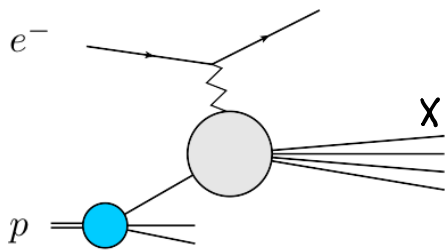


IX. Applications of SCET

The formalism we have developed in this course has widespread applications in collider physics, heavy-flavor physics and other fields. Some important examples are shown below (along with some key references).

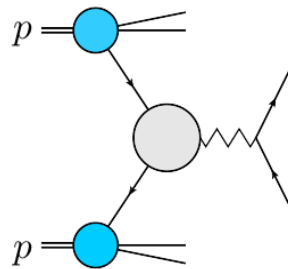
Collider physics:



DIS for $x \rightarrow 1$

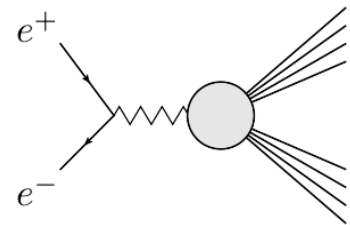
$$M_X^2 = Q^2 \frac{1-x}{x} \ll Q^2$$

Becher, MN: hep-ph/0605050
Becher, MN, Pecjak: "/0607228



DY production
soft additional
radiation

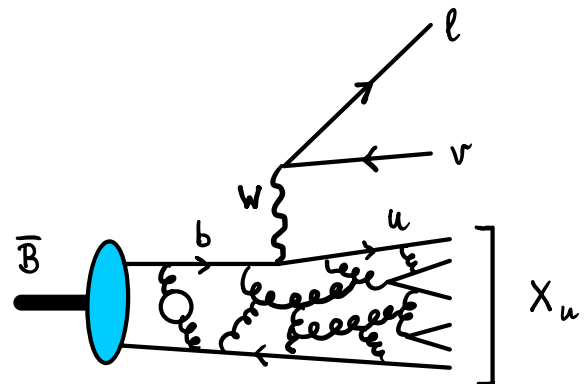
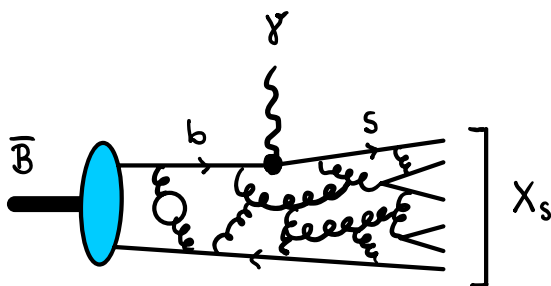
Becher, MN, Xu:
hep-ph/0710.0680



$e^+e^- \rightarrow 2 \text{ jets}$
event shapes

Lee, Sterman: hep-ph/0611061
Becher, Schwartz: 0803.0342

Flavor physics:



Inclusive decays $\bar{B} \rightarrow X_s \gamma$ and $\bar{B} \rightarrow X_u \ell \bar{\nu}$ in the kinematic region where $M_X^2 \ll m_B^2$

Bauer, Pirjol, Stewart: hep-ph/0109045
Bosch, Lange, MN, Paz: hep-ph/0402094

In all of these processes the relevant modes are collinear or ultra-soft and there are at most two collinear directions ($\vec{n}^*$ and $\vec{\bar{n}}^*$). Jet processes at hadron colliders are more complicated, since they require introducing $(2 + n_{\text{jets}})$ collinear directions $\vec{n}_i^*$, where the first two refer to the beams.

In this lecture we discuss the process

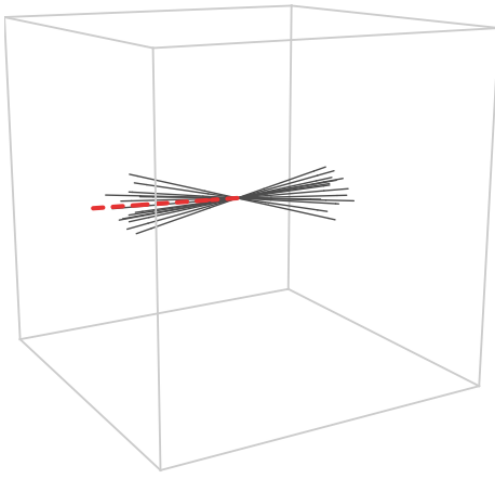
$$e^+e^- \rightarrow 2 \text{ jets} \quad \hookrightarrow \text{see 1803.04310 by Becher}$$

in more detail. Rather than defining the jets through some complicated jet algorithm ($\hookrightarrow$ non-global logs, see 1508.06645, 1605.02737 for a treatment in SCET)

