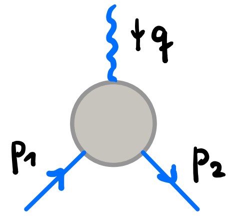
VI. The Sudakov Form Factor in SCET

We now return to the case of the off-shell Sudakov form factor in QCD,



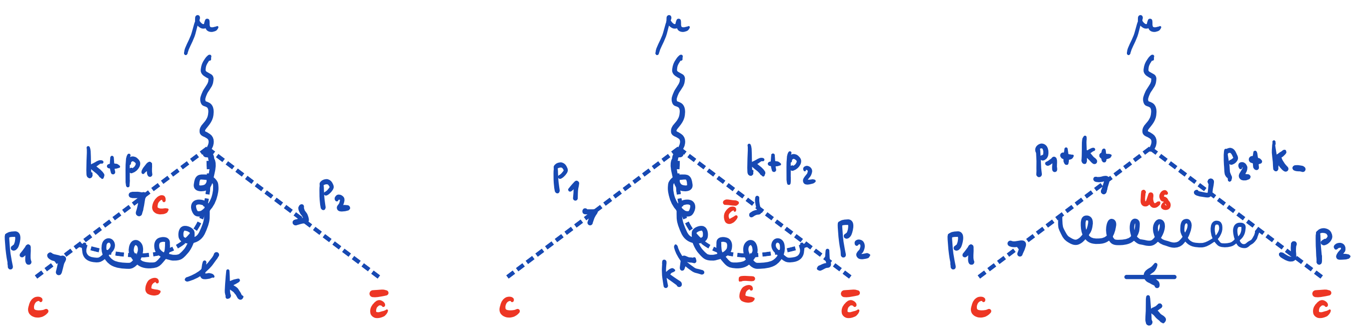
$|q^2| \gg |p_i^2|$

$(m_i = 0)$

this time performing the calculation in SCET (and not ignoring the numerator terms). We thus evaluate the quark matrix element

$$C_V(Q^2)\,(\bar{\xi}_{\bar{n}} W_{\bar{c}})\,\gamma_\perp^\mu\,(W_c^\dagger \xi_n)$$

at one-loop order, where $C_V(Q^2) = 1 + \mathcal{O}(\alpha_s)$ is the hard matching coefficient. The relevant graphs are:



+ WFR

please study the details of this calculation as a reading assignment

Adding up all pieces, we find at one-loop order:

$$\langle q(p_2)|\,(\bar{\xi}_{\bar{n}} W_{\bar{c}})\,\gamma_\perp^\mu\,(W_c^\dagger \xi_n)\,|q(p_1)\rangle$$

$$= \bar{u}_{\bar{n}}(p_2)\,\gamma_\perp^\mu\,u_n(p_1)$$

$$\times \Big\{ 1 + \frac{C_F \alpha_s}{4\pi} \Big[\frac{2}{\epsilon^2} + \frac{2}{\epsilon}\Big(\ln\frac{\mu^2}{P_1^2} + \ln\frac{\mu^2}{P_2^2} - \ln\frac{\mu^2 Q^2}{P_1^2 P_2^2}\Big) + \frac{3}{\epsilon}$$

$$+ \ln^2\frac{\mu^2}{P_1^2} + \ln^2\frac{\mu^2}{P_2^2} - \ln^2\frac{\mu^2 Q^2}{P_1^2 P_2^2}$$

$$+ \frac{3}{2}\ln\frac{\mu^2}{P_1^2} + \frac{3}{2}\ln\frac{\mu^2}{P_2^2} + 9 - \frac{5\pi^2}{6} \Big]\Big\}$$

This is a matrix element of a "bare" SCET operator, which still needs to be renormalized. The fact that the coefficient of the $1/\epsilon$ pole depends on the collinear and ultra-soft (and hence low-energy) scales appears to be troublesome at first sight, since the counterterms removing the divergences must be of UV nature. On second look, however, we see that

$$\ln\frac{\mu^2}{P_1^2} + \ln\frac{\mu^2}{P_2^2} - \ln\frac{\mu^2 Q^2}{P_1^2 P_2^2} = \ln\frac{\mu^2}{Q^2}$$

only depends on the hard scale Q^2! It is thus consistent to interpret these poles as UV divergences.

