Introduction to Neutrino Oscillations

Shun Zhou
(IHEP, Beijing)

A. History of Neutrino Oscillations

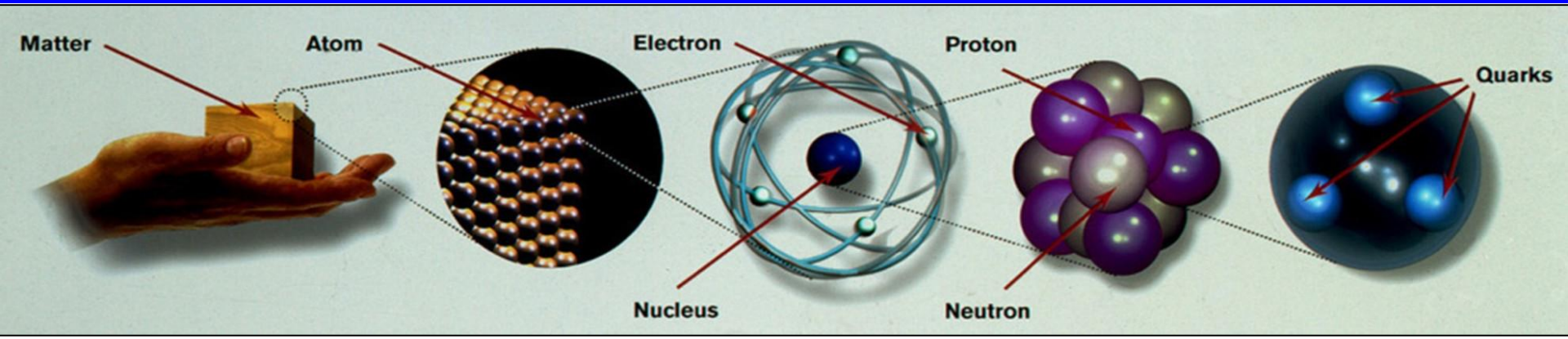
B. Neutrino Oscillations in Vacuum

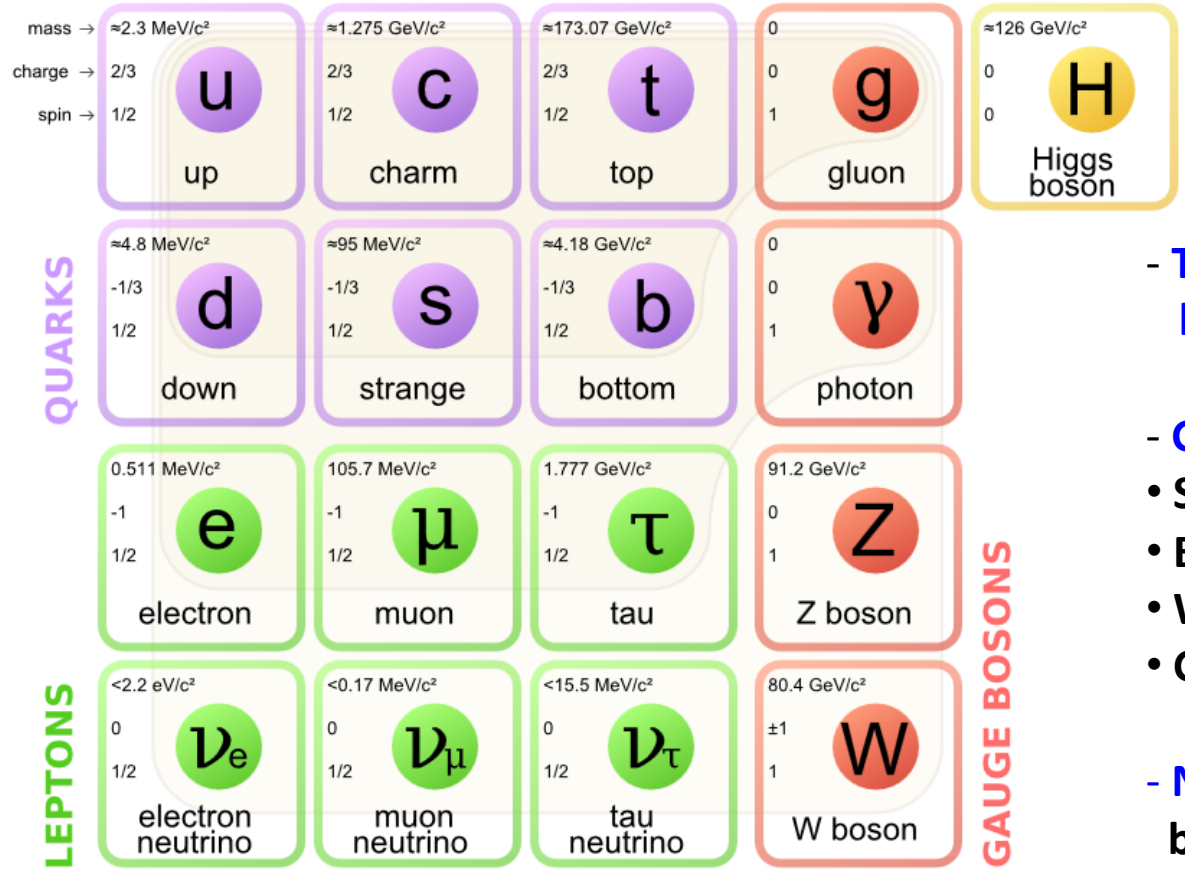
C. Neutrino Oscillations in Matter

The First China-Europe Neutrino Astronomy Summer School

September 14-23, 2026

Part A: History of Neutrino Oscillations





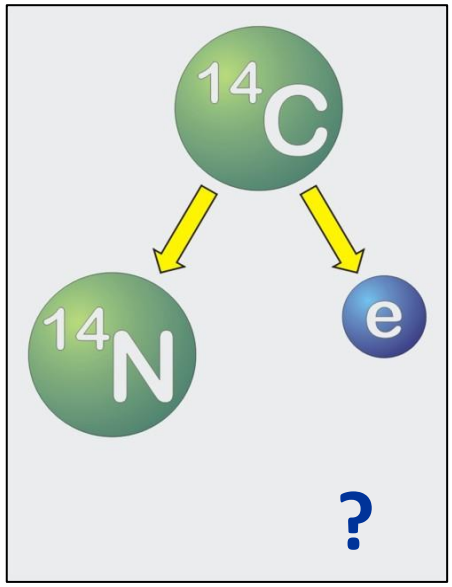
Standard Model of Elementary Particles

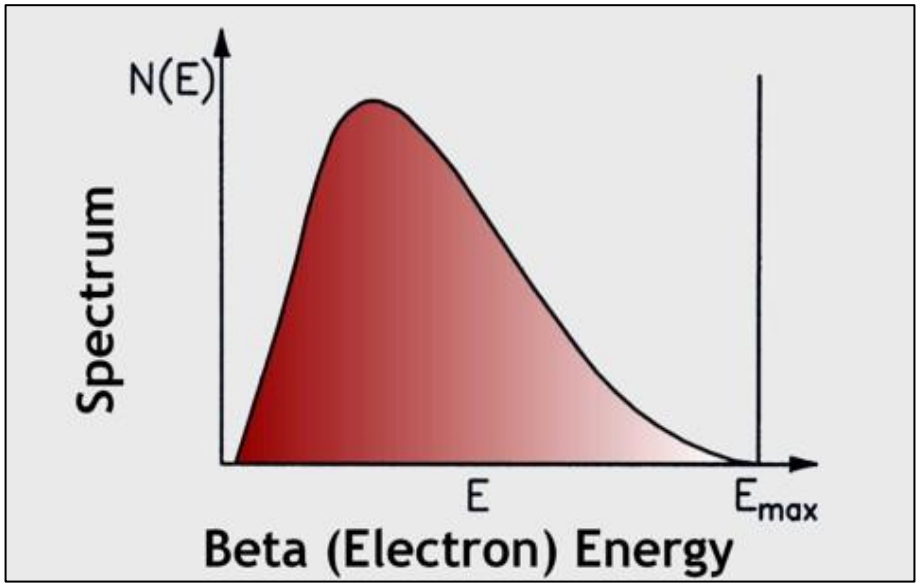
- **Three generations of quarks and leptons**-building blocks of matter

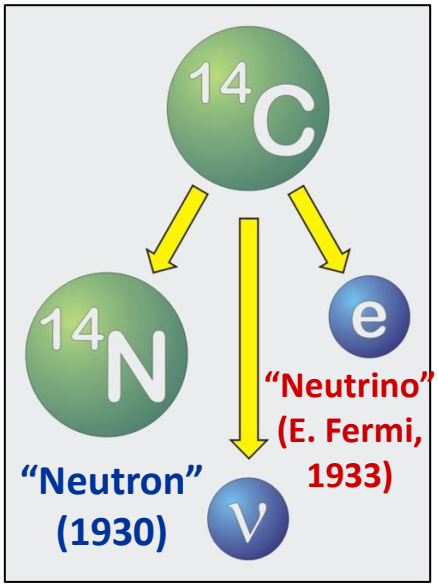
- **Gauge bosons as force carriers**
  - Strong Interaction (8 gluons)
  - Electromagnetic Interaction (γ)
  - Weak Interaction (W and Z)
  - Gravitation (Graviton?)

- **Neutrinos are massive particles**-beyond the Standard Model

Pauli's Neutrino Hypothesis







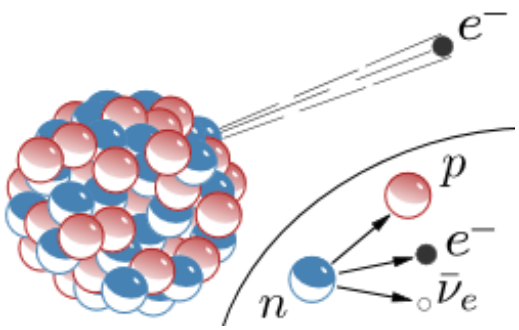
Niels Bohr
(1885-1962)
Nobel Prize 1922

Energy not conserved in the quantum domain?

Wolfgang Pauli
(1900–1958)
Nobel Prize 1945

there exist in the nuclei electrically neutral particles, that I call neutrons

Detecting Neutrinos from Nuclear Explosions?



Bethe and Peierls (1934)

$$\bar{\nu}_e + p \rightarrow n + e^+ \quad \text{Cross section } \sigma < 10^{-44} \text{ cm}^2$$

"It is therefore absolutely impossible to observe the processes of this kind with the neutrinos created in nuclear transformations."

Hans A. Bethe
(1906–2005)
Nobel Prize 1967

The combination of four protons and two electrons can occur essentially only in two ways. The first mechanism starts with the combination of two protons to form a deuteron with positron emission, *viz.*

$$H + H = D + \epsilon^+. \qquad (1)$$

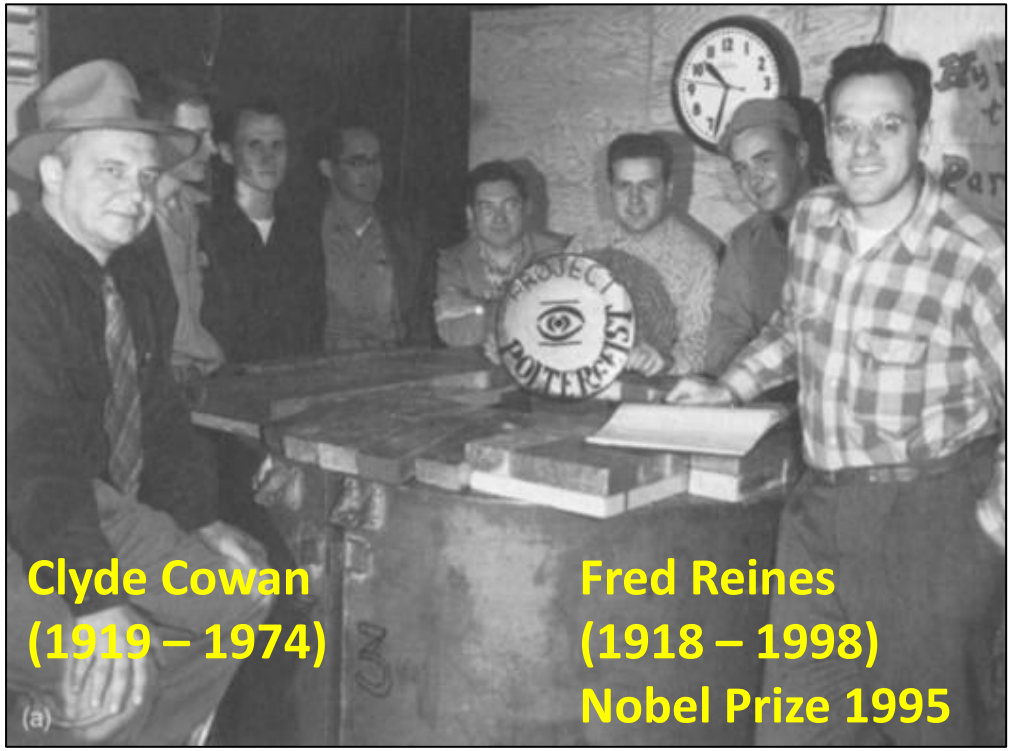
The deuteron is then transformed into He^4 by further capture of protons; these captures occur very rapidly compared with process (1). The second mechanism uses carbon and nitrogen as catalysts, according to the chain reaction

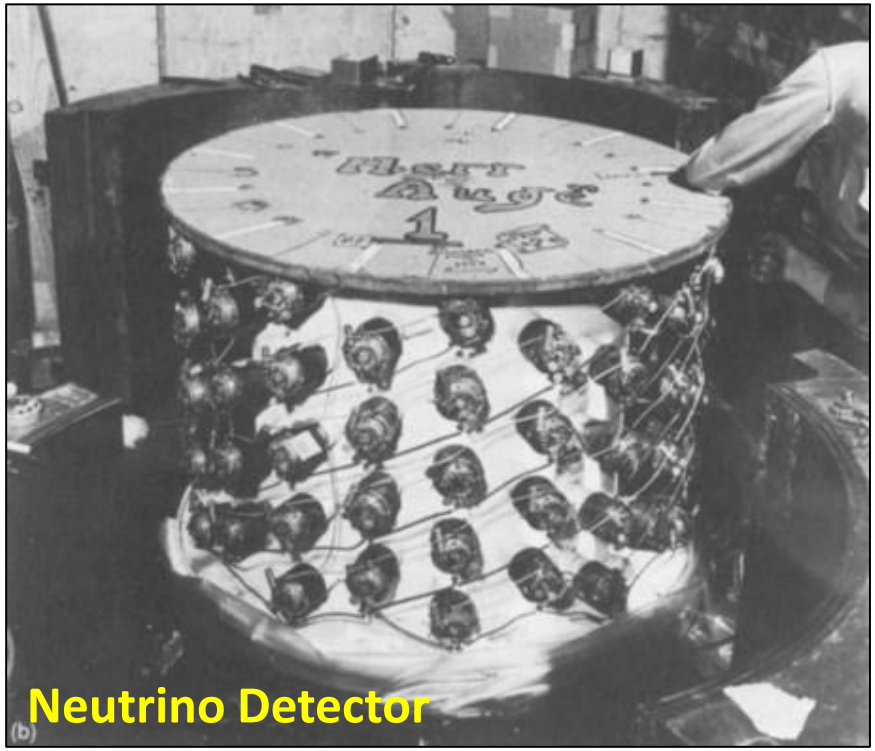
$$\begin{aligned} &C^{12} + H = N^{13} + \gamma, && N^{13} = C^{13} + \epsilon^+ \\ &C^{13} + H = N^{14} + \gamma, \\ &N^{14} + H = O^{15} + \gamma, && O^{15} = N^{15} + \epsilon^+ \\ &N^{15} + H = C^{12} + He^4. \end{aligned} \qquad (2)$$

No neutrinos in Bethe's classic paper on nuclear reactions in stars (1939)

Reines to Fermi (1951): neutrino detector near 'A-bombs'?