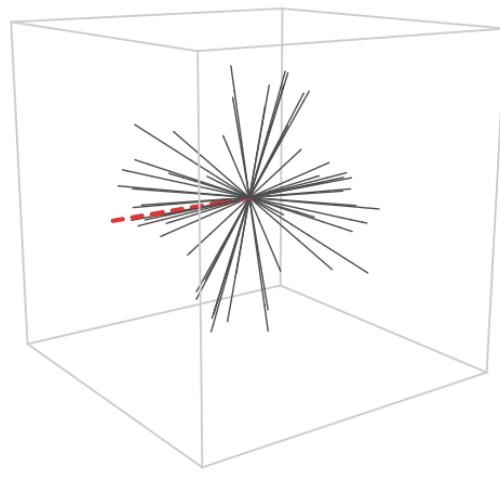
we consider an event shape, which characterizes the geometry of an event and measures how "pencil-like" it is. The prototypical event shape is thrust:

$$T = \frac{1}{Q} \max_{\vec{n}_T} \sum_i |\vec{n}_T \cdot \vec{p}_i| \quad \text{thrust axis (in CMS)}$$

Here $Q = \sqrt{s} = \sum_i |\vec{p}_i|$ is the total CMS energy (massless particles). The event shape thrust varies between $T_{\text{max}} = 1$ (perfect alignment of two jets) and $T_{\text{min}} = 1/2$ (completely spherical event).



$$T = 0.998 \quad (\tau = 0.002)$$



$$T = 0.65 \quad (\tau = 0.35)$$

One defines $\tau = 1 - T$ to measure the departure from the perfect 2-jet limit. We are interested in the region where $\tau \ll 1$.

Thrust is soft and collinear safe, meaning that its value does not change under exactly collinear splittings:

$$\vec{P}_i \rightarrow \vec{P}_{i,a} + \vec{P}_{i,b} ; \quad \vec{P}_{i,a} \parallel \vec{P}_{i,b}$$

and infinitely soft emissions:

$$\vec{P}_i \rightarrow \vec{P}_{i,a} + \vec{P}_{i,b} ; \quad |\vec{P}_{i,b}| \rightarrow 0$$

This property ensures that the cross section $d\sigma/d\tau$ is free of IR divergences. However, in the 2-jet limit $\tau \ll 1$ the cross section receives large double-logarithmic corrections $\sim (\alpha_s \ln^2 \tau)^n$, which need to be

resummed to all orders of perturbation theory. A region analysis shows that the relevant modes are:

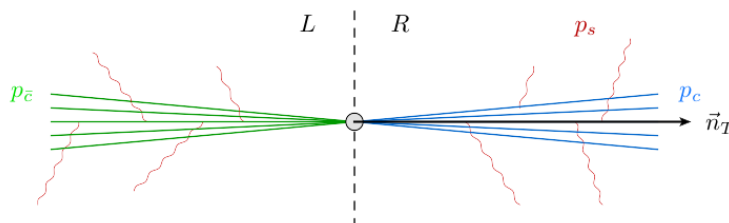
hard	Q^2	← integrate out	
(anti-) collinear	τQ^2	}	SCET with $\lambda = \tau^{1/2} \ll 1$
ultra-soft	$\tau^2 Q^2$		

We choose the thrust axis to define the reference vectors:

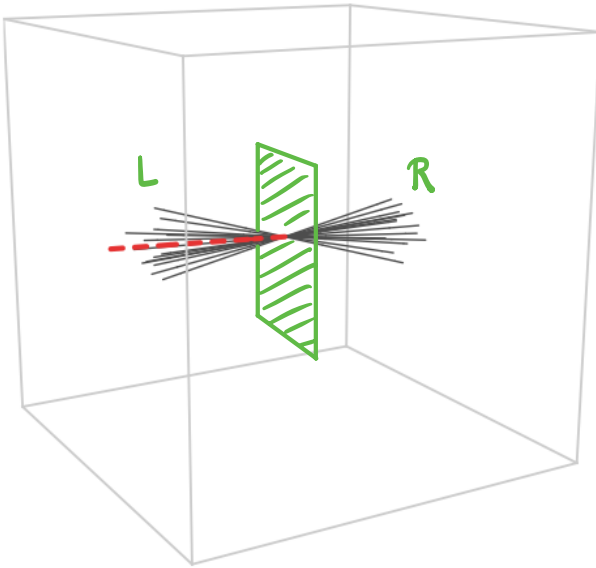
$$n^\mu = (1, \vec{n}_T) \quad , \quad \bar{n}^\mu = (1, -\vec{n}_T)$$

