> terms of $\mathcal{O}(\lambda^5)$ and higher contain at least one ultra-soft field.

From now on we focus on the leading-order SCET Lagrangian:

$$\mathcal{L}_{SCET} = \mathcal{L}_c + \mathcal{L}_{us} + \mathcal{L}_{c+us}$$

with:

$$\mathcal{L}_c = \bar{\xi}_n \frac{\bar{\not n}}{2} i n\cdot D_c \xi_n + \bar{\eta}_n \frac{\not n}{2} i \bar{n}\cdot D_c \eta_n$$
$$+ \bar{\xi}_n i \not D_c^{\perp} \eta_n + \bar{\eta}_n i \not D_c^{\perp} \xi_n + (\text{pure glue terms}) \sim \lambda^4$$

$$\mathcal{L}_{c+us} = \bar{\xi}_n \frac{\bar{\not n}}{2} g_s n\cdot A_{us} \xi_n + (\text{pure glue terms}) \sim \lambda^4$$

$$\mathcal{L}_{us} = \bar{q}_{us} i \not D_{us} q_{us} + (\text{pure glue terms}) \sim \lambda^8$$

The leading-order action is:

$$S_{SCET} = \underbrace{\int d^4x}_{\lambda^{-4}} \underbrace{(\mathcal{L}_c + \mathcal{L}_{c+us})}_{\lambda^4} + \underbrace{\int d^4x}_{\lambda^{-8}} \underbrace{\mathcal{L}_{us}}_{\lambda^8} \sim \lambda^0$$

Since the field ξ_n contains the large components of the collinear spinor field, we can use it to describe collinear quarks and integrate out the power-suppressed field η_n in the generating functional. The functional determinant is just an irrelevant (divergent) constant. The resulting Lagrangian is obtained by using the solution of the classical equation of motion:

$$\frac{\delta \mathcal{L}_c}{\delta \bar{\eta}_n} = 0 \quad \Rightarrow \quad \underbrace{\frac{\not{n}}{2}\, i\bar{n}\cdot D_c}_{1}\, \underbrace{\eta_n}_{\lambda^3} + \underbrace{i\not{D}_c^\perp}_{\lambda}\, \underbrace{\xi_n}_{\lambda^2} = 0$$

To solve this equation for η_n we introduce an auxilliary regulator $i\delta$ to obtain:

$$\underbrace{\frac{\not{\bar{n}}\not{n}}{4}}\,(i\bar{n}\cdot D_c + i\delta)\,\eta_n = -\frac{\not{\bar{n}}}{2}\, i\not{D}_c^\perp\, \xi_n$$

replace by: $P_{\bar{n}} + P_n = 1$

↑ vanishes when acting on η_n

$$\Rightarrow \quad \boxed{\eta_n = -\frac{1}{\underbrace{i\bar{n}\cdot D_c}_{O(\lambda^0)} + i\delta}\,\frac{\not{\bar{n}}}{2}\, i\not{D}_c^\perp\, \xi_n}$$

↑ arbitrary sign, since pole is unphysical

Inserting the above solution into our expression for $\mathcal{L}_c$, we find:

$$\mathcal{L}_c = \bar{\xi}_n \frac{\not{\bar{n}}}{2}\, i n\cdot D_c\, \xi_n - \bar{\xi}_n\, i\not{D}_c^\perp\, \frac{\not{\bar{n}}}{2}\, \frac{1}{i\bar{n}\cdot D_c + i\delta}\, i\not{D}_c^\perp\, \xi_n$$

$$+ \text{(pure glue terms)}$$

The inverse of a derivative is an integral, but what is the inverse of a covariant derivative?

To define the above expression properly, we introduce the <u>collinear Wilson line</u>:

large component $\bar{n}\cdot A_c \sim \lambda^0$

light-like direction $\bar{n}^\mu$

$$\boxed{W_c(x) = P \exp\left(i g_s \int_{-\infty}^{0} dt\ \bar{n}\cdot A_c(x+t\bar{n})\right)}$$