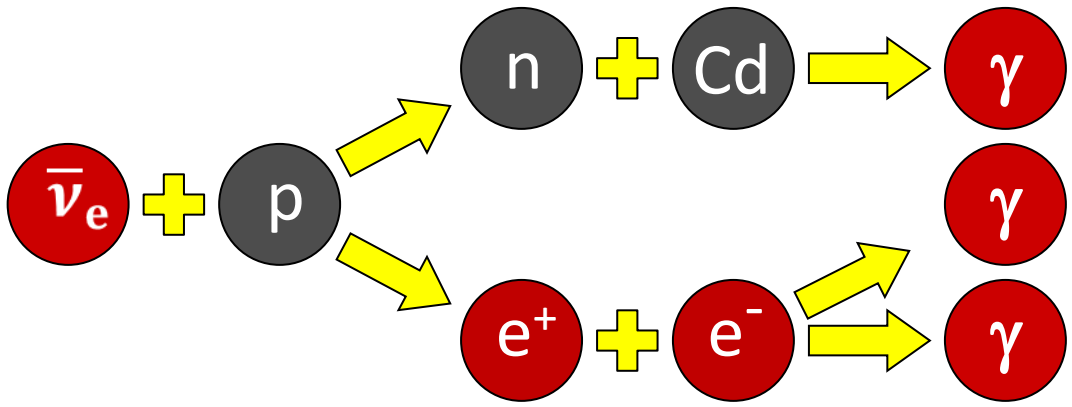
Discovery of Reactor Neutrinos 1954-1956





Gave up
'A-bombs'

Hanford
Nuclear
Reactor

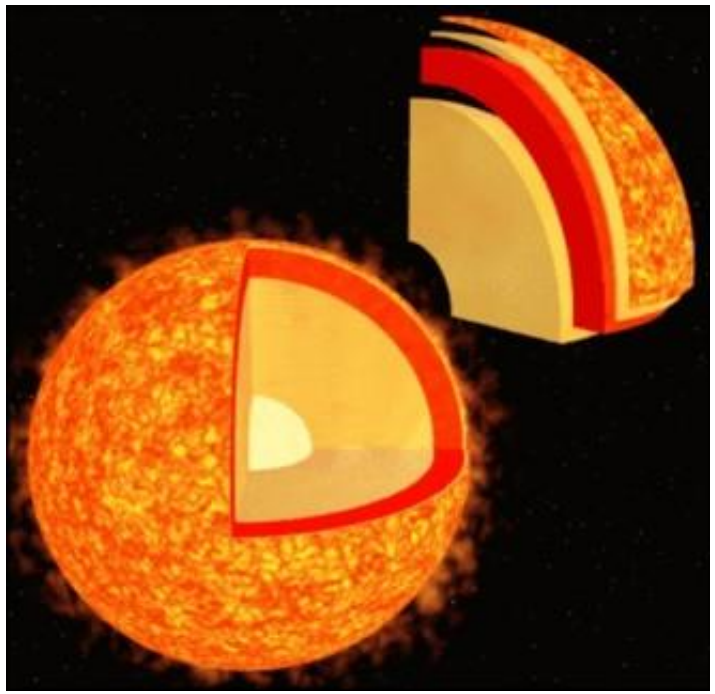


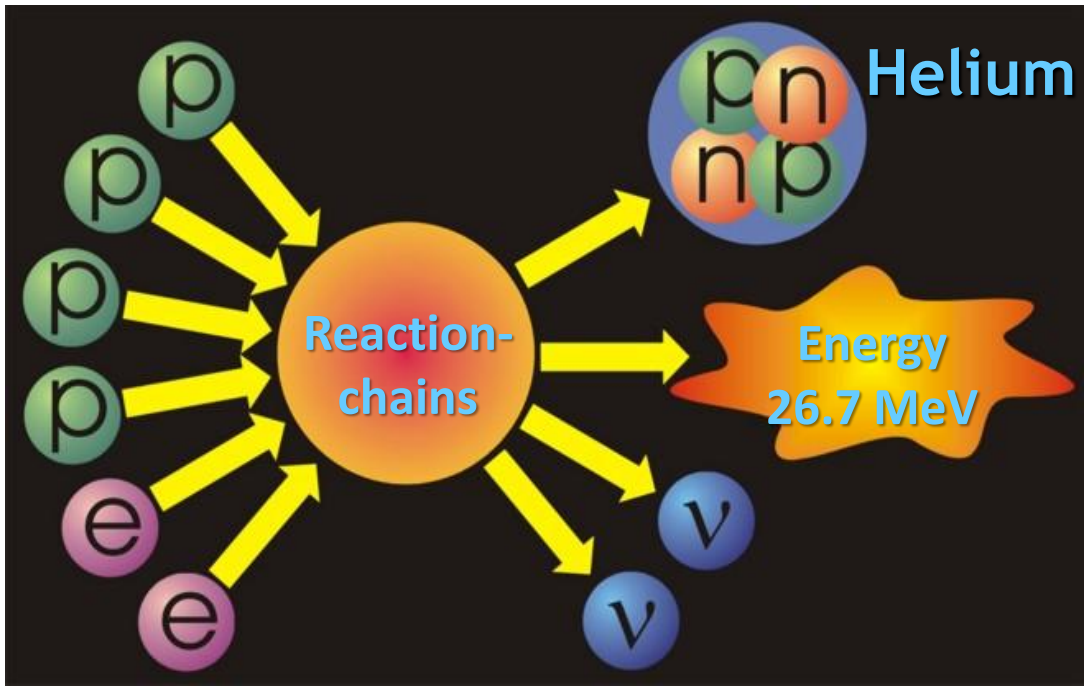
three Gammas

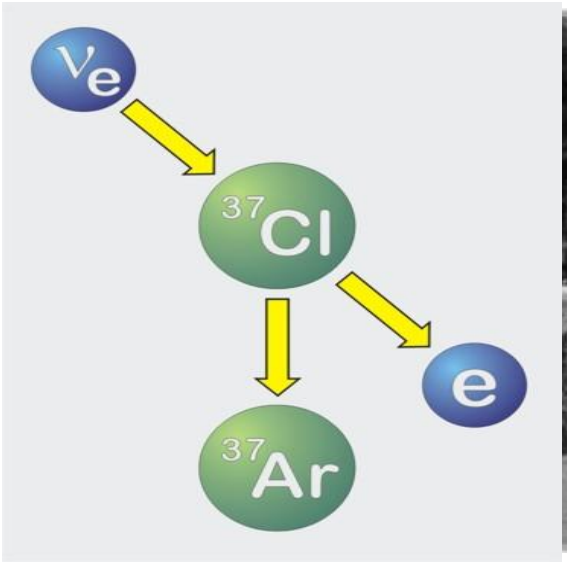
5 μs delay of γ
from n capture
on Cadmium

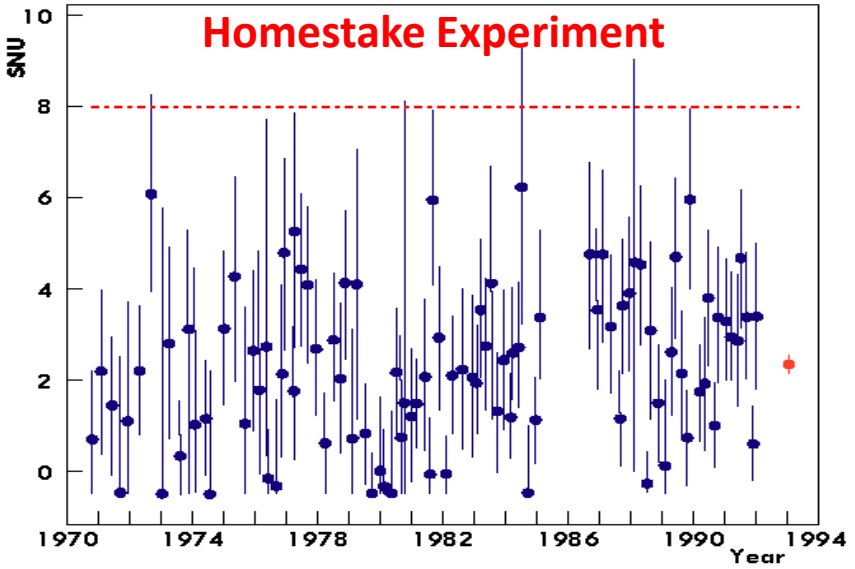
time coincidence

Discovery of Solar Neutrinos (1968)

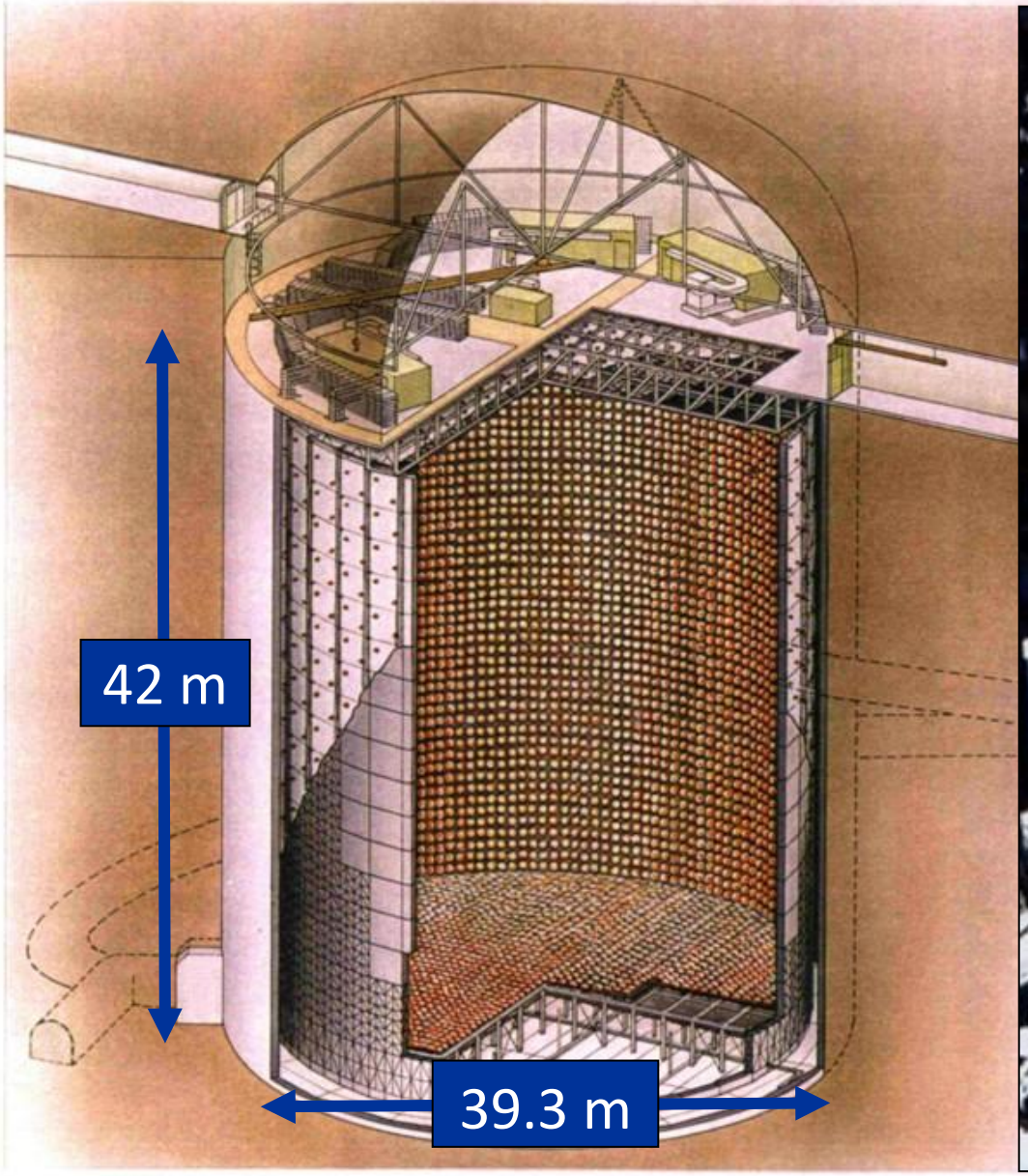


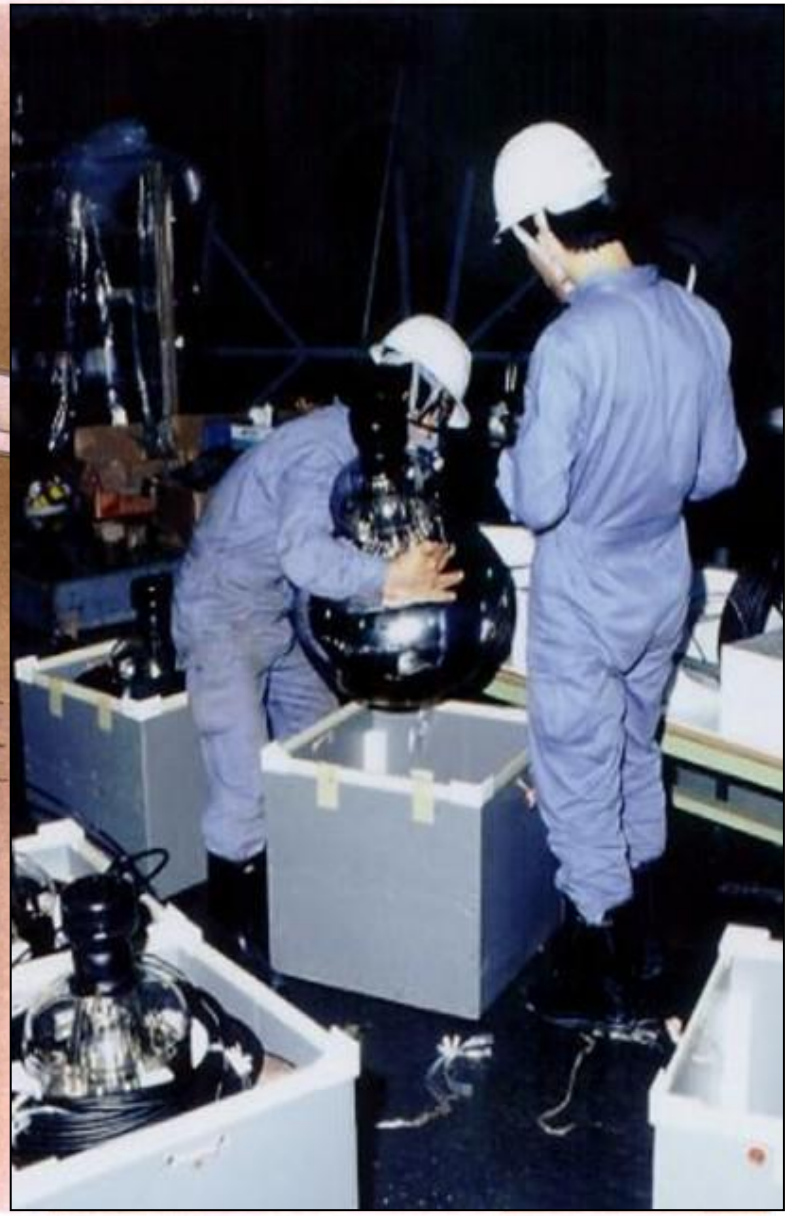




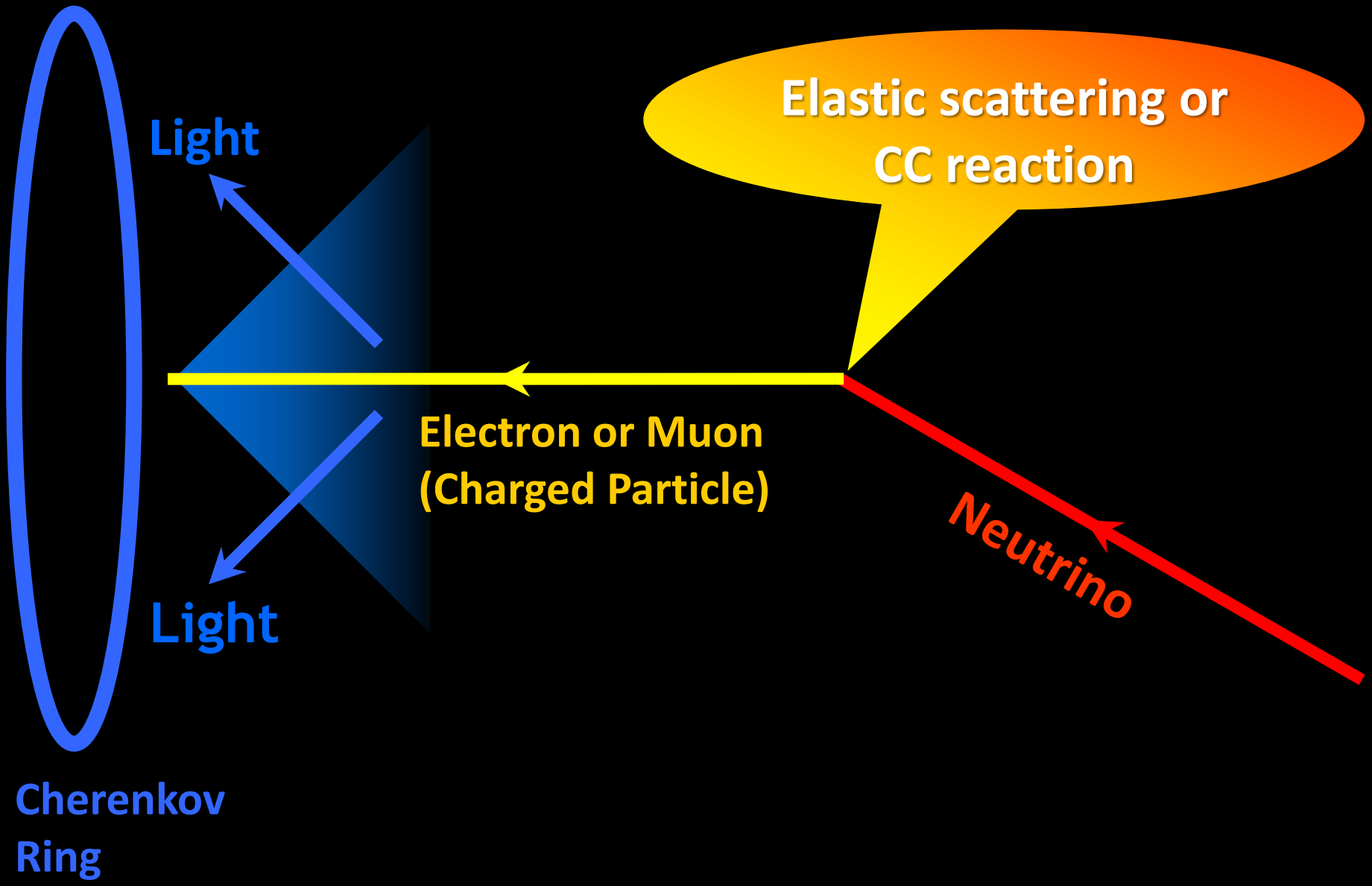


Super-Kamiokande Neutrino Detector (since 1996)