We define the renormalized SCET current operator as:

$$V^{\mu,\,\text{bare}}_{SCET} = Z_V(\mu)\, V^{\mu}_{SCET}(\mu)$$

The matrix elements of the renormalized current operator are finite, i.e., free of $1/\epsilon$ poles. In the $\overline{MS}$ scheme, one obtains:

$$\boxed{Z_V(\mu) = 1 + \frac{C_F \alpha_s}{4\pi}\left[\frac{2}{\epsilon^2} + \frac{2}{\epsilon}\ln\frac{\mu^2}{Q^2} + \frac{3}{\epsilon}\right] + \mathcal{O}(\alpha_s^2)}$$

The matrix element of the renormalized current is then given by:

$$\langle q(p_2)|\, V^{\mu}_{SCET}(\mu)\, | q(p_1)\rangle$$

$$= \bar{u}_{\bar{n}}(p_2)\,\gamma^{\mu}_{\perp}\, u_n(p_1)$$

$$\times \left\{ 1 + \frac{C_F \alpha_s}{4\pi}\left[\ln^2\frac{\mu^2}{p_1^2} + \ln^2\frac{\mu^2}{p_2^2} - \ln^2\frac{\mu^2 Q^2}{p_1^2\, p_2^2} + \frac{3}{2}\ln\frac{\mu^2}{p_1^2} + \frac{3}{2}\ln\frac{\mu^2}{p_2^2} + 9 - \frac{5\pi^2}{6}\right]\right\}$$

Derivation of the Wilson coefficient:

The matching relation for the renormalized vector current takes the form:

$$\bar{\psi}\gamma^{\mu}\psi \;\to\; C_V(Q^2,\mu)\, V^{\mu}_{SCET}(\mu)$$

We can derive the one-loop expression for the hard matching coefficient $C_V(Q^2, \mu)$ by comparing the above result for the renormalized SCET matrix element with the result for the Sudakov form factor in QCD, which reads:

$$\langle q(p_2) | \bar{\psi} \gamma^\mu \psi | q(p_1) \rangle = \bar{u}(p_2) \gamma^\mu u(p_1)$$

$$\times \left\{ 1 + \frac{C_F \alpha_s}{4\pi} \left[-2 \ln \frac{Q^2}{p_1^2} \ln \frac{Q^2}{p_2^2} + \frac{3}{2} \left(\ln \frac{Q^2}{p_1^2} + \ln \frac{Q^2}{p_2^2} \right) - \frac{2\pi^2}{3} + 1 \right] \right\}$$

$$+ \mathcal{O}(p_i^2 / Q^2)$$

Matching the two expressions, we find:

$$\boxed{C_V(Q^2, \mu) = 1 + \frac{C_F \alpha_s}{4\pi} \left[-\ln^2 \frac{Q^2}{\mu^2} + 3 \ln \frac{Q^2}{\mu^2} - 8 + \frac{\pi^2}{6} \right]}$$

$\rightarrow$ only depends on hard scale Q^2 ✓

VII. RG Evolution Equations

The scale dependence of the renormalized SCET current operator is determined by the differential equation:

$$\mu \frac{d}{d\mu} V^\mu_{SCET}(\mu) = - \Gamma_V(Q^2, \mu)\, V^\mu_{SCET}(\mu)$$

One can show that, to all orders of perturbation theory, the "anomalous dimension" Γ_V is given by:

coefficient of $1/\varepsilon$ pole

$$\boxed{\Gamma_V(Q^2, \mu) = -2\alpha_s \frac{\partial}{\partial \alpha_s} Z_V^{[1]}(\mu)}$$

(see e.g. my Les Houches lectures in 1901.06573)

Using our result from above, we find at one-loop order:

$$\Gamma_V(Q^2, \mu) = - \frac{C_F \alpha_s}{\pi} \left(\ln \frac{\mu^2}{Q^2} + \frac{3}{2} \right) + \mathcal{O}(\alpha_s^2)$$

The appearance of a logarithm of μ^2 in an anomalous dimension is a new feature of SCET. It is characteristic of Sudakov problems, in which the perturbation series contains two powers of logarithms for each power of α_s. One can show that to all orders:

$$\boxed{\Gamma_V(Q^2, \mu) = - \Gamma_{cusp}(\alpha_s) \ln \frac{\mu^2}{Q^2} + \gamma_V(\alpha_s)}$$

single log !