This objects satisfies the identity:

$$\left[i\,\bar{n}\cdot D_c\ W_c(x) \right] = 0$$

Proof:

$$i\bar{n}\cdot\partial_x\ W_c(x) = i g_s \left[i\bar{n}\cdot\partial_x \int_{-\infty}^{0} dt\ \bar{n}\cdot A_c(x+t\bar{n}) \right] W_c(x)$$

$$= -g_s \left[\bar{n}\cdot\partial_x \int_{-\infty}^{\frac{n\cdot x}{2}} dt'\ \bar{n}\cdot A_c\left(\frac{\bar{n}\cdot x}{2} n + x_\perp + t'\bar{n}\right)\right] W_c(x)$$

$$= -g_s\ \bar{n}\cdot A_c(x)\ W_c(x) \qquad \text{q.e.d.}$$

It follows that:

$$\Rightarrow \quad i\bar{n}\cdot D_c\ W_c(x)\ \phi(x) = W_c(x)\ i\bar{n}\cdot\partial\ \phi(x)$$

arb. function

As a differential operator, this implies:

$$\boxed{W_c^\dagger(x)\ i\bar{n}\cdot D_c\ W_c(x) = i\bar{n}\cdot\partial}$$

This in turn yields:

$$\frac{1}{i\bar{n}\cdot D_c + i\delta} = W_c \frac{1}{i\bar{n}\cdot\partial + i\delta} W_c^\dagger$$

↳ proof: apply $W_c^\dagger \, i\bar{n}\cdot D_c \ldots W_c$ on both sides

The second term in the Lagrangian can now be written in the form:

$$\bar{\xi}_n \, i\not{D}_c^\perp \frac{\not{\bar{n}}}{2} \frac{1}{i\bar{n}\cdot D_c + i\delta} i\not{D}_c^\perp \xi_n(x)$$

$$= \bar{\xi}_n \, i\not{D}_c^\perp W_c \frac{\not{\bar{n}}}{2} \frac{1}{i\bar{n}\cdot\partial + i\delta} W_c^\dagger \, i\not{D}_c^\perp \xi_n(x)$$

$$= (\bar{\xi}_n \, i\not{D}_c^\perp W_c)(x) \frac{\not{\bar{n}}}{2} (-i)\int_{-\infty}^{0} dt \, (W_c^\dagger \, i\not{D}_c^\perp \xi_n)(x+t\bar{n})$$

↳ check:

$$i\bar{n}\cdot\partial_x (-i)\int_{-\infty}^{0} dt\, \phi(x+t\bar{n}) = \bar{n}\cdot\partial_x \int_{-\infty}^{\frac{n\cdot x}{2}} dt' \, \phi\left(\frac{\bar{n}\cdot x}{2} n + x_\perp + t'\bar{n}\right) = \phi(x)$$

Note that the lower integration limit "$-\infty$" is appropriate for our choice of the "$i\delta$" regulator. If the collinear fields $(W_c^\dagger \, i\not{D}_c^\perp \xi_n)(x+t\bar{n})$ carry total momentum p_c, the t-integral gives:

$$(-i)\int_{-\infty}^{0} dt \; e^{-ip_c\cdot(x+t\bar{n})}$$

regulator to ensure convergence at $t\to-\infty$

$$= (-i)\, e^{-ip_c\cdot x} \int_{-\infty}^{0} dt \; e^{-i(\bar{n}\cdot p_c + i\delta)\, t} = \frac{e^{-ip_c\cdot x}}{\bar{n}\cdot p_c + i\delta}$$

This indeed corresponds to the action of the inverse differential operator:

$$\frac{1}{i\bar{n}\cdot\partial + i\delta}\, e^{-ip_c\cdot x}$$

This leads to the final form of the leading-order SCET Lagrangian:

$$\boxed{\begin{aligned}\mathcal{L}_c(x) &= \bar{\xi}_n \frac{\not{\bar{n}}}{2}\, in\cdot D_c\, \xi_n(x) \\ &\quad + \left(\bar{\xi}_n\, i\not{D}_c^{\perp} W_c\right)(x)\, \frac{\not{\bar{n}}}{2}\, i\int_{-\infty}^{0} dt\, \left(W_c^{\dagger}\, i\not{D}_c^{\perp}\, \xi_n\right)(x+t\bar{n}) \\ &\quad + (\text{pure glue terms}) \qquad \text{non-local !}\end{aligned}}$$

Important comment:

If we had chosen a regulator $-i\delta$ on page 6, we would have obtained an alternative result with:

$$i\int_{-\infty}^{0} dt \;\longrightarrow\; -i\int_{0}^{\infty} dt$$

In most applications both forms give equivalent result, but in some cases it is important to be more careful. In general, one should use:

- initial-state collinear particle: $i\int_{-\infty}^{0} dt$
- final-state collinear particle: $-i\int_{0}^{\infty} dt$