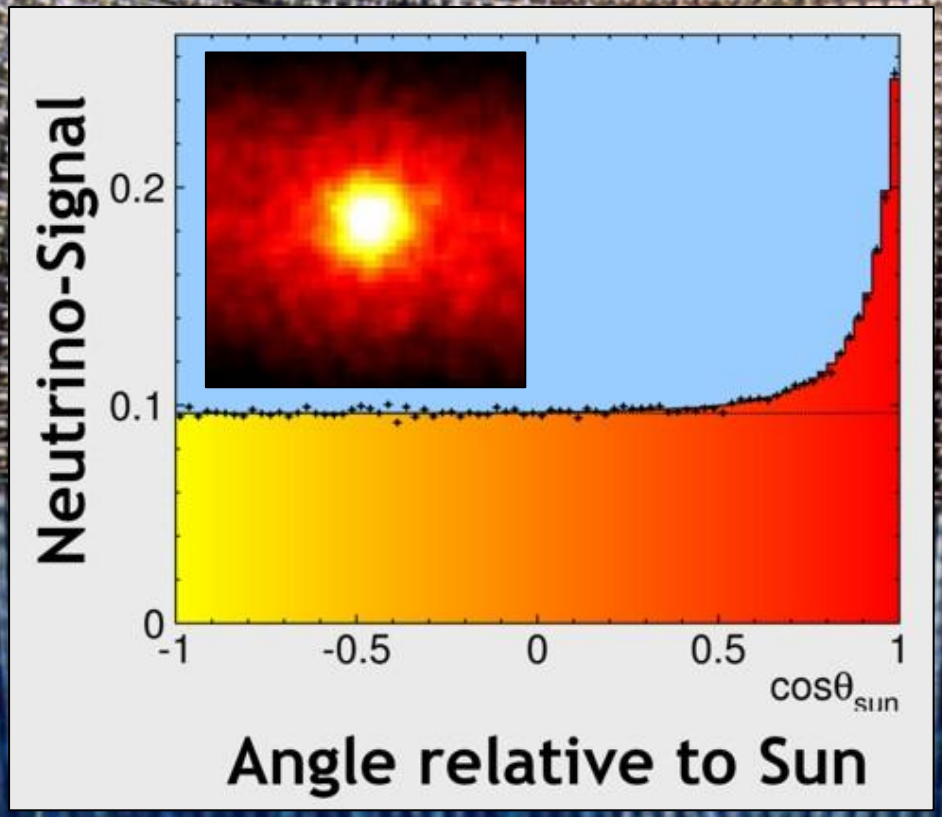


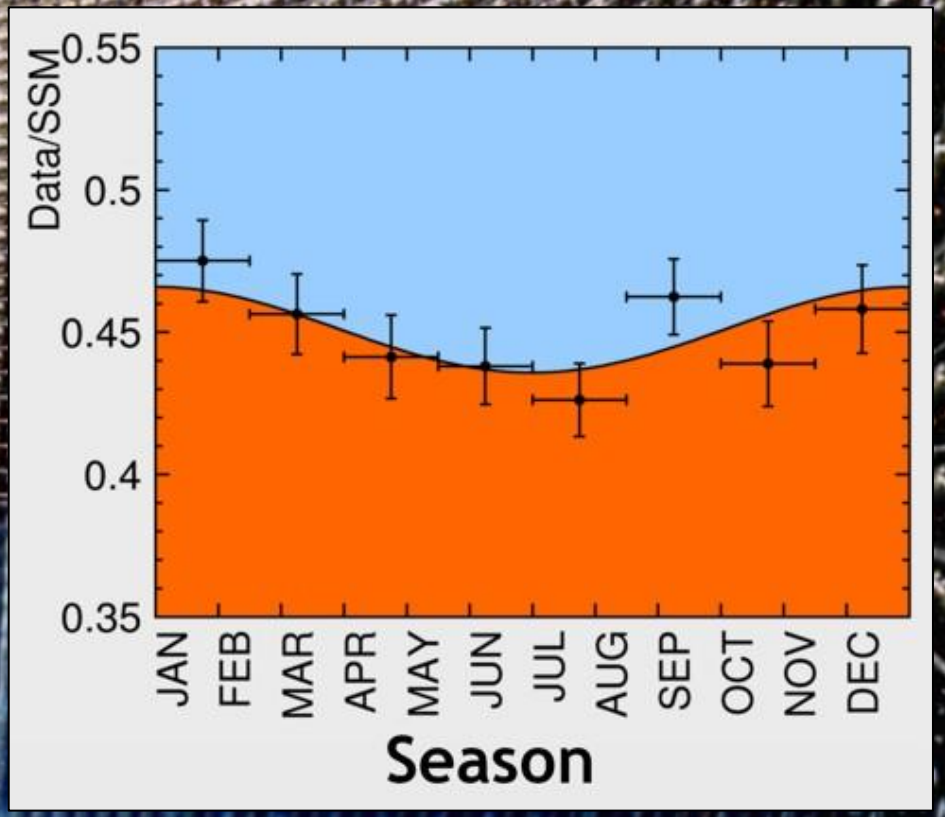


Cherenkov Detectors



Super-Kamiokande : Image of the Sun in Neutrino Light





Neutrino Sources in Nature

✓

Particle Accelerators

Nuclear Reactors

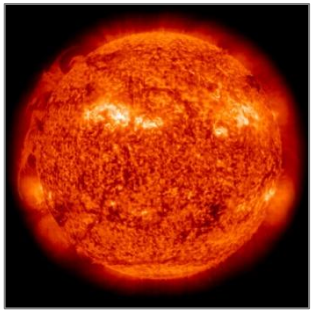
✓

✓

Cosmic rays
@ the Earth
Atmosphere

Natural radioactivity
in the Earth Crust

✓

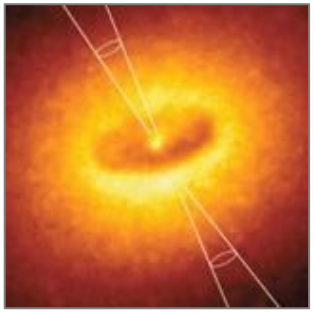


Nuclear Fusion
in the Sun

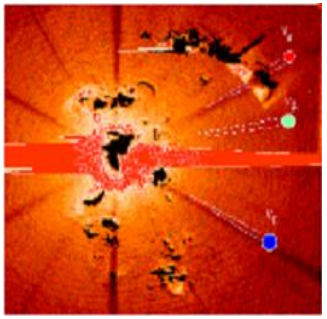
✓

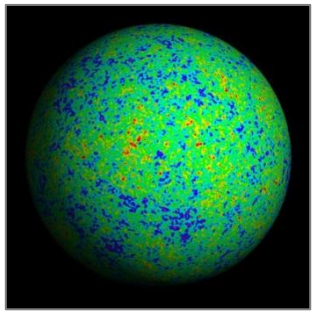
Supernova Explosion
SN 1987A

✓



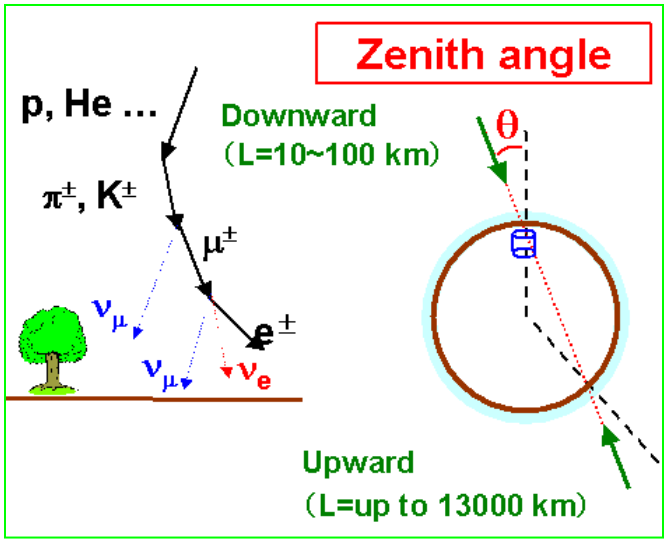
Cosmic Accelerators ?
PeV neutrino events
@ IceCube



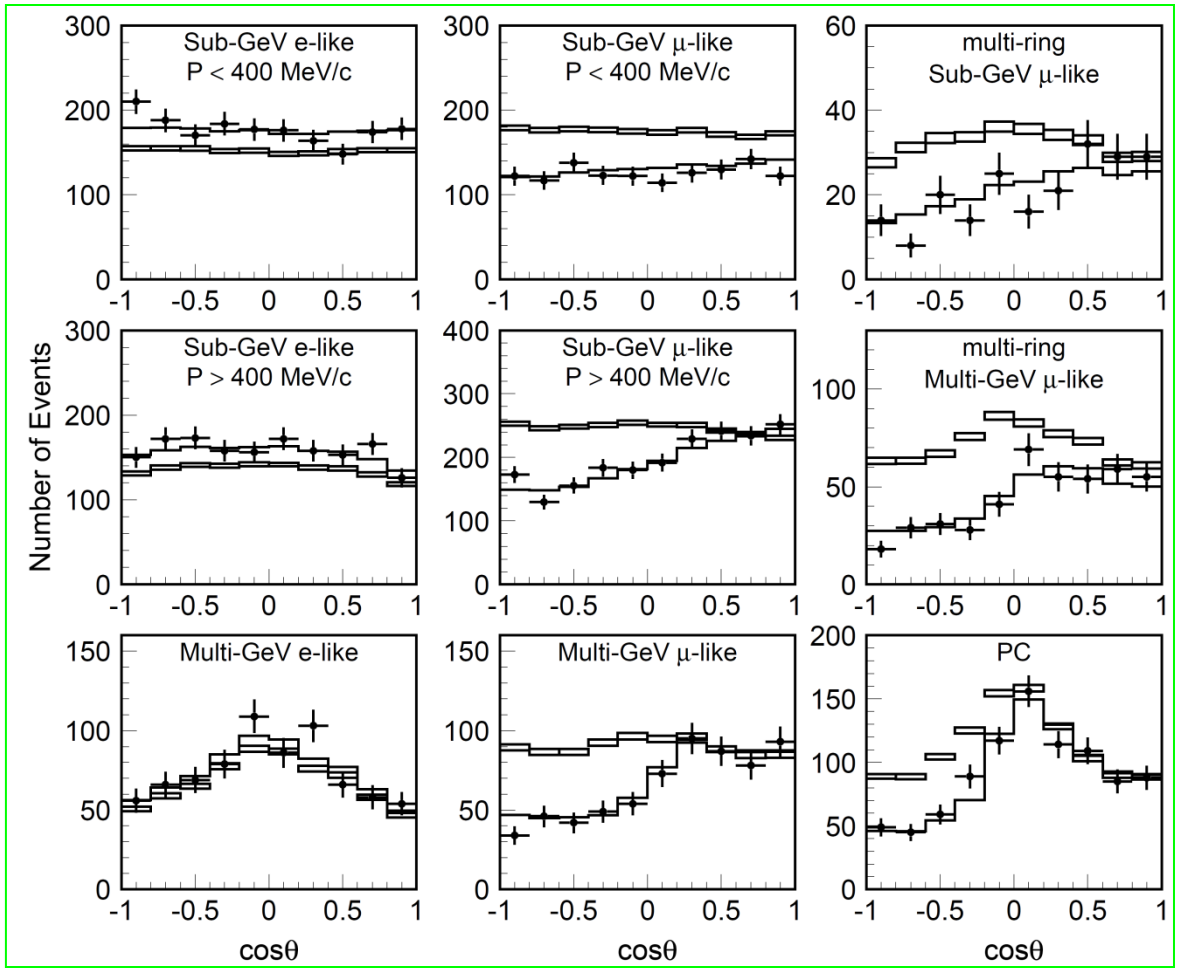


Big Bang
(Indirect
Evidence)