By definition, thrust is additive and we can separate the sum over particles into sums in the various sectors of SCET:

$$\begin{aligned}
 \tau Q &= \sum_i (|\vec{p}_i| - |\vec{n}_T \cdot \vec{p}_i|) && \begin{array}{cc} \vec{n}_T \cdot \vec{p}_i > 0 & \vec{n}_T \cdot \vec{p}_i < 0 \\ \text{right movers} & \text{left movers} \end{array} \\
 &= \sum_i \underbrace{n \cdot p_{c,i}}_{\substack{\uparrow \\ \text{collinear} \\ \text{particles}}} + \sum_i \underbrace{\bar{n} \cdot p_{\bar{c},i}}_{\substack{\uparrow \\ \text{anti-collinear} \\ \text{particles}}} + \sum_i \underbrace{n \cdot p_{s,i}^R}_{\substack{\uparrow \\ \text{soft particles}}} + \sum_i \underbrace{\bar{n} \cdot p_{s,i}^L}_{\substack{\uparrow \\ \text{soft particles}}} \\
 &\equiv \underbrace{n \cdot p_{X_c}}_{\lambda^2} + \underbrace{\bar{n} \cdot p_{X_{\bar{c}}}}_{\lambda^2} + \underbrace{n \cdot p_{X_s}^R}_{\lambda^2} + \underbrace{\bar{n} \cdot p_{X_s}^L}_{\lambda^2} \sim \lambda^2 \sim \tau
 \end{aligned}$$



The definition of the thrust axis ensures that the total transverse momentum in each hemisphere vanishes:



$$P_X^{L,\perp} = P_{X_{\bar{c}}}^{\perp} + P_{X_S}^{L,\perp} = 0$$

$$P_X^{R,\perp} = P_{X_c}^{\perp} + P_{X_S}^{R,\perp} = 0$$

$\lambda \qquad \lambda^2$

$$\Rightarrow P_{X_c}^{\perp} = 0 = P_{X_{\bar{c}}}^{\perp}$$

up to power corrections

It follows that, at leading power:

$$\begin{aligned} M_R^2 &= (P_{X_c} + P_{X_S}^R)^2 = P_{X_c}^2 + \bar{n} \cdot P_{X_c} \, n \cdot P_{X_S}^R + \dots \\ &= \bar{n} \cdot P_{X_c} (n \cdot P_{X_c} + n \cdot P_{X_S}^R) + \dots \\ &= Q \left(\underbrace{n \cdot P_{X_c}}_{\lambda^2 Q} + \underbrace{n \cdot P_{X_S}^R}_{\lambda^2 Q} \right) + \dots \end{aligned}$$

$$\begin{aligned} M_L^2 &= (P_{X_{\bar{c}}} + P_{X_S}^L)^2 = n \cdot P_{X_{\bar{c}}} (\bar{n} \cdot P_{X_{\bar{c}}} + \bar{n} \cdot P_{X_S}^L) + \dots \\ &= Q \left(\underbrace{\bar{n} \cdot P_{X_{\bar{c}}}}_{\lambda^2 Q} + \underbrace{\bar{n} \cdot P_{X_S}^L}_{\lambda^2 Q} \right) + \dots \end{aligned}$$

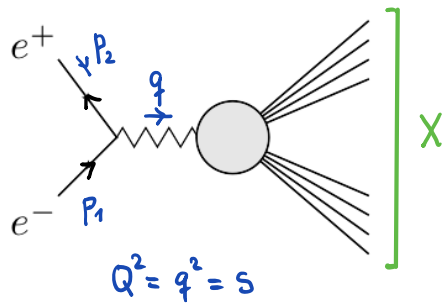
Up to power corrections, we thus obtain:

$$\tau Q^2 = M_L^2 + M_R^2 = \underbrace{P_{X_c}^2}_{\lambda^2 Q^2} + \underbrace{P_{X_{\bar{c}}}^2}_{\lambda^2 Q^2} + Q \left(\underbrace{n \cdot P_{X_S}^R}_{\lambda^2 Q} + \underbrace{\bar{n} \cdot P_{X_S}^L}_{\lambda^2 Q} \right)$$

The fact that thrust is additive in the collinear and ultra-soft contributions is important to establish factorization. The differential cross section is given by:

$$\frac{d\sigma}{d\tau} = \frac{1}{2Q^2} \sum_X |\mathcal{M}(e^+e^- \rightarrow \gamma^* \rightarrow X)|^2 (2\pi)^4 \delta^{(4)}(p_X - q) \delta(\tau - \tau(X))$$

