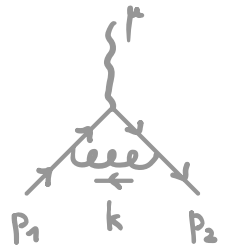


Region analysis of the Sudakov form factor:

Let us see how this procedure works in a concrete example. Every region we identify (hard, collinear, anti-collinear, ...) will later be associated with either a Wilson coefficient (hard region) or a field in the low-energy EFT.

We decompose the scalar one-loop integral

$$I = i \pi^{-D/2} \mu^{2\epsilon} \int d^D k \frac{2 P_1 \cdot P_2}{(k^2 + i0) [(k+p_1)^2 + i0] [(k+p_2)^2 + i0]}$$



into regions where the loop momentum k^μ is hard, collinear, and anti-collinear.

a) hard region: $k^\mu \sim (1, 1, 1) Q$

Expansions of propagators:

$$\begin{aligned} (k+p_1)^2 &= k^2 + 2k \cdot p_1 + p_1^2 \\ &= \underset{1}{k^2} + \underset{1}{2k \cdot} \left(\underset{\lambda^2}{(n \cdot p_1) \frac{\bar{n}}{2}} + \underset{1}{\bar{n} \cdot p_1 \frac{n}{2}} + \underset{\lambda}{p_1^\perp} \right) + \underset{\lambda^2}{p_1^2} \\ &= k^2 + (n \cdot k)(\bar{n} \cdot p_1) + \underset{\uparrow \text{drop}}{\sigma(\lambda)} \simeq (k+p_1)^2 \\ (k+p_2)^2 &= k^2 + (\bar{n} \cdot k)(n \cdot p_2) + \underset{\downarrow}{\sigma(\lambda)} \simeq (k+p_2)^2 \end{aligned}$$

with:

$$p_{1-}^\mu = (\bar{n} \cdot p_1) \frac{n^\mu}{2}, \quad p_{2+}^\mu = (n \cdot p_2) \frac{\bar{n}^\mu}{2} \quad (\text{null vectors})$$

This gives:

Taylor expansion of $2p_1 \cdot p_2$

$$\begin{aligned} I_h &= i \pi^{-D/2} \mu^{2\epsilon} \int d^D k \frac{2p_{1-} \cdot p_{2+}}{(k^2 + i0) [(k+p_1)^2 + i0] [(k+p_2)^2 + i0]} \\ &= \Gamma(1+\epsilon) \left[\frac{1}{\epsilon^2} + \frac{1}{\epsilon} \ln \frac{\mu^2}{Q^2} + \frac{1}{2} \ln^2 \frac{\mu^2}{Q^2} - \frac{\pi^2}{6} + \mathcal{O}(\epsilon) \right] \end{aligned}$$

↳ appearance of double and single poles in ϵ
(IR divergences, since integral is UV-finite)

↳ result depends on hard scale Q^2 only (besides the factorization scale μ)

b) collinear region: $k^\mu \sim (\lambda^2, 1, \lambda) Q$

Expansions of propagators:

$$(k+p_1)^2 = k^2 + 2k \cdot p_1 + p_1^2$$

$$\begin{aligned} &= k^2 + \underbrace{n \cdot k}_{\lambda^2} \underbrace{\bar{n} \cdot p_1}_{\lambda^2} + \underbrace{\bar{n} \cdot k}_{1} \underbrace{n \cdot p_1}_{1} + 2 \underbrace{k_\perp \cdot p_{1\perp}}_{\lambda \lambda} + \underbrace{p_1^2}_{\lambda^2} \sim \lambda^2 \end{aligned}$$

↳ nothing to expand

$$\begin{aligned}
 (k+p_2)^2 &= \underbrace{k^2}_{\lambda^2} + \underbrace{n \cdot k}_{\lambda^2} \underbrace{\bar{n} \cdot p_2}_{\lambda^2} + \underbrace{\bar{n} \cdot k}_1 \underbrace{n \cdot p_2}_1 + \underbrace{2k_{\perp} \cdot p_{2\perp}}_{\lambda \lambda} + \underbrace{p_2^2}_{\lambda^2} \sim \lambda^0 \\
 &= \bar{n} \cdot k \, n \cdot p_2 + \underbrace{O(\lambda^2)}_{\substack{\uparrow \\ \text{drop}}} \simeq 2k \cdot p_{2+}
 \end{aligned}$$