Discovery of Neutrino Oscillations

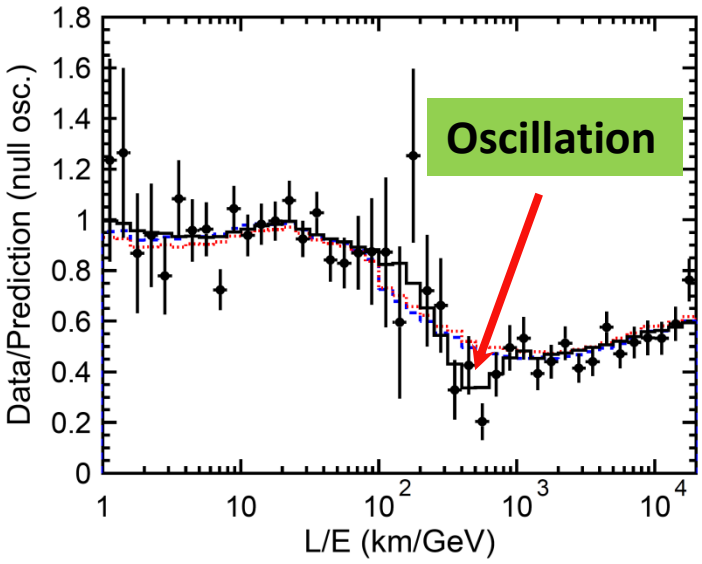


Angular Distribution of Events



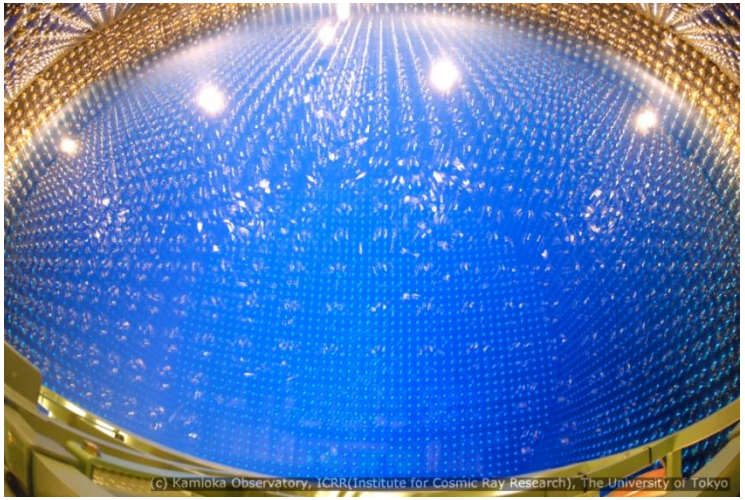
~13000 km ~500 km ~15 km

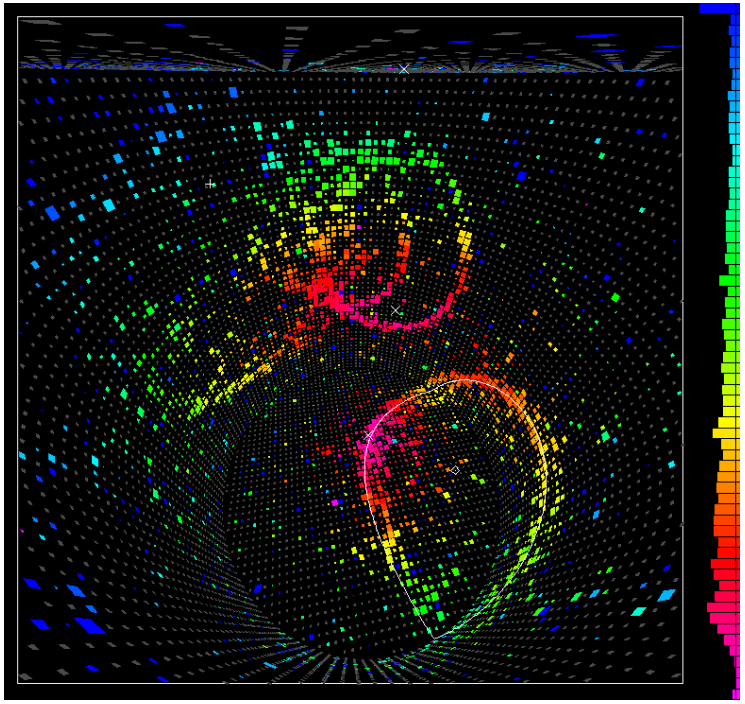
L/E Significance



Super-Kamiokande 1998, 2005

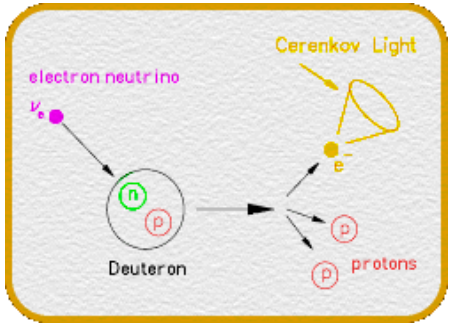
Discovery of Neutrino Oscillations

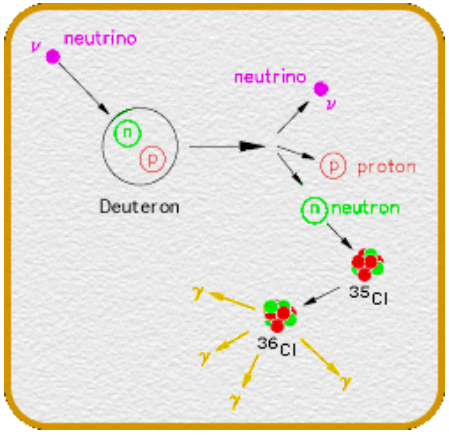


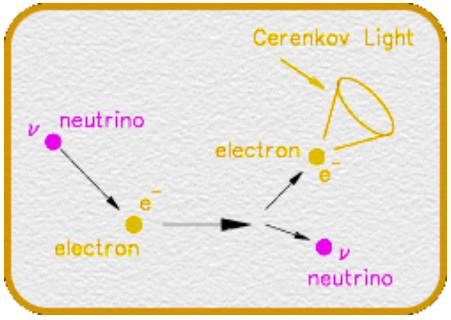


Super-Kamiokande Experiment

$$\nu_\alpha + e^- \rightarrow \nu_\alpha + e^-$$





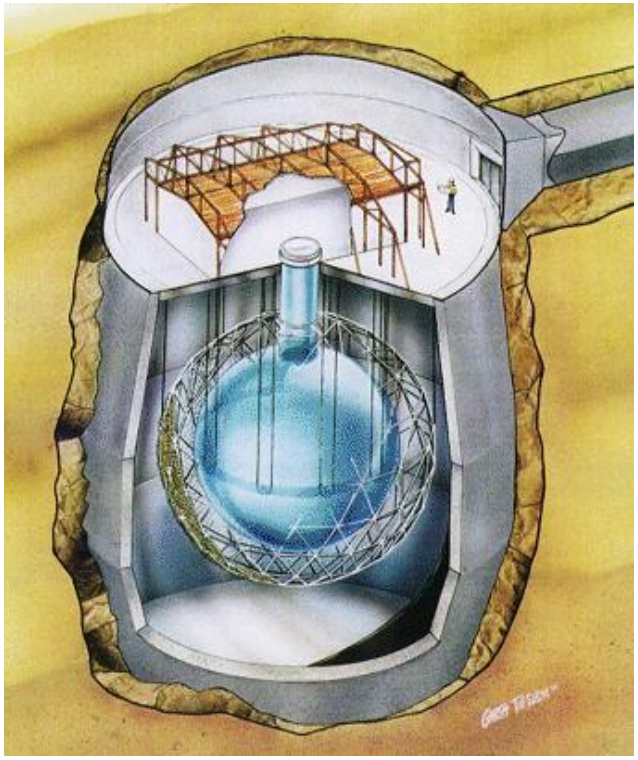


SNO Experiment

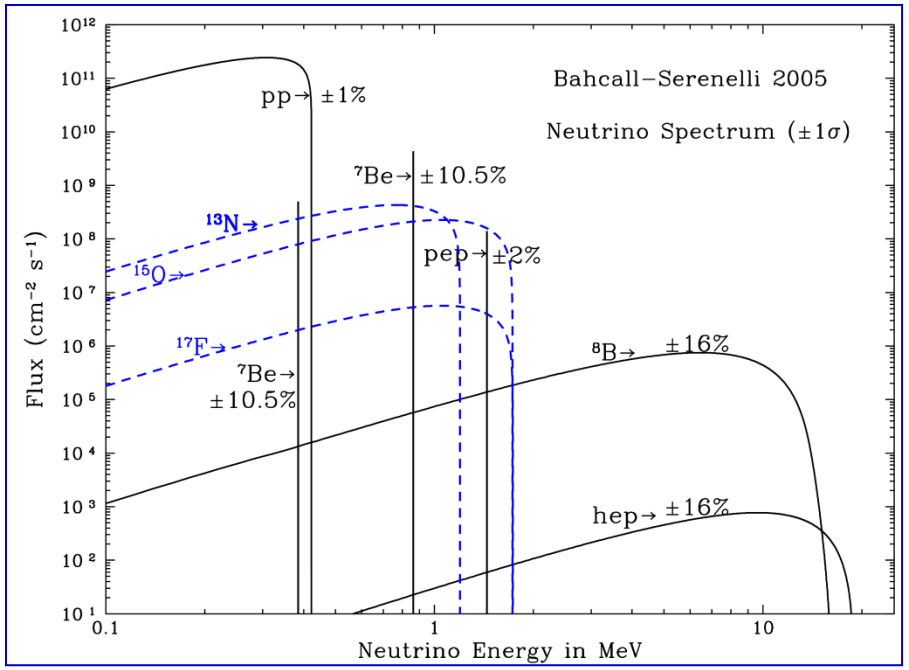
$$\text{CC}: \quad \nu_e + d \rightarrow p + p + e^-$$

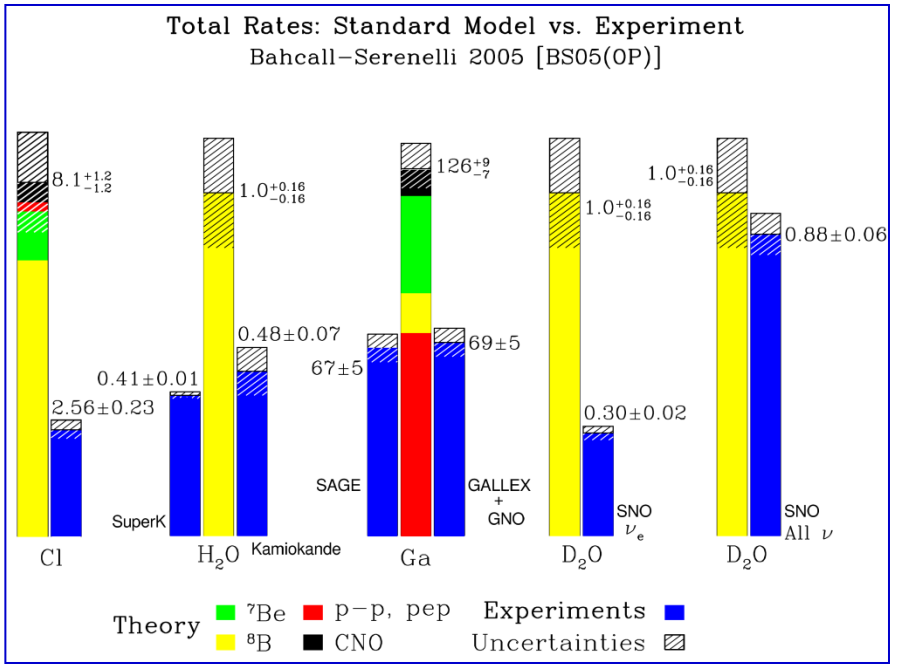
$$\text{NC}: \quad \nu_\alpha + d \rightarrow p + n + \nu_\alpha$$

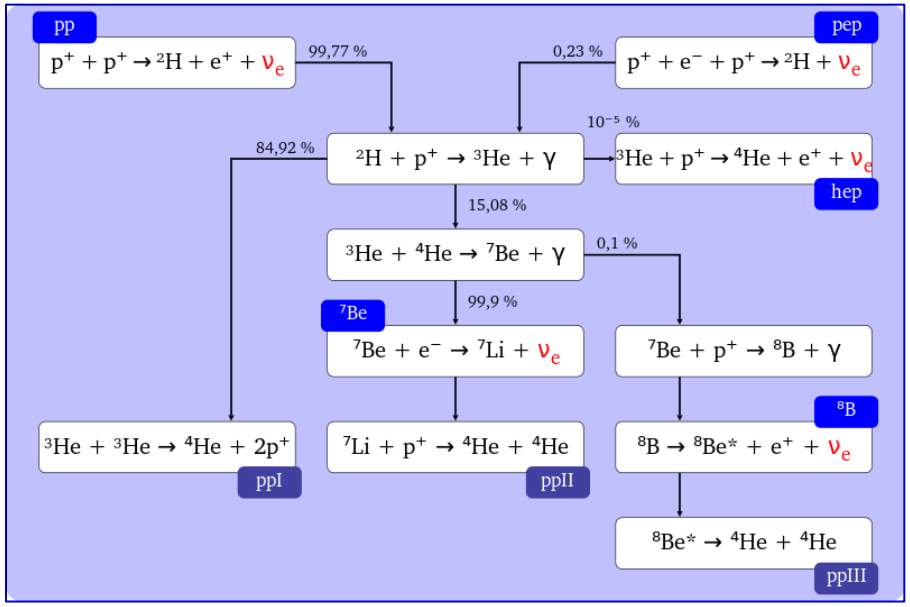
$$\text{ES}: \quad \nu_\alpha + e^- \rightarrow \nu_\alpha + e^-$$