↑
thrust of final state X



$$L^{\mu\nu} H_{\mu\nu}$$

Leptonic tensor:

$$L^{\mu\nu} \equiv \sum_q \frac{Q_q^2 e^4}{Q^4} (p_1^\mu p_2^\nu + p_2^\mu p_1^\nu - p_1 \cdot p_2 g^{\mu\nu})$$

(Q_q : quark electric charge)

The hadronic tensor is:

$$H_{\mu\nu} = \sum_X \langle 0 | J_\nu^\dagger(0) | X \rangle \langle X | J_\mu(0) | 0 \rangle (2\pi)^4 \delta^{(4)}(p_X - q) \delta(\tau - \tau(X))$$

↑
vector current

We wish to compute the differential cross section in the angle θ of the thrust axis $\vec{n}_T$ with respect to the momentum $\vec{p}_1$ of the electron. To do so, we insert:

$$1 = \int d^3\vec{n} \delta^{(3)}(\vec{n} - \vec{n}_T) = \int_0^{2\pi} d\varphi \int d\cos\theta \delta^{(2)}(\vec{n}^\perp)$$

↑
contains $\delta(|\vec{n}| - 1)$

$$= 2\pi \left(\frac{Q}{2}\right)^2 \int d\cos\theta \underbrace{\delta^{(2)}(p_{Xc}^\perp)}_{\text{required by definition of thrust axis}} ; \quad \vec{p}_{Xc} = \frac{Q}{2} \vec{n} \quad \text{up to power cons.}$$

↑
orthogonal to thrust axis

required by definition
of thrust axis

Combining this δ -function with the δ -function from momentum conservation gives:

$$\begin{aligned}\delta^{(4)}(p_X - q) \delta^{(2)}(p_{X_c}^\perp) &= \delta^{(4)}(p_{X_c} + p_{X_{\bar{c}}} + p_{X_s} - q) \delta^{(2)}(p_{X_c}^\perp) \\ &= 2 \delta(\bar{n} \cdot p_{X_c} - Q) \delta(n \cdot p_{X_{\bar{c}}} - Q) \delta^{(2)}(p_{X_c}^\perp) \delta^{(2)}(p_{X_{\bar{c}}}^\perp) + \dots\end{aligned}$$

We finally introduce new variables

$$M_c^2 = p_{X_c}^2, \quad M_{\bar{c}}^2 = p_{X_{\bar{c}}}^2, \quad \omega = n \cdot p_{X_s}^R + \bar{n} \cdot p_{X_s}^L$$

$$\Rightarrow \tau Q^2 = M_c^2 + M_{\bar{c}}^2 + Q\omega$$

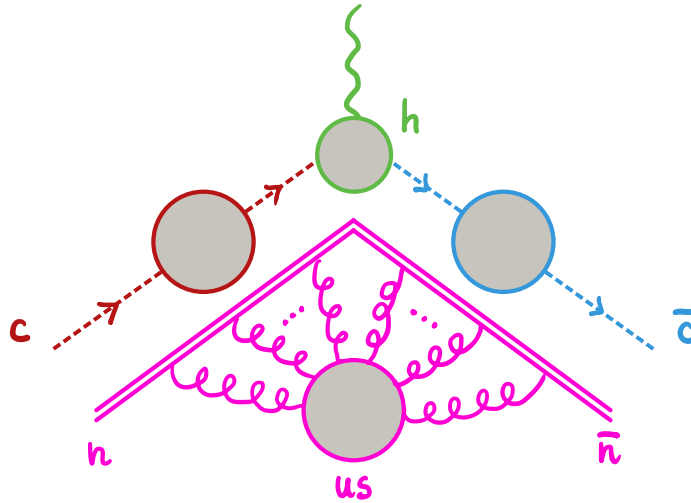
by multiplying the hadronic tensor with the dummy integral:

$$1 = \int dM_c^2 \delta(M_c^2 - p_{X_c}^2) \int dM_{\bar{c}}^2 \delta(M_{\bar{c}}^2 - p_{X_{\bar{c}}}^2) \int d\omega \delta(\omega - n \cdot p_{X_s}^R - \bar{n} \cdot p_{X_s}^L)$$