From the fact that the vector current in full QCD is not renormalized (Noether's theorem), it then follows that:

$$\mu \frac{d}{d\mu} \left[C_V(Q^2, \mu)\, V^\mu_{SCET}(\mu) \right] = 0$$

$$\Downarrow$$

$$\boxed{\mu \frac{d}{d\mu} C_V(Q^2, \mu) = \left[\Gamma_{cusp}(\alpha_s) \ln \frac{Q^2}{\mu^2} + \gamma_V(\alpha_s) \right] C_V(Q^2, \mu)}$$

("renormalization-group equation")

Γ_{cusp} is called the light-like cusp anomalous dimension. It is known to 4-loop order in QCD. The quantity γ_V is known at 3-loop order. At leading order we have:

$$\Gamma_{cusp}(\alpha_s) = \frac{C_F \alpha_s}{\pi} \,, \qquad \gamma_V(\alpha_s) = -\frac{3}{2} \frac{C_F \alpha_s}{\pi}$$

The general solution to the RGE is:

$$C_V(Q^2, \mu) = C_V(Q^2, \mu_h) \quad \text{"initial condition"}$$

$$\times \exp\left[\int_{\mu_h}^{\mu} \frac{d\mu'}{\mu'} \left(\Gamma_{cusp}(\alpha_s(\mu')) \ln \frac{Q^2}{\mu'^2} + \gamma_V(\alpha_s(\mu')) \right) \right]$$

At the "hard matching scale" $\mu_h^2 \approx Q^2$, the initial condition $C_V(Q^2, \mu_h)$ for the Wilson coefficient is free of large logarithms and can be calculated reliably using perturbation theory. For instance, with $\mu_h = Q$ we have:

$$C_V(Q^2, \mu_h) = 1 + \frac{C_F \alpha_s}{4\pi}\left(-8 + \frac{\pi^2}{6}\right) + \mathcal{O}(\alpha_s^2)$$

The above solution can then be used to evolve the Wilson coefficients to scales $\mu \ll Q$ in such a way that the large logarithms

$$\alpha_s^n \ln^k \frac{Q^2}{\mu^2} \quad ; \quad k \leq 2n$$

are resummed to all orders in α_s.

(↳ see problem set 2 for more details)

VIII. Decoupling of Ultra-Soft Gluons

Ultra-soft gluons couple to collinear fields through the eikonal interaction:

$$\bar{\xi}_n(x)\,\frac{\not{\bar{n}}}{2}\left(i n\cdot\partial + g_s\, n\cdot A_c(x) + g_s\, n\cdot A_{us}(x_-)\right)\xi_n(x)$$

This coupling can be removed by a <u>field redefinition</u>, i.e.

new fields

$$\xi_n(x) \longrightarrow S_n(x_-)\,\xi_n^{(0)}(x)$$

$$A_c^\mu(x) \longrightarrow S_n(x_-)\,A_c^{\mu(0)}(x)\,S_n^\dagger(x_-)$$

$$W_c(x) \longrightarrow S_n(x_-)\,W_c(x)\,S_n^\dagger(x_-)$$

where

note: n^μ, <u>not</u> $\bar{n}^\mu$!

$$\boxed{S_n(x) = \mathcal{P}\exp\left(i g_s \int_{-\infty}^{0} dt\; n\cdot A_{us}(x+nt)\right)} \qquad \text{(unitary)}$$

is a soft Wilson line along the light-like direction n^μ. Note that this has the form of an ultra-soft gauge transformation with $U_s(x) = S_n(x)$. Using the property

$$\left(i n\cdot\partial + g_s\, n\cdot A_{us}(x_-)\right) S_n(x_-) = S_n(x_-)\, i n\cdot\partial$$

which follows from $\left[i n\cdot D_{us}\, S_n(x)\right] = 0$, one finds that:

$$\bar{\xi}_n(x)\,\frac{\bar{n}\!\!\!/}{2}\left(in\cdot\partial + g_s\,n\cdot A_c(x) + g_s\,n\cdot A_{us}(x_-)\right)\xi_n(x)$$

$$\rightarrow\ \bar{\xi}_n^{(0)}(x)\,\frac{\bar{n}\!\!\!/}{2}\,S_n^\dagger(x_-)\,S_n(x_-)\left(in\cdot\partial + g_s\,n\cdot A_c^{(0)}(x)\right)\xi_n^{(0)}(x)$$

$$=\ \bar{\xi}_n^{(0)}(x)\,\frac{\bar{n}\!\!\!/}{2}\,in\cdot D_c^{(0)}(x)\,\xi_n^{(0)}(x)$$