This gives:

$$\begin{aligned}
 I_c &= i \pi^{-D/2} \mu^{2\epsilon} \int d^D k \frac{2p_{1+} \cdot p_{2+}}{(k^2 + i0) [(k+p_1)^2 + i0] [2k \cdot p_{2+} + i0]} \\
 &= \Gamma(1+\epsilon) \left[-\frac{1}{\epsilon^2} - \frac{1}{\epsilon} \ln \frac{\mu^2}{P_1^2} - \frac{1}{2} \ln^2 \frac{\mu^2}{P_1^2} + \frac{\pi^2}{6} + O(\epsilon) \right] \\
 &\quad (P_1^2 \equiv -p_1^2)
 \end{aligned}$$

↪ appearance of double and single poles in ϵ
(IR divergences, since integral is UV-finite)

↪ result depends on collinear scale P_1^2 only

c) anti-collinear region: $k^\mu \sim (1, \lambda^2, \lambda) Q$

We find an analogous contribution:

$$\begin{aligned}
 I_{\bar{c}} &= \Gamma(1+\epsilon) \left[-\frac{1}{\epsilon^2} - \frac{1}{\epsilon} \ln \frac{\mu^2}{P_2^2} - \frac{1}{2} \ln^2 \frac{\mu^2}{P_2^2} + \frac{\pi^2}{6} + O(\epsilon) \right] \\
 &\quad (P_2^2 \equiv -p_2^2)
 \end{aligned}$$

⇒ sum of the three contributions:

$$\begin{aligned}
 & I_h + I_c + I_{\bar{c}} \\
 &= \Gamma(1+\epsilon) \left[-\frac{1}{\epsilon^2} + \frac{1}{\epsilon} \left(\ln \frac{\mu^2}{Q^2} - \ln \frac{\mu^2}{P_1^2} - \ln \frac{\mu^2}{P_2^2} \right) \right. \\
 &\quad \left. + \frac{1}{2} \ln^2 \frac{\mu^2}{Q^2} - \frac{1}{2} \ln^2 \frac{\mu^2}{P_1^2} - \frac{1}{2} \ln^2 \frac{\mu^2}{P_2^2} + \frac{\pi^2}{6} + O(\epsilon) \right]
 \end{aligned}$$

Surprisingly, this does not reproduce the exact result on p.7 (lecture 1), and uncanceled IR divergences remain. It follows that we have failed to identify (at least) one relevant region. Combining the three logs in the coefficient of the $1/\epsilon$ pole, we get:

$$[\text{above}] = -\frac{1}{\epsilon^2} - \frac{1}{\epsilon} \ln \frac{\mu^2 Q^2}{P_1^2 P_2^2} + \dots$$

↪ suggest that missing region corresponds to scale:

$$\frac{P_1^2 P_2^2}{Q^2} \sim \lambda^4 Q^2 \ll \text{collinear scale } P_i^2 \sim \lambda^2 Q^2$$

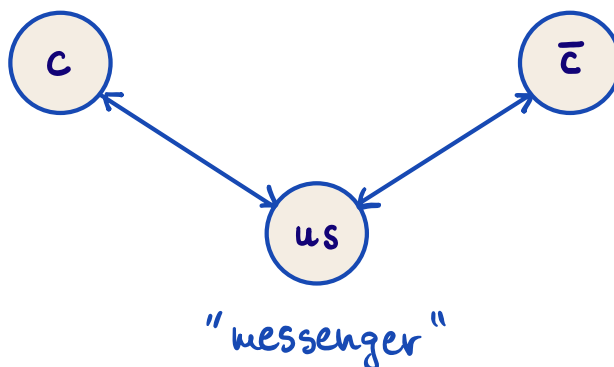
d) ultra-soft contribution:

There is a strong physics reason suggesting that we need another mode (corresponding to a momentum region) in the low-energy effective theory. An EFT built out of collinear and anti-collinear particles would contain two disjoint sectors, because no vertices connecting both types of particles are allowed:

$$\begin{array}{c} p_c^\mu + p_{\bar{c}}^\mu \sim (1, 1, \lambda) \text{ hard!} \\ (\lambda^2, 1, \lambda) \quad (1, \lambda^2, \lambda) \end{array}$$

Physically, it would be strange if the two jets could not interact in the low energy theory, since they need to neutralize their color. The "largest" on-shell mode that can connect to both collinear and anti-collinear particles without taking them far off-shell is the ultra-soft mode:

$$p_{us}^\mu \sim (\lambda^2, \lambda^2, \lambda^2) \Rightarrow p_{us}^2 \sim \lambda^4 Q^2$$



Let us evaluate the ultra-soft contribution to the Sudakov form factor: $k^\mu \sim (\lambda^2, \lambda^2, \lambda^2) Q$

$$(k+p_1)^2 = \underbrace{k^2}_{\lambda^4} + \underbrace{n \cdot k}_{\lambda^2} \underbrace{\bar{n} \cdot p_1}_1 + \underbrace{\bar{n} \cdot k}_{\lambda^2} \underbrace{n \cdot p_1}_{\lambda^2} + \underbrace{2 k_\perp \cdot p_{1\perp}}_{\lambda^2 \lambda} + \underbrace{p_1^2}_{\lambda^2} \sim \lambda^2$$

$$= n \cdot k \bar{n} \cdot p_1 + p_1^2 + \mathcal{O}(\lambda^3) \approx 2 k \cdot p_{1-} + p_1^2$$

↑ drop
↓

$$(k+p_2)^2 = \bar{n} \cdot k n \cdot p_2 + p_2^2 + \mathcal{O}(\lambda^3) \approx 2 k \cdot p_{2+} + p_2^2$$

This gives:

$$I_{us} = i \pi^{-D/2} \mu^{2\epsilon} \int d^D k \frac{2 p_{1-} p_{2+}}{(k^2 + i0) (2 k \cdot p_{1-} + p_1^2 + i0) (2 k \cdot p_{2+} + p_2^2 + i0)}$$

$$= \Gamma(1+\epsilon) \left[\frac{1}{\epsilon^2} + \frac{1}{\epsilon} \ln \frac{\mu^2 Q^2}{p_1^2 p_2^2} + \frac{1}{2} \ln^2 \frac{\mu^2 Q^2}{p_1^2 p_2^2} + \frac{\pi^2}{6} + \mathcal{O}(\epsilon) \right]$$

Adding this result to the expression on page 2, we find:

$$I_h + I_c + I_{\bar{c}} + I_{us}$$

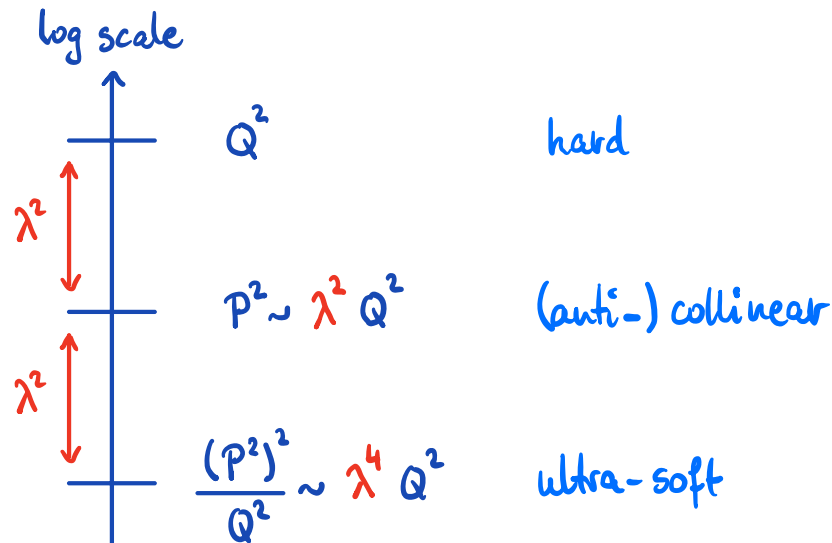
$$= \underbrace{\frac{1}{2} \ln^2 \frac{\mu^2}{Q^2}}_{\text{hard}} - \underbrace{\frac{1}{2} \ln^2 \frac{\mu^2}{p_1^2}}_{\text{collinear}} - \underbrace{\frac{1}{2} \ln^2 \frac{\mu^2}{p_2^2}}_{\text{anti-collinear}} + \underbrace{\frac{1}{2} \ln^2 \frac{\mu^2 Q^2}{p_1^2 p_2^2}}_{\text{ultra-soft}} + \frac{\pi^2}{3} + \mathcal{O}(\epsilon)$$