This leads to:

$$\begin{aligned}\frac{d\sigma}{d\tau d\cos\theta} &= \frac{\pi}{2} L_{\mu\nu} |C_V(-Q^2 - i0, \mu)|^2 \int dM_c^2 \int dM_{\bar{c}}^2 \int d\omega \delta(\tau - \overbrace{\frac{M_c^2 + M_{\bar{c}}^2 + Q\omega}{Q^2}}^{\tau(x)}) \\ &\quad \times \sum_{X_c} \langle 0 | \chi_{c,\delta}^a(0) | X_c \rangle \langle X_c | \bar{\chi}_{c,\alpha}^b | 0 \rangle \delta(M_c^2 - p_{X_c}^2) \delta^{(2)}(p_{X_c}^\perp) \delta(\bar{n} \cdot p_{X_c} - Q) \\ &\quad \times \sum_{X_{\bar{c}}} \langle 0 | \bar{\chi}_{\bar{c},\gamma}^d(0) | X_{\bar{c}} \rangle \langle X_{\bar{c}} | \chi_{\bar{c},\beta}^e | 0 \rangle \delta(M_{\bar{c}}^2 - p_{X_{\bar{c}}}^2) \delta^{(2)}(p_{X_{\bar{c}}}^\perp) \delta(n \cdot p_{X_{\bar{c}}} - Q) \\ &\quad \times \sum_{X_s} \langle 0 | [S_n^\dagger S_{\bar{n}}]_{da} | X_s \rangle \langle X_s | [S_{\bar{n}}^\dagger S_n]_{be} | 0 \rangle \delta(\omega - n \cdot p_{X_s}^R - \bar{n} \cdot p_{X_s}^L) \\ &\quad \times (2\pi)^4 (\gamma_\perp^\mu)_{\alpha\beta} (\gamma_\perp^\nu)_{\gamma\delta}\end{aligned}$$

Here $\chi_c \equiv W_c^{(0)\dagger} \xi_n^{(0)}$ and $\chi_{\bar{c}} = W_{\bar{c}}^{(0)\dagger} \xi_{\bar{n}}^{(0)}$ are the gauge-invariant collinear building blocks after decoupling of the ultra-soft gluons, and $S_n, S_{\bar{n}}$ are the soft Wilson lines.



At this stage, one defines jet and soft functions via the matrix elements in the different sectors of SCET.

Jet functions:

$$\sum_{X_c} \langle 0 | \chi_{c,\delta}^a(0) | X_c \rangle \langle X_c | \bar{\chi}_{c,\alpha}^b | 0 \rangle \delta(M_c^2 - p_{X_c}^2) \delta^{(2)}(p_{X_c}^\perp) \delta(\bar{n} \cdot p_{X_c} - Q)$$

$$= \frac{\delta^{ab}}{2(2\pi)^3} \left[\frac{\not{n}}{2} \right]_{\delta\alpha} J(M_c^2)$$

↑
color conservation

since $\not{n} \xi_n^{(0)} = 0$

Note that this is the spectral representation of a collinear quark propagator (c.f. Section 7.1 in Peskin & Schroeder).

At lowest order, X_c are single quark states and we find:

$$\int \frac{d^3 \vec{p}}{(2\pi)^3} \underbrace{\sum_s u_{n,s}(p,s) \bar{u}_{n,\alpha}(p,s) \delta^{ab} \delta(M_c^2) \delta^{(2)}(p_\perp) \delta(2p^3 - Q)}_{\left(\frac{\not{n} \not{p}}{4} \not{p} \frac{\not{n} \not{p}}{4}\right)_{\delta\alpha} = \left(\frac{\not{n}}{2}\right)_{\delta\alpha} \underline{\bar{n} \cdot p}}$$

$$= \frac{\delta^{ab}}{2(2\pi)^3} \left(\frac{\not{n}}{2}\right)_{\delta\alpha} \delta(M_c^2)$$

It follows that:

$$J(M^2) = \delta(M^2) + \mathcal{O}(\alpha_s)$$

This jet function is known to 3-loop order in QCD.

↳ Becher, MN: hep-ph/0603140 (2-loop)

Brüser, Liu, Stahlhofen: 1804.09722 (3-loop)

At 1-loop order, one finds: (hep-ph/0402094)

$$J(M^2) = \delta(M^2) + \frac{C_F \alpha_s}{4\pi} \left[(7 - \pi^2) \delta(M^2) + 4 \left(\frac{\ln M^2/\mu^2}{M^2} \right)_*^{[\mu^2]} - 3 \left(\frac{1}{M^2} \right)_*^{[\mu^2]} \right] + \mathcal{O}(\alpha_s^2)$$

generalized plus distributions

The jet function in the anti-collinear sector is defined as:

$$\sum_X \langle 0 | \bar{\chi}_{\bar{c},\gamma}^d(0) | X_{\bar{c}} \rangle \langle X_{\bar{c}} | \chi_{\bar{c},\beta}^e | 0 \rangle \delta(M_{\bar{c}}^2 - p_{X_{\bar{c}}}^2) \delta^{(2)}(p_{X_{\bar{c}}}^\perp) \delta(n \cdot p_{X_{\bar{c}}} - Q)$$

$$= \frac{\delta^{de}}{2(2\pi)^3} \left[\frac{\not{n}}{2} \right]_{\beta\gamma} J(M_{\bar{c}}^2)$$