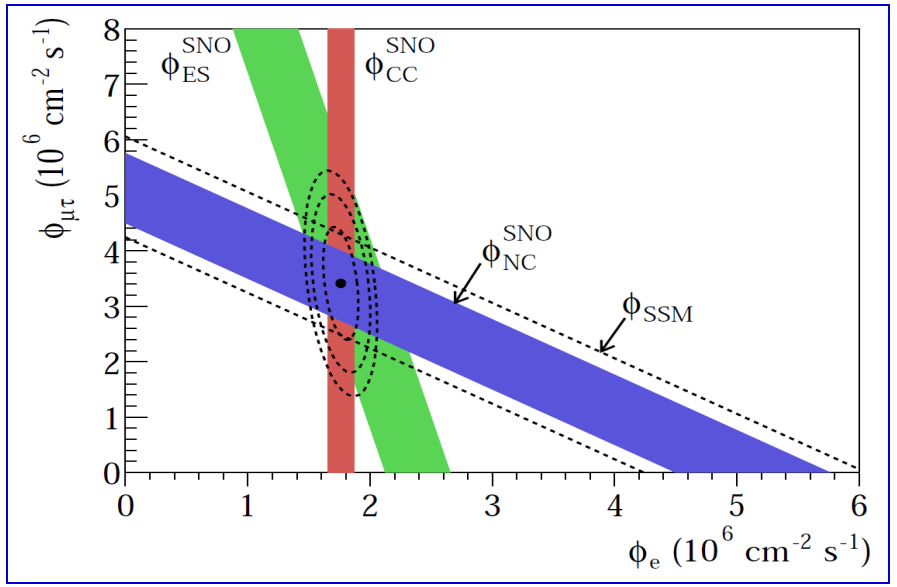


Solar Neutrino Experiments



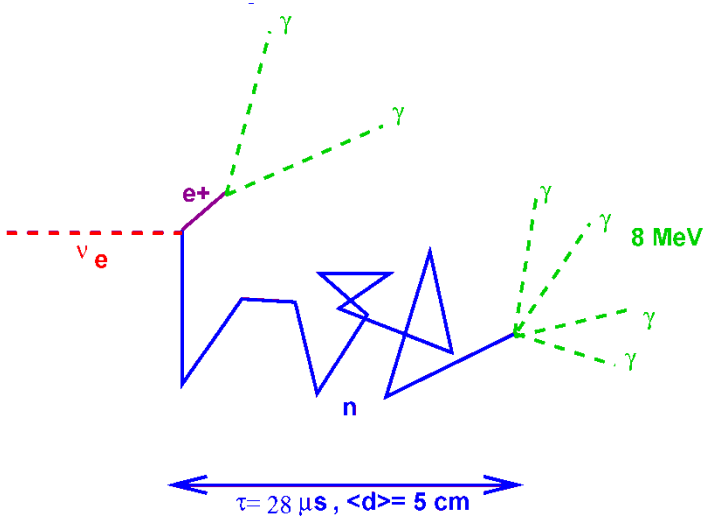


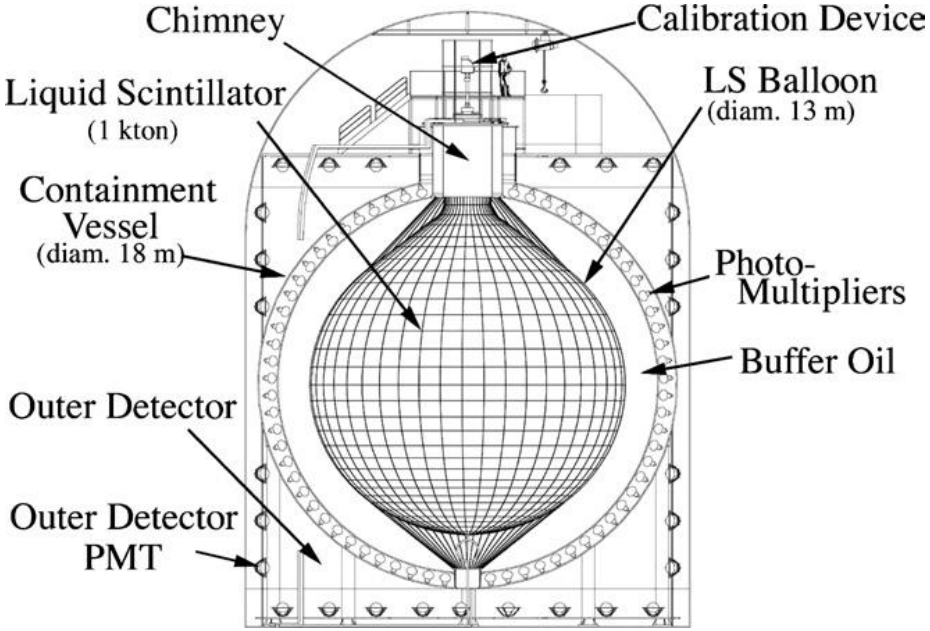


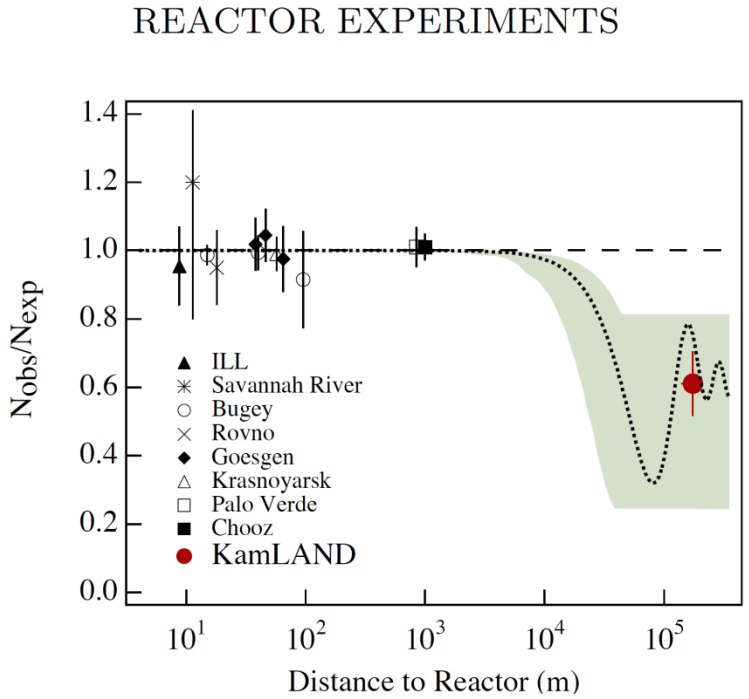


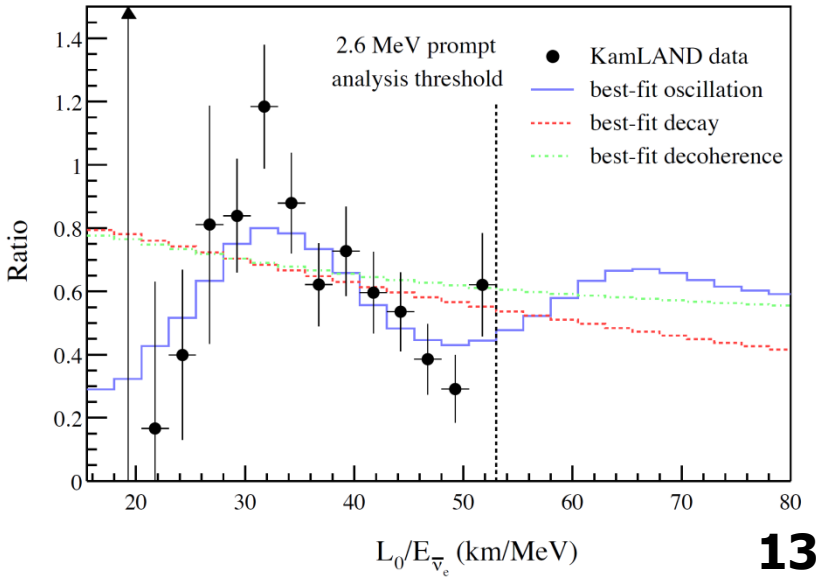
Reactor Neutrino Experiments

$$\overline{\nu}_e + p \rightarrow e^+ + n$$

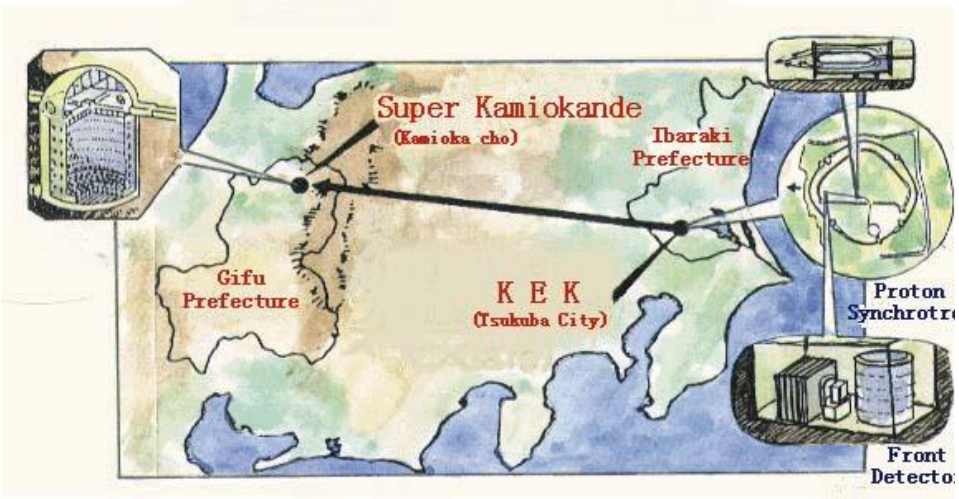


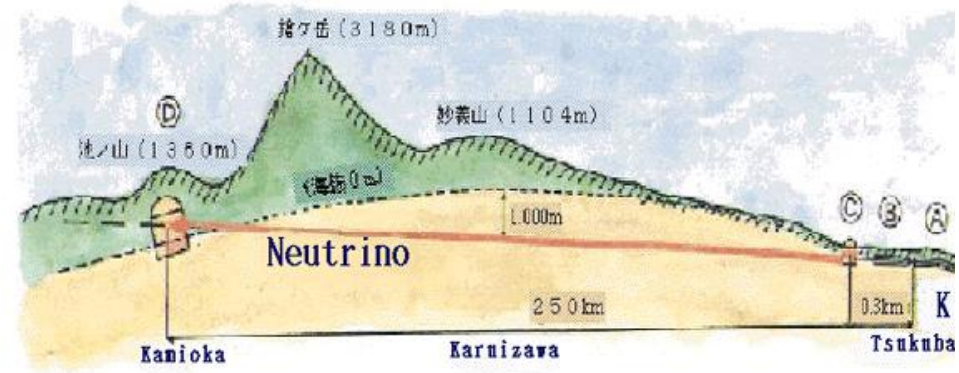






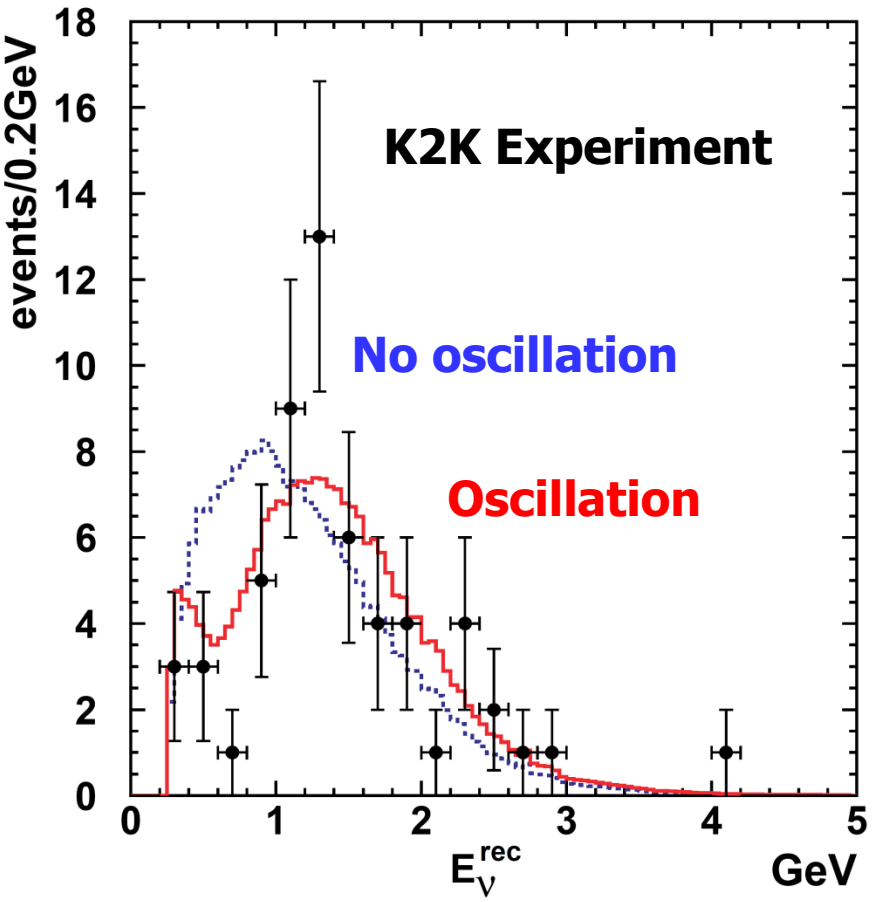
Accelerator Neutrino Experiments





From KEK to Super-Kamiokande

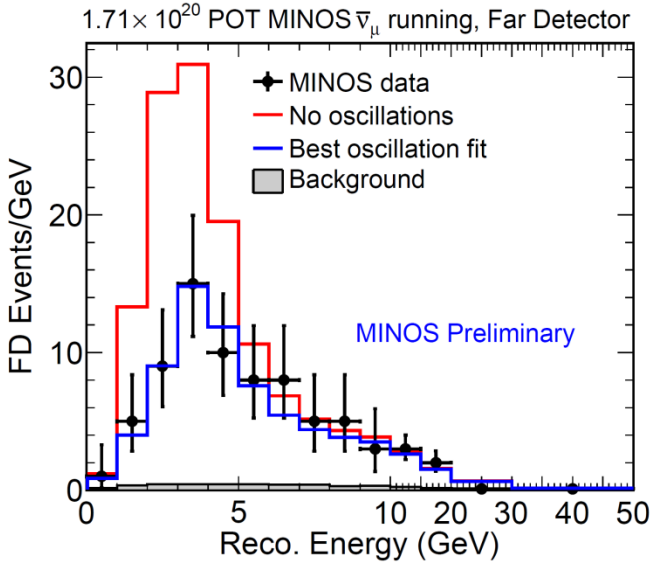
E ~ 1.3 GeV, L ~ 250 km

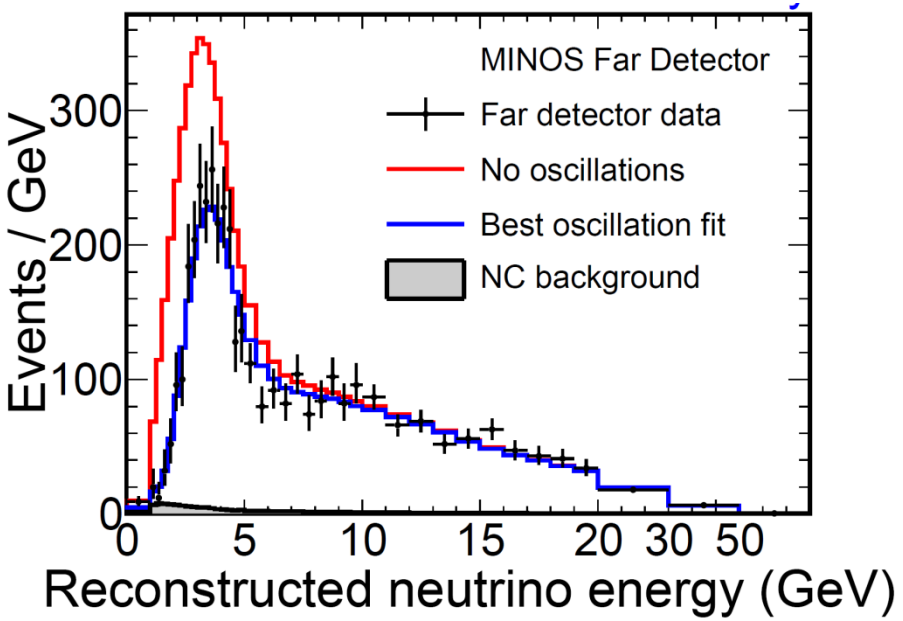


Accelerator Neutrino Experiments

MINOS Experiment

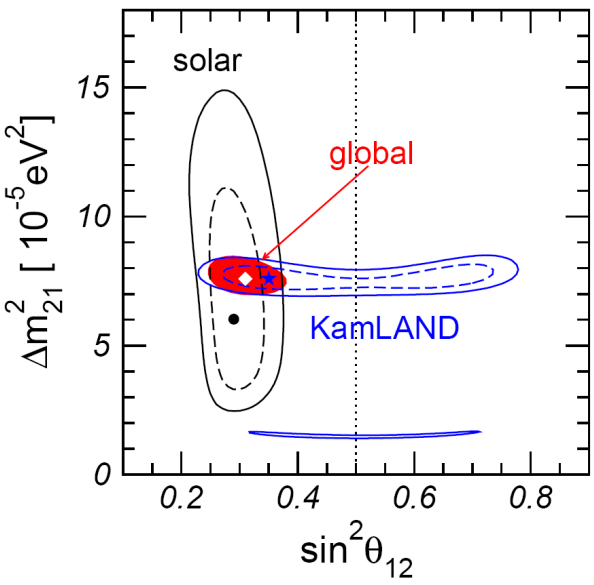
E ~ 3 GeV, L ~ 735 km

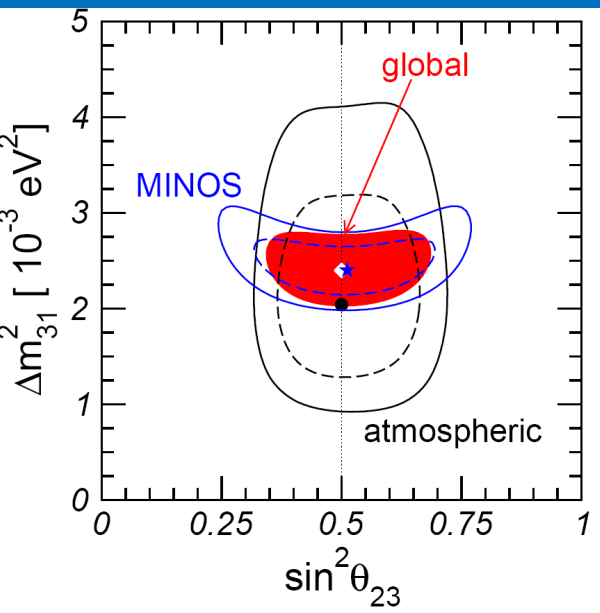


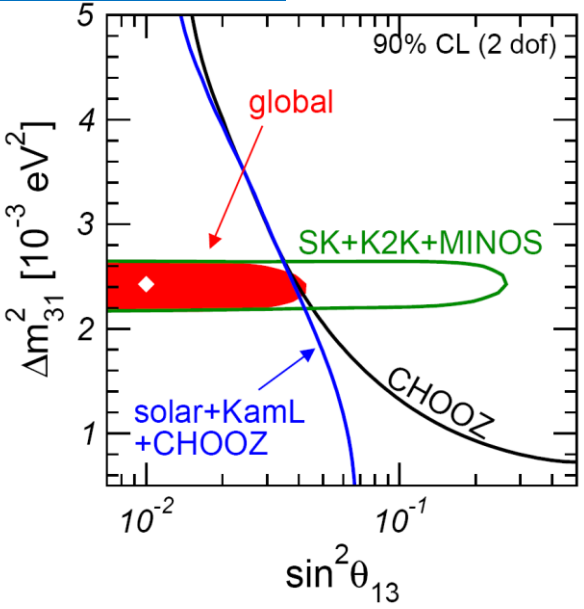


Story of the Smallest Mixing Angle

Global Analysis of Neutrino Oscillation Experiments







Schwetz et al., NJP 2008

No direct evidence for a nonzero θ_{13}

parameter	best fit	2σ	3σ
Δm^2_{21} [10^{-5}eV^2]	$7.65^{+0.23}_{-0.20}$	7.25–8.11	7.05–8.34
$\lvert\Delta m^2_{31}\rvert$ [10^{-3}eV^2]	$2.40^{+0.12}_{-0.11}$	2.18–2.64	2.07–2.75
$\sin^2\theta_{12}$	$0.304^{+0.022}_{-0.016}$	0.27–0.35	0.25–0.37
$\sin^2\theta_{23}$	$0.50^{+0.07}_{-0.06}$	0.39–0.63	0.36–0.67
$\sin^2\theta_{13}$	$0.01^{+0.016}_{-0.011}$	≤ 0.040	≤ 0.056