$$= \frac{1}{2} \ln \frac{Q^2}{p_1^2} \ln \frac{Q^2}{p_2^2} + \frac{\pi^2}{3} + \mathcal{O}(\epsilon)$$

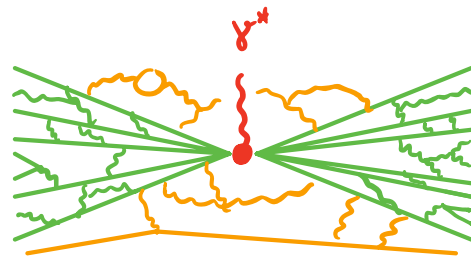
This agrees with the original expression on p. 7 of lecture 1!

Comments:

The decomposition of Sudakov double logarithms into a sum of logarithms depending on a single physical scale requires the presence of three correlated scales:



The ultra-soft scale is physical and characterizes soft exchanges between the two jets:



{ ultra-soft exchanges
are needed for
color neutralization

The process can be calculated in perturbative QCD only if the ultra-soft scale is much larger than Λ_{QCD} .

↳ else, need nonperturbative soft functions

III. Effective Lagrangian of Soft-Collinear Effective Theory

Our goal is to construct an effective Lagrangian built out of collinear, anti-collinear and ultra-soft quark and gluon fields (and ghost fields, but we will not write them out explicitly). Momentum conservation allows the following interactions involving different modes:

$$\underbrace{\phi_c \dots \phi_c}_{n_c \geq 2} \quad \underbrace{\phi_{us} \dots \phi_{us}}_{n_{us} \geq 1} \quad \checkmark$$

$$\underbrace{\phi_{\bar{c}} \dots \phi_{\bar{c}}}_{n_{\bar{c}} \geq 2} \quad \underbrace{\phi_{us} \dots \phi_{us}}_{n_{us} \geq 1} \quad \checkmark$$

$$p_c^\mu \sim Q(\lambda^2, 1, \lambda)$$

$$p_{\bar{c}}^\mu \sim Q(1, \lambda^2, \lambda)$$

$$p_{us}^\mu \sim Q(\lambda^2, \lambda^2, \lambda^2)$$

but not:

$$\underbrace{\phi_c \dots \phi_c}_{n_c \geq 1} \quad \underbrace{\phi_{\bar{c}} \dots \phi_{\bar{c}}}_{n_{\bar{c}} \geq 1} \quad \underbrace{\phi_{us} \dots \phi_{us}}_{n_{us} \geq 0}$$

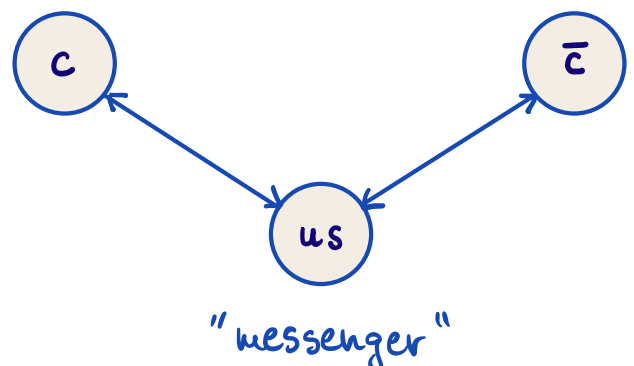
forbidden (hard interaction)

$$\phi_c \underbrace{\phi_{us} \dots \phi_{us}}_{n_{us} \geq 1}, \quad \phi_{\bar{c}} \underbrace{\phi_{us} \dots \phi_{us}}_{n_{us} \geq 1}$$

momentum not conserved

It follows that:

$$\begin{aligned} \mathcal{L}_{\text{SCET}}^{(c, \bar{c}, us)} &= \mathcal{L}_c + \mathcal{L}_{us} + \mathcal{L}_{\bar{c}} \\ &\quad + \mathcal{L}_{c+us} + \mathcal{L}_{\bar{c}+us} \end{aligned}$$



It suffices to study the Lagrangian of collinear and ultra-soft fields.