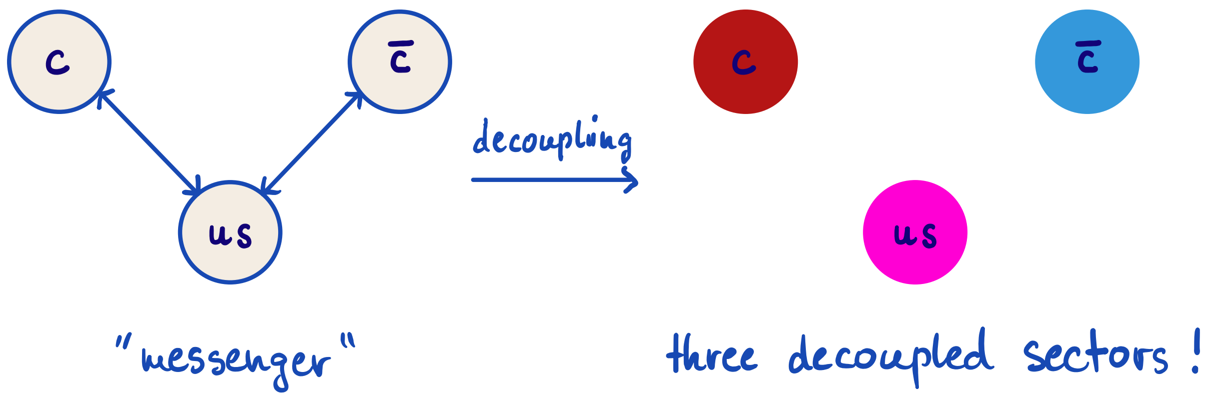
This field redefinition thus removes the ultra-soft gluon field from the leading-order SCET Lagrangian! The same trick also works for the pure gluon terms in the Lagrangian.

The "ultra-soft decoupling transformation" is the ke to deriving factorization theorems in SCET! It does not imply that ultra-soft interactions disappear entirely. Rather, it means that, as far as their couplings to ultra-soft gluon are concerned, collinear particles behave like Light-Like Wilson Lines. The ultra-soft gluons reappear when we consider external operators (such as currents) built out of two or more types of collinear fields.

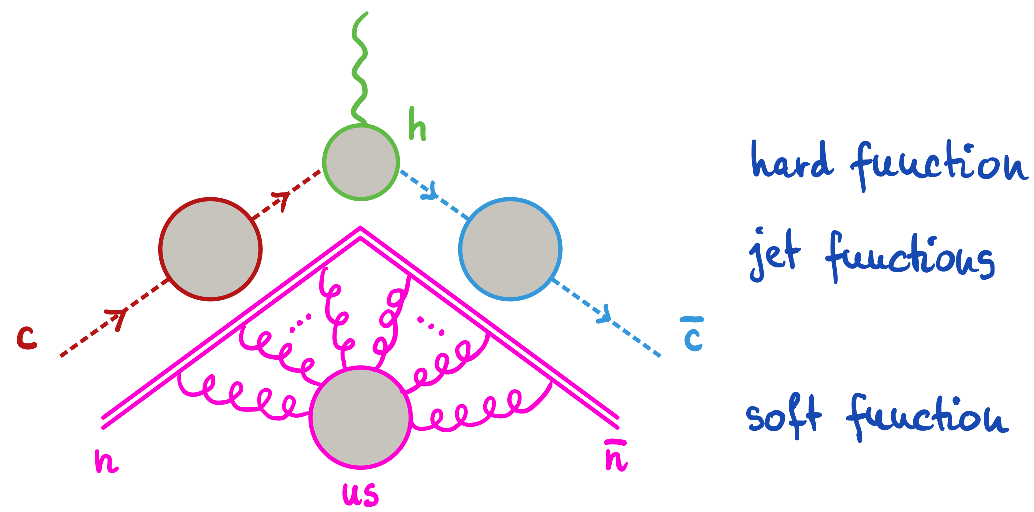
For the example of the 2-jet vector current, the decoupling transformation implies:

$$\bar\psi\gamma^\mu\psi \to C_V(Q^2)\,(\bar\xi_{\bar n} W_{\bar c})(0)\,\gamma_\perp^\mu\,(W_c^\dagger \xi_n)(0)$$

$$\to C_V(Q^2)\,(\bar\xi_{\bar n}^{(0)} W_{\bar c}^{(0)})(0)\,\gamma_\perp^\mu\, S_{\bar n}^\dagger(0)\, S_n(0)\,(W_c^{(0)\dagger}\xi_n^{(0)})(0)$$



The Sudakov form factor then factorizes into four distinct objects each characterized by a single scale:



This is an example of a <u>factorization formula</u>.

In a second reading assignment to this lecture we discuss an example of a factorization theorem for a concrete physical process.