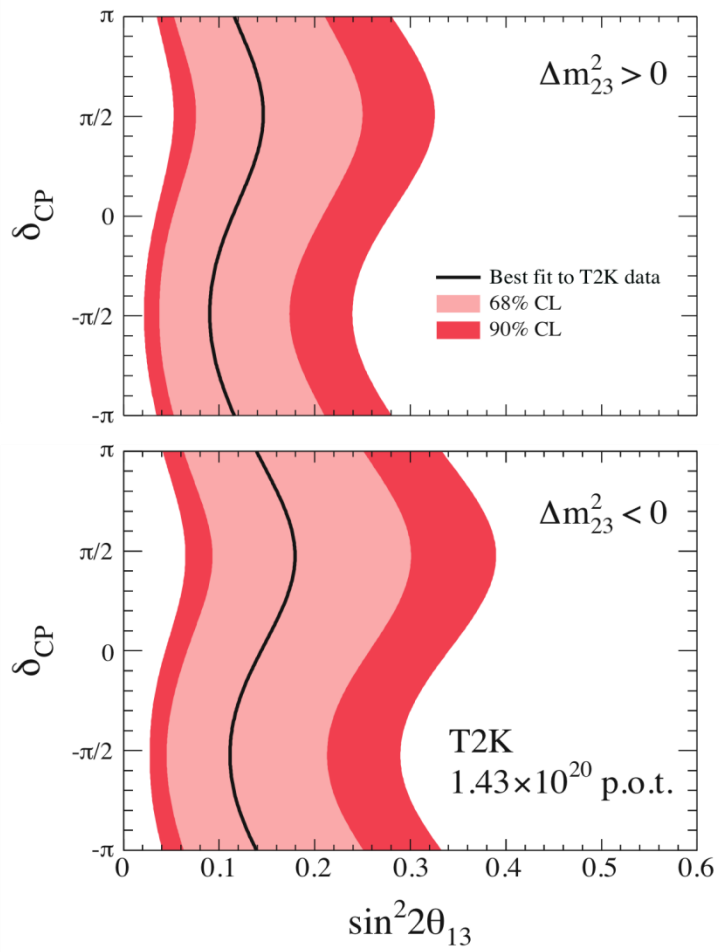
0.9σ

θ_{12}=33.5°, θ_{23}=45°, θ_{13}=5.7°

Story of the Smallest Mixing Angle

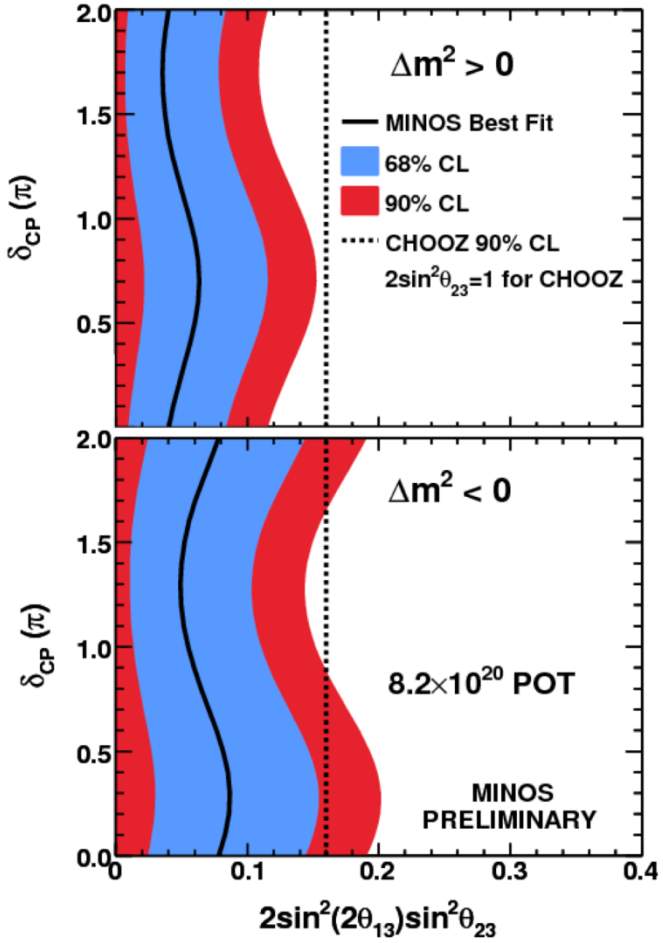
June 2011 breakthrough: Appearance results from T2K and MINOS

T2K



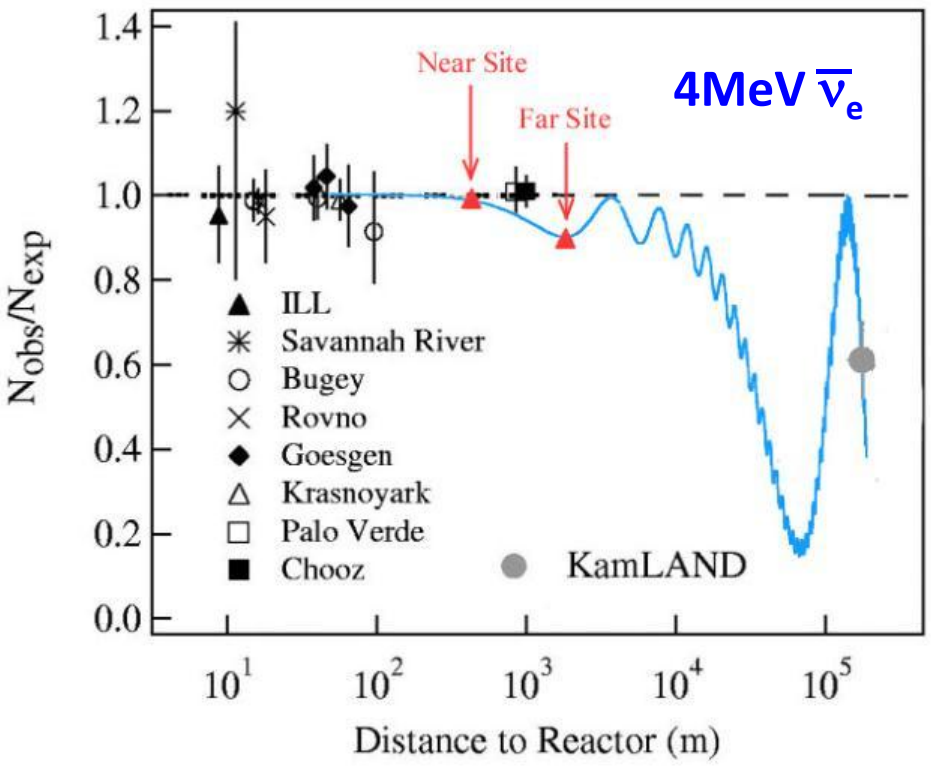
2.5σ

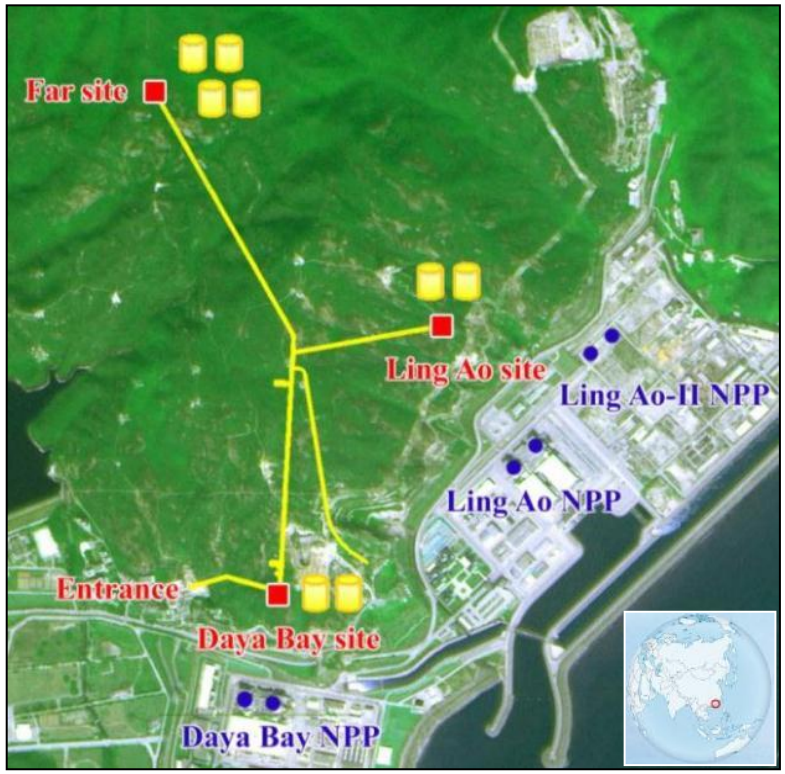
MINOS



1.7σ

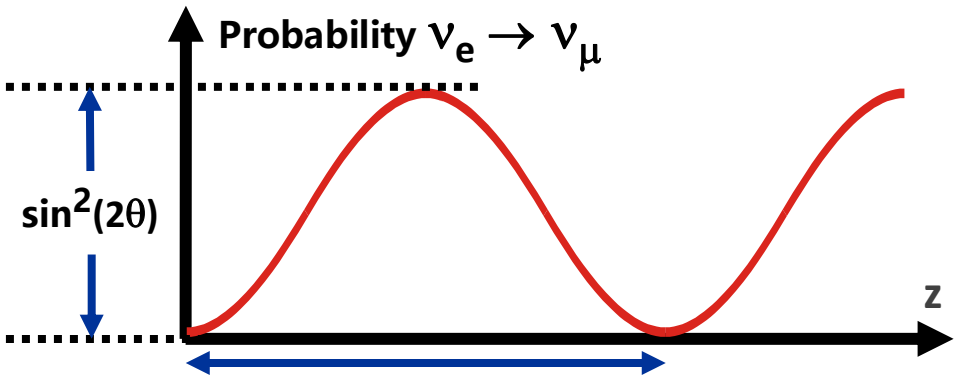
Story of the Smallest Mixing Angle





Dec. 2011	Double Chooz (far detector):	$\sin^2 \theta_{13} = 0.022 \pm 0.013$	1.7σ
Mar. 2012	Daya Bay (near + far detectors):	$\sin^2 \theta_{13} = 0.024 \pm 0.004$	5.2σ
Apr. 2012	RENO (near + far detectors):	$\sin^2 \theta_{13} = 0.029 \pm 0.006$	4.9σ

Neutrino Oscillations



Oscillation Length

$$\frac{4\pi E}{\Delta m^2} = 2.5\ \mathrm{m}\left(\frac{E}{\mathrm{MeV}}\right)\left(\frac{\mathrm{eV}^2}{\Delta m^2}\right)$$

Pontecorvo-Maki-Nakagawa-Sakata Matrix

$$\begin{pmatrix}\nu_e\\ \nu_\mu\\ \nu_\tau\end{pmatrix} = \begin{pmatrix}U_{e1} & U_{e2} & U_{e3}\\ U_{\mu 1} & U_{\mu 2} & U_{\mu 3}\\ U_{\tau 1} & U_{\tau 2} & U_{\tau 3}\end{pmatrix}\begin{pmatrix}\nu_1\\ \nu_2\\ \nu_3\end{pmatrix}$$

Standard Parametrization

3-flavor Mixing

$$\begin{pmatrix}1 & 0 & 0\\ 0 & \cos\theta_{23} & \sin\theta_{23}\\ 0 & -\sin\theta_{23} & \cos\theta_{23}\end{pmatrix}\begin{pmatrix}\cos\theta_{13} & 0 & \sin\theta_{13}e^{-i\delta}\\ 0 & 1 & 0\\ -\sin\theta_{13}e^{i\delta} & 0 & \cos\theta_{13}\end{pmatrix}\begin{pmatrix}\cos\theta_{12} & \sin\theta_{12} & 0\\ -\sin\theta_{12} & \cos\theta_{12} & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}e^{i\rho} & 0 & 0\\ 0 & e^{i\sigma} & 0\\ 0 & 0 & 1\end{pmatrix}$$

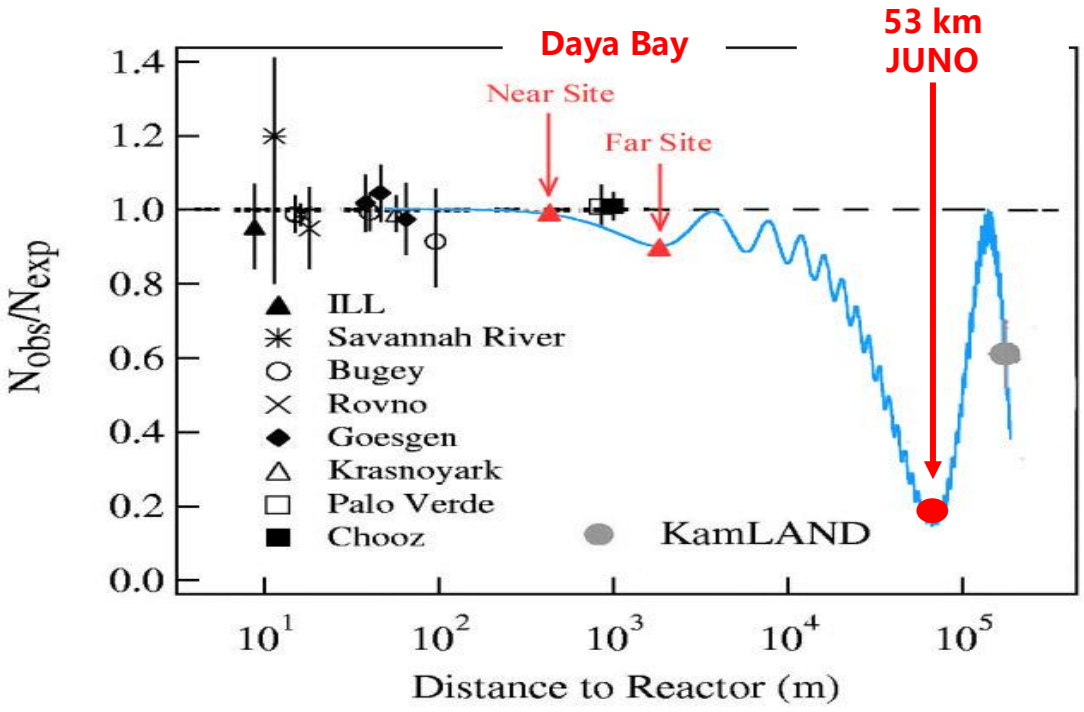
Atmospheric Accelerator
$\theta_{23} \sim 45°$

Reactor Accelerator
$\theta_{13} \sim 9°$

Solar Reactor
$\theta_{12} \sim 34°$

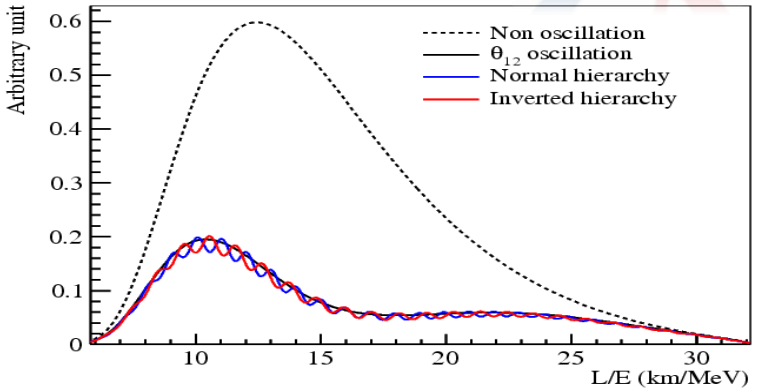
0νββ?