Collinear quark field:

The spinor of a highly energetic (along z -axis), light fermion satisfies:

$$\not{p} u_s(p) = m u_s(p) ; \quad m \ll E = p^0 \simeq p^3$$

$$\Rightarrow \not{n} u_s(p) \simeq 0 \quad (\text{up to } m/E \text{ corrections})$$

We can identify the large and small components of such a spinor using the projection operators:

$$P_n = \frac{\not{n} \not{n}}{4} , \quad P_{\bar{n}} = \frac{\not{\bar{n}} \not{\bar{n}}}{4} \quad (\bar{P}_n = \gamma^0 P_n^\dagger \gamma^0 = P_{\bar{n}})$$

with:

$$P_n + P_{\bar{n}} = \frac{\{\not{n}, \not{\bar{n}}\}}{4} = \frac{2n \cdot \bar{n}}{4} = 1$$

$$P_n^2 = P_n , \quad P_{\bar{n}}^2 = P_{\bar{n}} , \quad P_n P_{\bar{n}} = 0 = P_{\bar{n}} P_n$$

We define:

$$\xi_n = P_n \psi_c , \quad \eta_n = P_{\bar{n}} \psi_c$$

modes restricted to collinear region
↙

$$\Rightarrow \not{n} \xi_n = 0 \quad (\text{suggest that } \xi_n \text{ will describe a collinear quark in SCET})$$

$$\not{\bar{n}} \eta_n = 0$$

To derive the power counting in λ , we consider (massless fermion):

$$\langle 0 | T \{ \psi_c(x) \bar{\psi}_c(0) \} | 0 \rangle = \int \frac{d^4 p}{(2\pi)^4} e^{-ip \cdot x} \frac{i \not{p}}{p^2 + i\epsilon}$$

$n \cdot p \frac{\not{n}}{2} + \bar{n} \cdot p \frac{\not{n}}{2} + \not{p}_\perp$

λ^4 1 $\frac{1}{\lambda^2} (\lambda^2, 1, \lambda)$

Hence:

$$\langle 0 | T \{ \xi_n(x) \bar{\xi}_n(0) \} | 0 \rangle = \int \frac{d^4 p}{(2\pi)^4} e^{-ip \cdot x} \frac{i \bar{n} \cdot p}{p^2 + i\epsilon} \frac{\not{n}}{2} \sim \lambda^2$$

λ^4 1 λ^{-2} $\frac{\not{n}}{2}$

$P_n \not{n} P_{\bar{n}} = \bar{n} \cdot p \frac{\not{n}}{2}$

$$\langle 0 | T \{ \eta_n(x) \bar{\eta}_n(0) \} | 0 \rangle = \int \frac{d^4 p}{(2\pi)^4} e^{-ip \cdot x} \frac{i n \cdot p}{p^2 + i\epsilon} \frac{\not{n}}{2} \sim \lambda^4$$

λ^4 1 λ^0 $\frac{\not{n}}{2}$

$$\langle 0 | T \{ \xi_n(x) \bar{\eta}_n(0) \} | 0 \rangle = \int \frac{d^4 p}{(2\pi)^4} e^{-ip \cdot x} \frac{i \not{p}_\perp}{p^2 + i\epsilon} P_n \gamma_\mu \bar{P}_{\bar{n}} \sim \lambda^3$$

λ^4 1 λ^{-1}

It follows that:

large components
small components

$\xi_n \sim \lambda, \quad \eta_n \sim \lambda^2$

In the next lecture, we will integrate out the small components η_n and use the field ξ_n to describe a collinear quark in SCET.