At this point, we obtain the following trace over Dirac matrices:

$$\text{tr} \left(\gamma_1^\mu \frac{\not{n}}{2} \gamma_1^\nu \frac{\not{\bar{n}}}{2} \right) = -g_{\perp}^{\mu\nu} n \cdot \bar{n} = -2 g_{\perp}^{\mu\nu}$$

Also, the four color indices in the ultra-soft matrix element get contracted in pairs.

Soft function:

$$S(\omega) = \frac{1}{N_c} \sum_{X_s} \langle 0 | [S_n^\dagger S_{\bar{n}}]_{ab} | X_s \rangle \langle X_s | [S_{\bar{n}}^\dagger S_n]_{ba} | 0 \rangle \delta(\omega - n \cdot p_{X_s}^R - \bar{n} \cdot p_{X_s}^L)$$

The prefactor $1/N_c$ has been introduced such that at leading order:

$$S(\omega) = \delta(\omega) + \mathcal{O}(\alpha_s)$$

↑
calculable only
if $\omega \gg \Lambda_{\text{QCD}}$

"shape function"

hep-ph/9311325
9902341

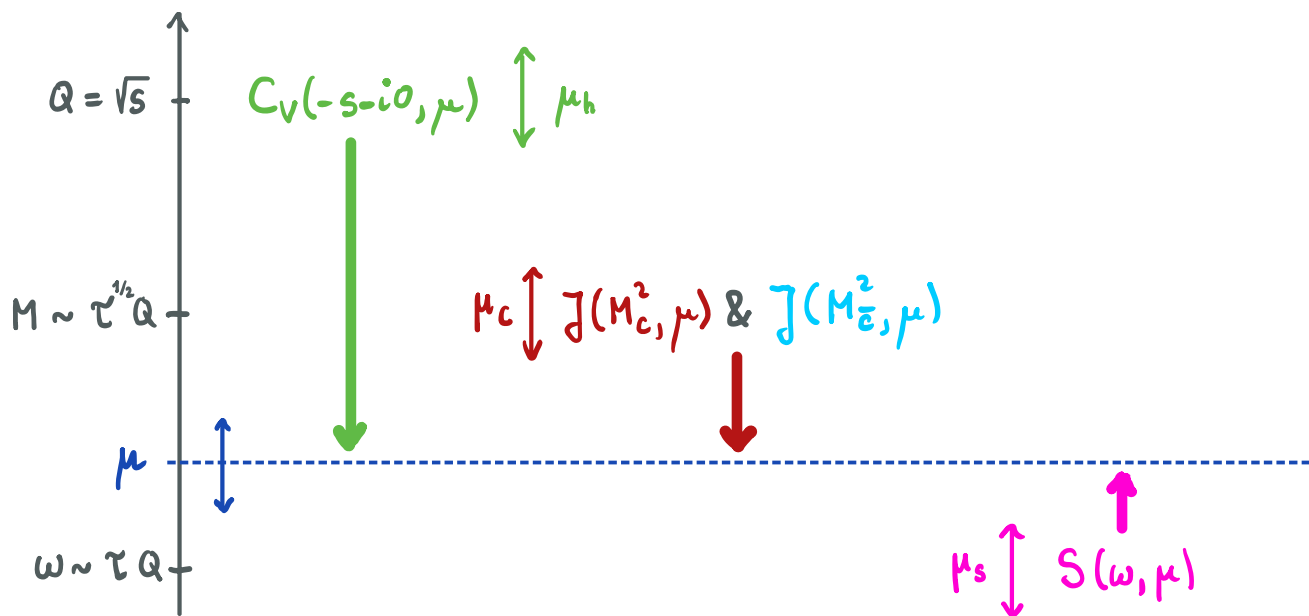
This function is known at 2-loop order in perturbation theory.

(Becher, Schwartz: 0803.0342) Note that for $\omega \sim \Lambda_{\text{QCD}}$, i.e.

$\tau \sim \frac{\Lambda_{\text{QCD}}}{Q}$, the shape function is a genuinely nonperturbative object, which must be extracted from data.