Neutrino Mass Ordering: JUNO



© JUNO, JPG, 2016

- 20 kton LS detector
- 3%/$\sqrt{E}$ resolution



Cores	YJ-C1	YJ-C2	YJ-C3	YJ-C4	YJ-C5	YJ-C6
Power (GW)	2.9	2.9	2.9	2.9	2.9	2.9
Baseline(km)	52.75	52.84	52.42	52.51	52.12	52.21
Cores	TS-C1	TS-C2	TS-C3	TS-C4	DYB	HZ
Power (GW)	4.6	4.6	4.6	4.6	17.4	17.4
Baseline(km)	52.76	52.63	52.32	52.20	215	265

Oscillation Probability:

$$P_{ee} = 1 - \sin^2 2\theta_{12}\cos^4\theta_{13}\sin^2\Delta_{21} - \frac{1}{2}\sin^2 2\theta_{13}[1 - \cos\Delta_*\cos\Delta_{21} + \cos 2\theta_{12}\sin\Delta_*\sin\Delta_{21}]$$

$$\Delta_* = \Delta_{31} + \Delta_{32} \qquad \Delta_{ji} = \frac{\left(m_j^2 - m_i^2\right)L}{4E}$$

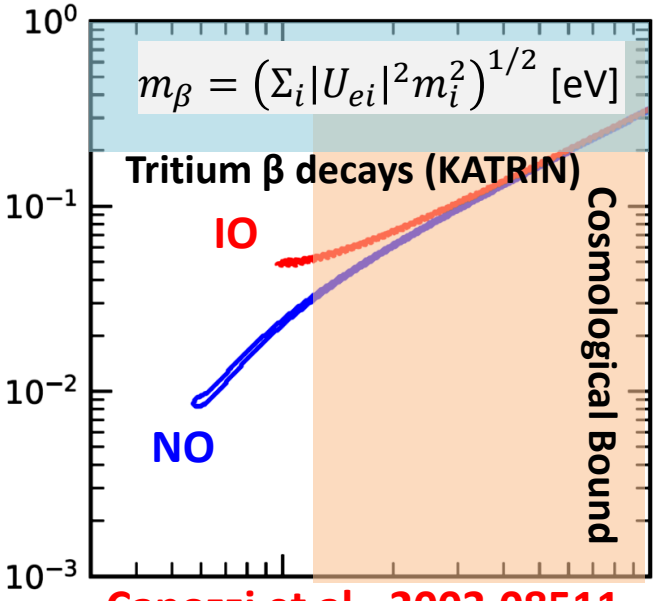
$$m_3 > m_2 > m_1$$
$$m_2 > m_1 > m_3$$

JUNO: Jiangmen, Guangdong

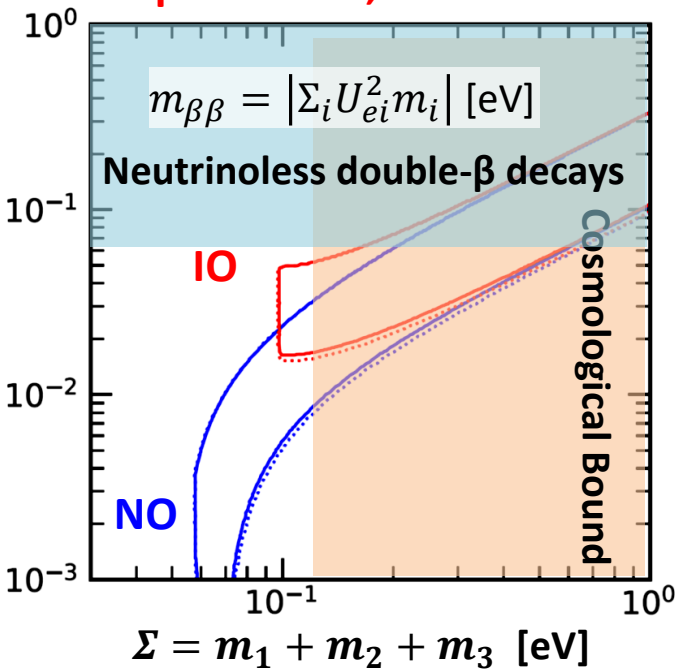
3σ ~4σ in 6 years

Li *et al.*, 1303.6733

Current Status of Absolute Neutrino Masses



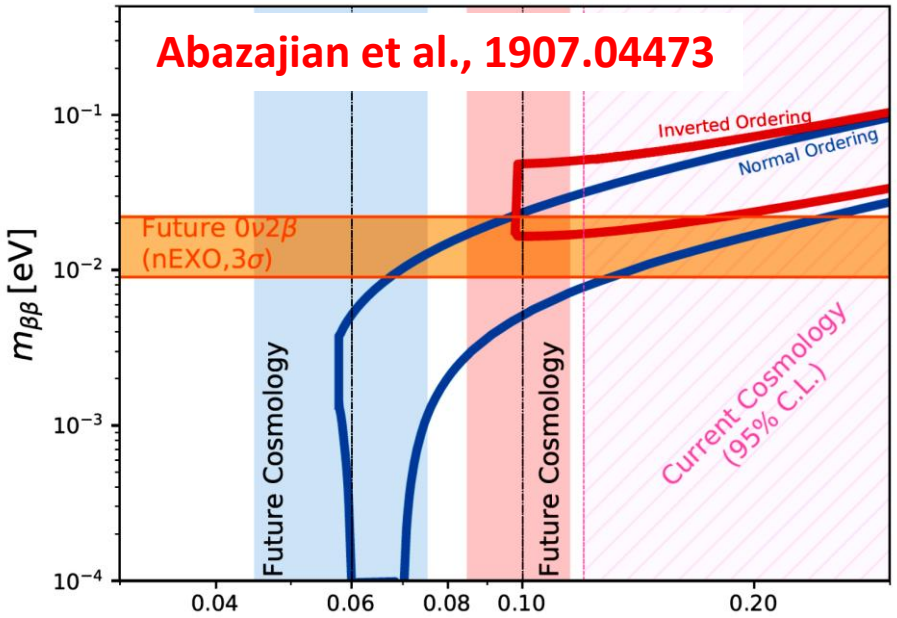
Capozzi et al., 2003.08511



$m_1 < m_2 < m_3$ (NO) or $m_3 < m_1 < m_2$ (IO)

Constraints on absolute neutrino masses

- **Tritium β decays (95% C.L.)**
  $m_\beta < 0.45$ eV (KATRIN 2025)
- **Neutrinoless double-β decays (90% C.L.)**
  $m_{\beta\beta} < (0.028 \sim 0.122)$ eV (KamLAND-Zen)
  $(0.078 \sim 0.239)$ eV (EXO-200)
  $(0.079 \sim 0.180)$ eV (GERDA-II)
  $(0.075 \sim 0.150)$ eV (CUORE)
- **Cosmological observations (95% probability)**
  $\Sigma < 0.12$ eV (Planck)

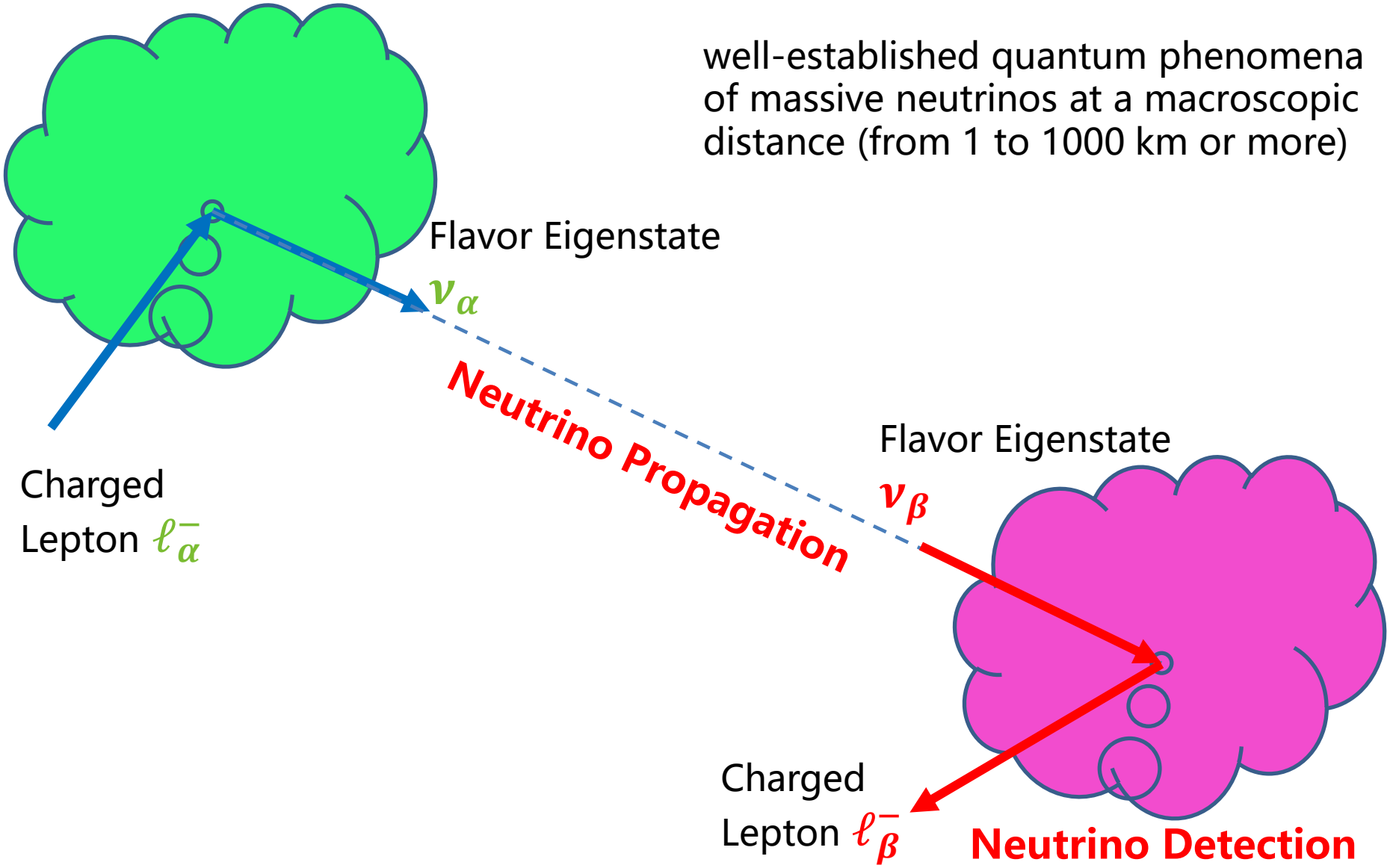


Part B: Neutrino Oscillation in Vacuum

Neutrino Production

Neutrino (flavor) oscillations:

well-established quantum phenomena of massive neutrinos at a macroscopic distance (from 1 to 1000 km or more)



History of Neutrino Oscillations

1930 neutrino hypothesis by Pauli

1934 an effective theory for weak interactions by Fermi

1956 electron antineutrinos $\bar{\nu}_e$ discovered by Cowan and Reines

1957 neutrino-antineutrino transitions proposed by Pontecorvo

1962 The 2nd family of neutrinos $\nu_\mu/\bar{\nu}_\mu$ by Lederman, Schwarz & Steinberger

1962 neutrino flavor transitions proposed by Maki, Nakagawa & Sakata

1968 solar neutrinos ν_e detected by Davis & solar neutrino problem

1998 deficit of atmospheric neutrinos $\nu_\mu/\bar{\nu}_\mu$ in Super-Kamiokande (Kajita)

2000 The 3rd family of neutrinos $\nu_\tau/\bar{\nu}_\tau$ discovered in DONUT

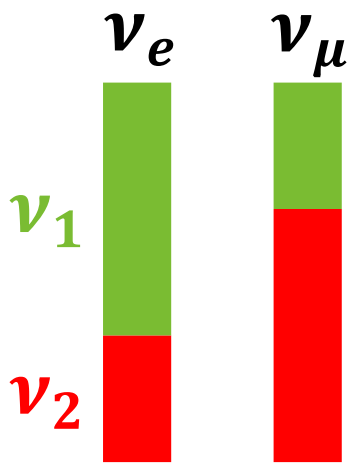
2001 solar ν_e and ν_μ/ν_τ neutrinos found in SNO (McDonald)

2002 KamLAND selects the LMA-MSW solution to solar neutrino problem

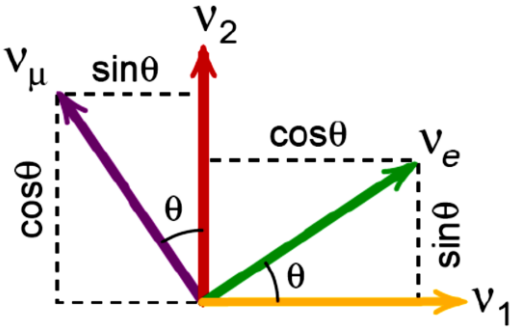
2004 K2K & MINOS confirms the disappearance of atmospheric neutrinos

2012 reactor antineutrino disappearance discovered by Daya Bay

Two-flavor Neutrino Oscillations



Neutrino **flavor eigenstates** are defined with respect to corresponding charged leptons, and they may not coincide with **mass eigenstates**



$$\nu_e = +\cos\theta\,\nu_1 + \sin\theta\,\nu_2 \qquad \nu_1 = +\cos\theta\,\nu_e - \sin\theta\,\nu_\mu$$
$$\nu_\mu = -\sin\theta\,\nu_1 + \cos\theta\,\nu_2 \qquad \nu_2 = +\sin\theta\,\nu_e + \cos\theta\,\nu_\mu$$

The oscillation probability in the plane-wave approximation: simple & useful

$$|\nu_e(0)\rangle = |\nu_e\rangle = \cos\theta|\nu_1\rangle + \sin\theta\,|\nu_2\rangle$$

$$|\nu_e(t)\rangle = \cos\theta\, e^{-iE_1t}|\nu_1\rangle + \sin\theta\, e^{-iE_2t}\,|\nu_2\rangle$$

$$= e^{-iE_1t}(\cos\theta\,|\nu_1\rangle + \sin\theta\, e^{-i\Delta Et}|\nu_2\rangle)$$

$$\Delta E = E_2 - E_1 = \sqrt{p^2+m_2^2} - \sqrt{p^2+m_1^2}$$
$$\approx \left(p + \frac{m_2^2}{2p^2}\right) - \left(p + \frac{m_1^2}{2p^2}\right) \equiv \frac{\Delta m^2}{2E}$$

$$\Delta m^2 \equiv m_2^2 - m_1^2 \qquad E \simeq p \gg m_1, m_2$$

The transitional amplitude

$$A = \langle\nu_\mu|\nu_e(t)\rangle = e^{-iE_1t}\sin\theta\cos\theta\left(e^{-i\Delta Et} - 1\right)$$

$$|\nu_\mu\rangle = -\sin\theta|\nu_1\rangle + \cos\theta\,|\nu_2\rangle$$

Two-flavor Neutrino Oscillations

The oscillation probability for the appearance channel

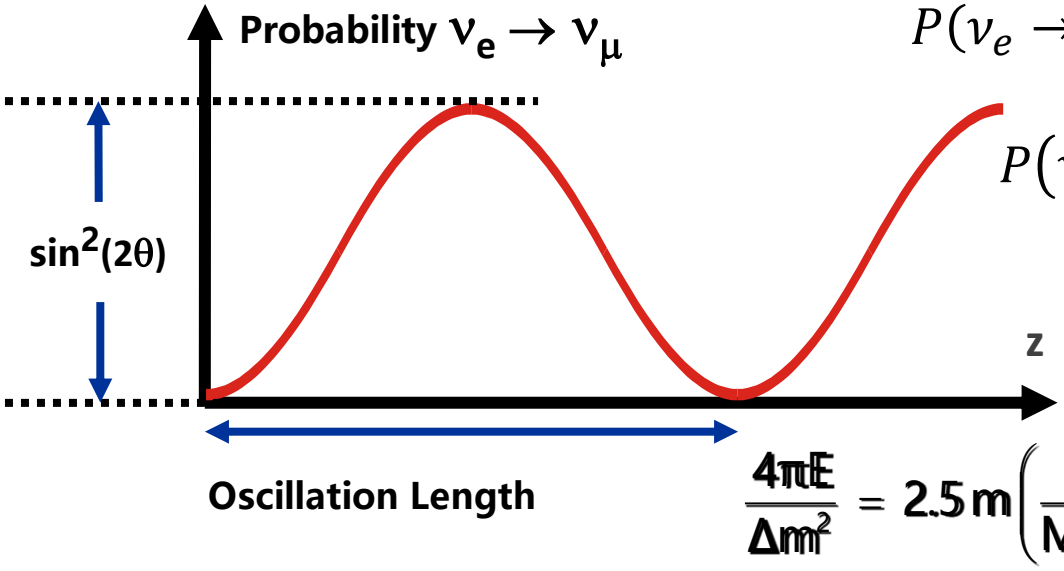
$$P(\nu_e \to \nu_\mu) \equiv \left|\langle \nu_\mu | \nu_e(t) \rangle\right|^2 = \left|\sin\theta\cos\theta\left(e^{-i\Delta E t} - 1\right)\right|^2$$

$$= 2\sin^2\theta\cos^2\theta\left(1 - \cos\frac{\Delta m^2 t}{2E}\right)$$

$$= \sin^2 2\theta \sin^2\frac{\Delta m^2 L}{4E}$$

$L \approx t$ **for neutrinos in the relativistic limit**

The survival probability for the disappearance channel



$$P(\nu_e \to \nu_e) \equiv 1 - \sin^2 2\theta \sin^2\frac{1.27\Delta m^2 L}{E}$$

$$P(\nu_e \to \nu_\mu) \equiv \sin^2 2\theta \sin^2\frac{1.27\Delta m^2 L}{E}$$

$$\frac{4\pi E}{\Delta m^2} = 2.5\,\mathrm{m}\left(\frac{E}{\mathrm{MeV}}\right)\left(\frac{\mathrm{eV}^2}{\Delta m^2}\right)$$