Resummation of large logarithms:

The theoretical prediction for the cross section is independent of the renormalization scale μ . However, for each fixed choice of μ there are large logs in at least some of the component functions C_V , J and S . The strategy is therefore to calculate these functions at their "natural" scales, and then evolve ("run") them to a common (and arbitrary) scale μ by solving their RG equations:



For the hard matching coefficient we have discussed the solution of the RG equation in lecture 5 (see pages 5-7). The RG equations for the jet and soft

functions are more complicated, e.g.:

(Becher, MN: hep-ph/0603140)

$$\mu \frac{d}{d\mu} J(p^2, \mu) = \left[-2\Gamma_{\text{cusp}}(\alpha_s) \ln \frac{p^2}{\mu^2} - 2\gamma_J(\alpha_s) \right] J(p^2, \mu) \\ + 2\Gamma_{\text{cusp}}(\alpha_s) \int_0^\infty dp'^2 \frac{J(p^2, \mu) - J(p'^2, \mu)}{p^2 - p'^2}$$

and similarly for the soft function.

The form of the cross section on p.11 and the RG equations simplify in Laplace space, where one defines:

$$\begin{aligned} \tilde{\sigma}(v) &\equiv \int_0^\infty d\tau e^{-v\tau} \frac{d\sigma}{d\tau} \\ &= \frac{4\pi Q_f^2 \alpha^2}{s} |C_V(-s-i0, \mu)|^2 \int_0^\infty dM_c^2 e^{-\frac{vM_c^2}{s}} J(M_c^2, \mu) \\ &\quad \times \int_0^\infty dM_{\bar{c}}^2 e^{-\frac{vM_{\bar{c}}^2}{s}} J(M_{\bar{c}}^2, \mu) \int_0^\infty d\omega e^{-\frac{v\omega}{\sqrt{s}}} S(\omega, \mu) \\ &\equiv \frac{4\pi Q_f^2 \alpha^2}{s} |C_V(-s-i0, \mu)|^2 \underbrace{\tilde{J}\left(\frac{v}{s}, \mu\right)}_{\sim (\tau s)^{-1}} \underbrace{\tilde{J}\left(\frac{v}{s}, \mu\right)}_{\sim (\tau s)^{-1}} \underbrace{\tilde{S}\left(\frac{v}{\sqrt{s}}, \mu\right)}_{\sim (\tau \sqrt{s})^{-1}} \end{aligned}$$

In Laplace space we obtain a product rather than a

convolution. Also, the RG equations take on a local form (Becher, MN: hep-ph/0605050) and one obtains:

$$\mu \frac{d}{d\mu} \tilde{J}\left(\frac{\sqrt{s}}{s}, \mu\right) = \left[-2\Gamma_{\text{cusp}}(\alpha_s) \left(\ln \frac{s}{\sqrt{s}\mu^2} - \gamma_E \right) - 2\gamma_J(\alpha_s) \right] \tilde{J}\left(\frac{\sqrt{s}}{s}, \mu\right)$$

$$\mu \frac{d}{d\mu} \tilde{S}\left(\frac{\sqrt{s}}{\sqrt{s}}, \mu\right) = \left[2\Gamma_{\text{cusp}}(\alpha_s) \left(\ln \frac{s}{\sqrt{s}^2\mu^2} - 2\gamma_E \right) + 2\gamma_S(\alpha_s) \right] \tilde{S}\left(\frac{\sqrt{s}}{\sqrt{s}}, \mu\right)$$

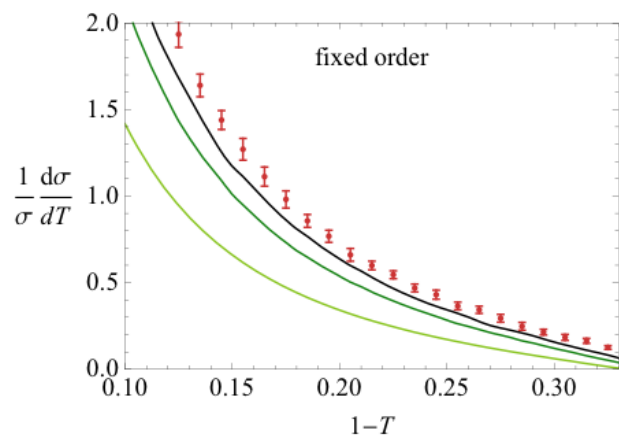
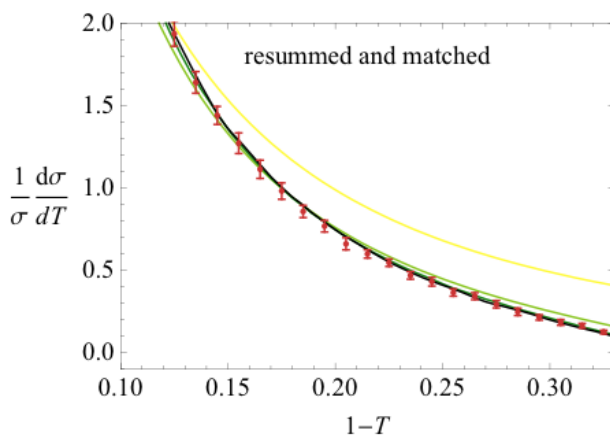
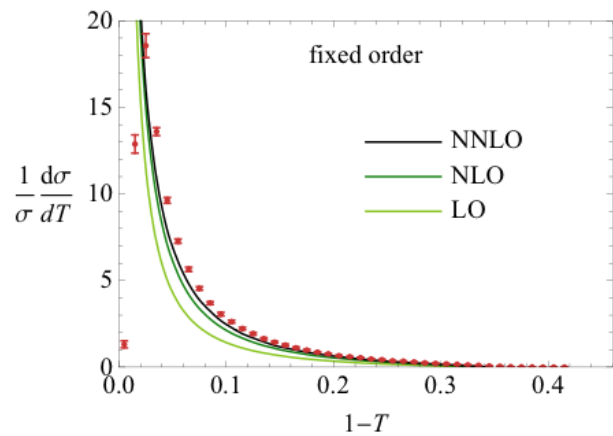
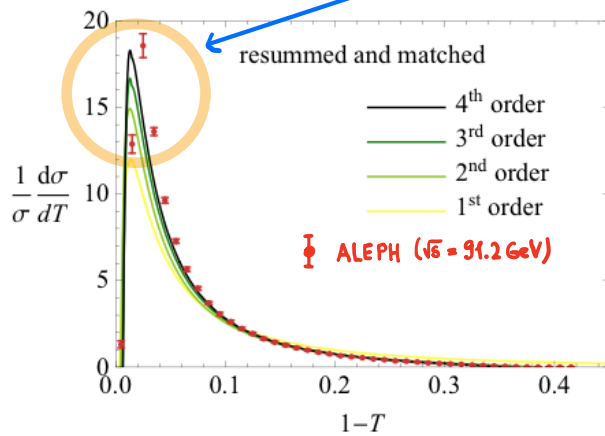
They can be solved using the same techniques as for the hard function. The fact that the cross section is RG invariant, $\mu \frac{d}{d\mu} \tilde{\sigma}(v) = 0$, implies:

$$\begin{aligned} & 2\Gamma_{\text{cusp}}(\alpha_s) \ln \frac{s}{\mu^2} + 2\gamma_V(\alpha_s) && |C_V|^2 \\ & - 4\Gamma_{\text{cusp}}(\alpha_s) \left(\ln \frac{s}{\sqrt{s}\mu^2} - \gamma_E \right) - 4\gamma_J(\alpha_s) && \tilde{J} \tilde{J} \\ & + 2\Gamma_{\text{cusp}}(\alpha_s) \left(\ln \frac{s}{\sqrt{s}^2\mu^2} - 2\gamma_E \right) + 2\gamma_S(\alpha_s) \stackrel{!}{=} 0 && \tilde{S} \\ \Rightarrow & \gamma_S(\alpha_s) = 2\gamma_J(\alpha_s) - \gamma_V(\alpha_s) \end{aligned}$$

This consistency condition is satisfied to all orders in α_s .

Comparison with data:

region where nonperturbative effects can be important



(Becher, Schwartz: 0803.0342)

The "matching" of a resummed SCET prediction (valid for $\tau \ll 1$) onto a prediction obtained using fixed-order perturbation theory (valid for $\tau = \mathcal{O}(1)$) is performed as follows:

$$\left(\frac{d\sigma}{d\tau} \right)_{\text{matched}} = \left(\frac{d\sigma}{d\tau} \right)_{\text{resummed}}^{\text{SCET}} + \left[\left(\frac{d\sigma}{d\tau} \right)_{\text{fixed order}} - \left(\frac{d\sigma}{d\tau} \right)_{\text{resummed}}^{\text{SCET}} \right]_{\text{reexpanded in powers of } \alpha_s}$$

← optimal prediction for $\tau \ll 1$

↑ subtraction to avoid double counting

optimal prediction for $\tau = \mathcal{O}(1)$ (vanishes for $\tau \rightarrow 0$)