Quantities	Units
Δm^2	eV^2
E	GeV (MeV)
L	km (m)

Exercise: Baseline for Daya Bay

Natural units: $\hbar = c = 1$

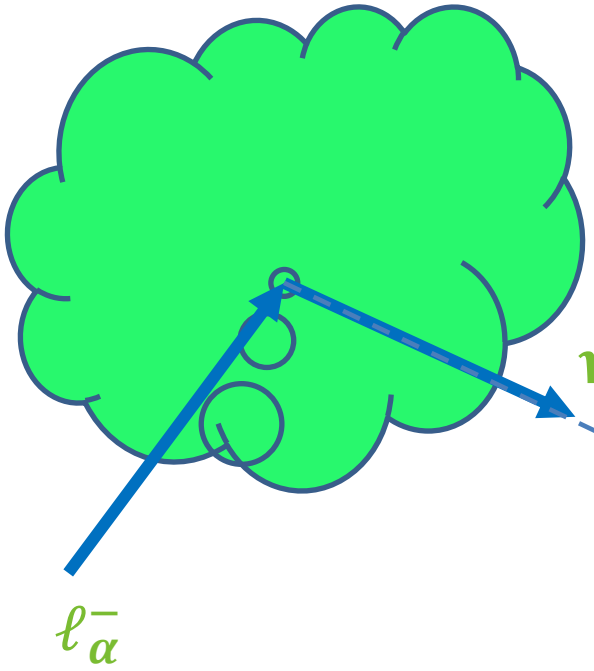
	Natural units	Realistic units
Phase factors	$\exp\left(-iE_{1,2}t\right)$	$\exp\left(-i\frac{E_{1,2}}{\hbar}t\right)$
Energies and momentum	$E_{1,2} = \sqrt{p^2 + m_{1,2}^2}$	$E_{1,2} = \sqrt{p^2c^2 + m_{1,2}^2c^4}$
Energy difference	$\Delta E = \frac{\Delta m^2}{2E}$	$\Delta E = \frac{\Delta m^2 c^3}{2p} = \frac{\Delta m^2 c^4}{2E}$
Time and distance	$t = L$	$t = \frac{L}{c}$
Oscillation argument	$\frac{1}{2}\Delta E t = \frac{\Delta m^2 L}{4E}$	$\frac{1}{2}\frac{\Delta E}{\hbar}t = \frac{c^3}{\hbar}\cdot\frac{\Delta m^2 L}{4E}$

(1) Derive the factor 1.27 in the formula of oscillation probabilities

(2) Daya Bay experiment is designed to observe the disappearance of reactor neutrinos, so it is better to put the far detector at the oscillation maximum for $\Delta m^2 = 2.5 \times 10^{-3}\ \mathrm{eV}^2$. Take $E = 4\ \mathrm{MeV}$ & estimate the optimal baseline length for Daya Bay.

Three-flavor Neutrino Oscillations

Neutrino Production



Charged-current Weak Interaction

$$\mathcal{L}_{\mathrm{cc}} = \frac{g}{\sqrt{2}} \overline{(e \quad \mu \quad \tau)_{\mathrm{L}}} \gamma^\mu V \begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{pmatrix}_{\mathrm{L}} W_\mu^- + \mathrm{h.c.}$$

Flavor Mixing Matrix or the PMNS Matrix

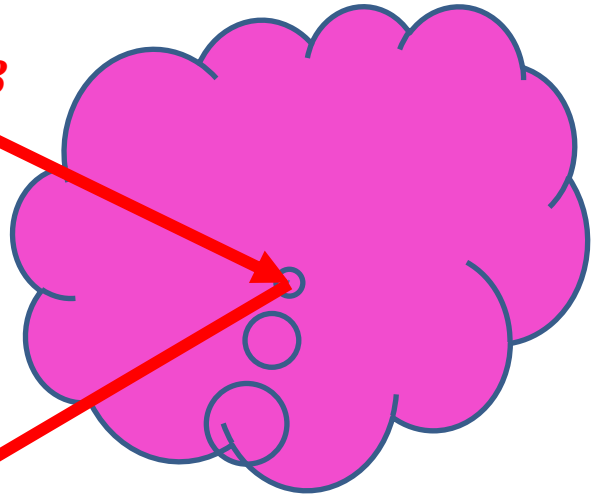
$$|\nu_\alpha(0)\rangle = |\nu_\alpha\rangle = \sum_i V_{\alpha i}^* |\nu_i\rangle$$

Neutrino Propagation

$$|\nu_\alpha(t)\rangle = \sum_i V_{\alpha i}^* e^{-iE_i t} |\nu_i\rangle \quad |\nu_\beta\rangle = \sum_j V_{\beta j}^* |\nu_j\rangle$$

$$A(\nu_\alpha \to \nu_\beta) = \langle \nu_\beta | \nu_\alpha(t) \rangle = \sum_i V_{\alpha i}^* V_{\beta i} e^{-iE_i t}$$

$$P(\nu_\alpha \to \nu_\beta) = |\langle \nu_\beta | \nu_\alpha(t) \rangle|^2 = \left| \sum_i V_{\alpha i}^* V_{\beta i} e^{-iE_i t} \right|^2$$



Neutrino Detection

Three-flavor Neutrino Oscillations

Derivation of the general formula of oscillation probabilities

Fritzsch & Minkowski, 76

$$P(\nu_\alpha \to \nu_\beta) = \left|\langle \nu_\beta | \nu_\alpha(t)\rangle\right|^2 = \left|\sum_i V_{\alpha i}^* V_{\beta i} e^{-iE_i t}\right|^2$$

$$= \sum_{i=1}^{3} \left|V_{\alpha i}^* V_{\beta i}\right|^2 + 2\sum_{i<j}^{3} \mathrm{Re}\left[V_{\alpha i}^* V_{\beta i} V_{\alpha j} V_{\beta j}^* e^{i(E_j - E_i)t}\right]$$

$$= \sum_{i=1}^{3} \left|V_{\alpha i}^* V_{\beta i}\right|^2 + 2\sum_{i<j}^{3} \mathrm{Re}\left[V_{\alpha i}^* V_{\beta i} V_{\alpha j} V_{\beta j}^*\right] \cos\frac{\Delta m_{ji}^2 L}{2E}$$

$$- 2\sum_{i<j}^{3} \mathrm{Im}\left[V_{\alpha i}^* V_{\beta i} V_{\alpha j} V_{\beta j}^*\right] \sin\frac{\Delta m_{ji}^2 L}{2E}$$

$$= \sum_{i=1}^{3} \left|V_{\alpha i}^* V_{\beta i}\right|^2 + 2\sum_{i<j}^{3} \mathrm{Re}\left[V_{\alpha i}^* V_{\beta i} V_{\alpha j} V_{\beta j}^*\right] \to \left|\sum_{i=1}^{3} V_{\alpha i}^* V_{\beta i}\right|^2 = \delta_{\alpha\beta}$$

$$- 4\sum_{i<j}^{3} \mathrm{Re}\left[V_{\alpha i}^* V_{\beta i} V_{\alpha j} V_{\beta j}^*\right] \sin^2\frac{\Delta m_{ji}^2 L}{4E} - 2\sum_{i<j}^{3} \mathrm{Im}\left[V_{\alpha i}^* V_{\beta i} V_{\alpha j} V_{\beta j}^*\right] \sin\frac{\Delta m_{ji}^2 L}{2E}$$

$$\Delta m_{ji}^2 \equiv m_j^2 - m_i^2$$

$$E \simeq p \gg m_i, m_j$$

Three-flavor Neutrino Oscillations

The general formula of oscillation probabilities

$$P(\nu_\alpha \to \nu_\beta) = \delta_{\alpha\beta} - 4\sum_{i<j} \mathrm{Re}\left[V_{\alpha i}V_{\beta j}V_{\alpha j}^{*}V_{\beta i}^{*}\right]\sin^2\frac{\Delta m_{ji}^2 L}{4E}$$

$$+ 2\sum_{i<j} \mathrm{Im}\left[V_{\alpha i}V_{\beta j}V_{\alpha j}^{*}V_{\beta i}^{*}\right]\sin\frac{\Delta m_{ji}^2 L}{2E}$$

$$= \delta_{\alpha\beta} - 4\sum_{i<j} \mathrm{Re}\left[V_{\alpha i}V_{\beta j}V_{\alpha j}^{*}V_{\beta i}^{*}\right]\sin^2\frac{\Delta m_{ji}^2 L}{4E}$$

Jarlskog Invariant

$$\mathrm{Im}\left[V_{\alpha i}V_{\beta j}V_{\alpha j}^{*}V_{\beta i}^{*}\right] \equiv \mathcal{J}\sum_{\gamma,k}\varepsilon_{\alpha\beta\gamma}\,\varepsilon_{ijk}$$

$$+ 8\mathcal{J}\sum_{\gamma}\varepsilon_{\alpha\beta\gamma}\sin\frac{\Delta m_{21}^2 L}{4E}\sin\frac{\Delta m_{32}^2 L}{4E}\sin\frac{\Delta m_{31}^2 L}{4E}$$

CP Violation

Exercise: try to demonstrate that the imaginary part can be expressed in terms of the Jarlskog invariant and derive the CP-violating term

CP Violation and Jarlskog Invariant

$$+2\sum_{i<j}^{3}\mathrm{Im}\left[V_{\alpha i}V_{\beta j}V_{\alpha j}^{*}V_{\beta i}^{*}\right]\sin\frac{\Delta m_{ji}^{2}L}{2E}$$

$$=+2\mathcal{J}\sum_{\gamma}\varepsilon_{\alpha\beta\gamma}\left(\sin\frac{\Delta m_{21}^{2}L}{2E}-\sin\frac{\Delta m_{31}^{2}L}{2E}+\sin\frac{\Delta m_{32}^{2}L}{2E}\right)$$

$$=-2\mathcal{J}\sum_{\gamma}\varepsilon_{\alpha\beta\gamma}\left(\sin\frac{\Delta m_{12}^{2}L}{2E}+\sin\frac{\Delta m_{31}^{2}L}{2E}+\sin\frac{\Delta m_{23}^{2}L}{2E}\right)$$

$$=+8\mathcal{J}\sum_{\gamma}\varepsilon_{\alpha\beta\gamma}\sin\frac{\Delta m_{12}^{2}L}{4E}\sin\frac{\Delta m_{31}^{2}L}{4E}\sin\frac{\Delta m_{23}^{2}L}{4E}$$

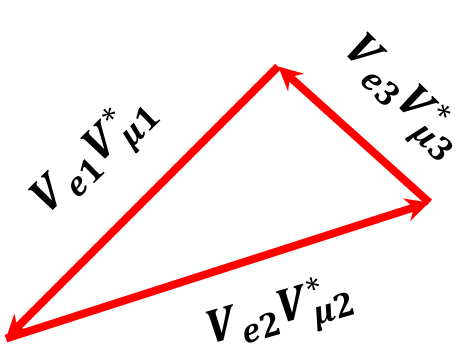
Jarlskog Invariant

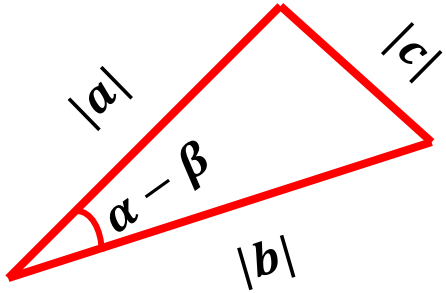
$$\begin{pmatrix} V_{e1} & V_{e2} & V_{e3} \\ V_{\mu 1} & V_{\mu 2} & V_{\mu 3} \\ V_{\tau 1} & V_{\tau 2} & V_{\tau 3} \end{pmatrix}^{*}$$

$$\mathcal{J}=\mathrm{Im}\left[V_{e1}V_{\mu 2}V_{e2}^{*}V_{\mu 1}^{*}\right]$$

Exercise: $\mathcal{J}$ is unique

Unitarity triangle and Jarlskog invariant





$$a=\left|V_{e1}V_{\mu 1}^{*}\right|e^{i\alpha}$$

$$b=\left|V_{e2}V_{\mu 2}^{*}\right|e^{i\beta}$$

$$c=\left|V_{e3}V_{\mu 3}^{*}\right|e^{i\gamma}$$

$$V_{e1}V_{\mu 1}^{*}+V_{e2}V_{\mu 2}^{*}+V_{e3}V_{\mu 3}^{*}=0$$

Exercise: prove $|\mathcal{J}|=2\times$ area of triangle

Oscillations of Antineutrinos

$$\mathcal{L}_{\rm cc} = \frac{g}{\sqrt{2}} \overline{(e \quad \mu \quad \tau)}_{\rm L} \gamma^{\mu} \boldsymbol{V} \begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{pmatrix}_{\rm L} W_{\mu}^{-} + \text{h. c.}$$

$$|\bar{\nu}_{\alpha}\rangle = \sum_i V_{\alpha i} |\bar{\nu}_i\rangle$$

$$\boldsymbol{V} \Rightarrow \boldsymbol{V}^*$$

The oscillation probabilities for antineutrinos

$$P(\bar{\nu}_{\alpha} \to \bar{\nu}_{\beta}) = \delta_{\alpha\beta} - 4 \sum_{i<j}^{3} \text{Re}\left[V_{\alpha i} V_{\beta j} V_{\alpha j}^* V_{\beta i}^*\right] \sin^2 \frac{\Delta m_{ji}^2 L}{4E}$$

$$\mathcal{J} \Rightarrow -\mathcal{J}$$

$$- 8\mathcal{J} \sum_{\gamma} \varepsilon_{\alpha\beta\gamma} \sin \frac{\Delta m_{21}^2 L}{4E} \sin \frac{\Delta m_{32}^2 L}{4E} \sin \frac{\Delta m_{31}^2 L}{4E}$$

CP violation for neutrino oscillations in vacuum

$$A_{\rm CP} = P(\nu_{\alpha} \to \nu_{\beta}) - P(\bar{\nu}_{\alpha} \to \bar{\nu}_{\beta}) = 16\mathcal{J} \sum_{\gamma} \varepsilon_{\alpha\beta\gamma} \sin \frac{\Delta m_{21}^2 L}{4E} \sin \frac{\Delta m_{32}^2 L}{4E} \sin \frac{\Delta m_{31}^2 L}{4E}$$

No CP asymmetries in the disappearance channels (via CPT theorem)

Discrete Symmetries

Basic formula:

$$P(\nu_\alpha \to \nu_\beta) = \delta_{\alpha\beta} - 4\sum_{i<j}^{3} \mathrm{Re}\left[V_{\alpha i}V_{\beta j}V_{\alpha j}^{*}V_{\beta i}^{*}\right]\sin^2\frac{\Delta m_{ji}^2 L}{4E}$$

$$+8\mathcal{J}\sum_{\gamma}\varepsilon_{\alpha\beta\gamma}\sin\frac{\Delta m_{21}^2 L}{4E}\sin\frac{\Delta m_{32}^2 L}{4E}\sin\frac{\Delta m_{31}^2 L}{4E}$$

CP transformation:

$V \Rightarrow V^*$

$\mathcal{J} \Rightarrow -\mathcal{J}$

$$P(\bar{\nu}_\alpha \to \bar{\nu}_\beta) = \delta_{\alpha\beta} - 4\sum_{i<j}^{3} \mathrm{Re}\left[V_{\alpha i}V_{\beta j}V_{\alpha j}^{*}V_{\beta i}^{*}\right]\sin^2\frac{\Delta m_{ji}^2 L}{4E}$$

$$-8\mathcal{J}\sum_{\gamma}\varepsilon_{\alpha\beta\gamma}\sin\frac{\Delta m_{21}^2 L}{4E}\sin\frac{\Delta m_{32}^2 L}{4E}\sin\frac{\Delta m_{31}^2 L}{4E}$$

T transformation:

$\alpha \Leftrightarrow \beta$

$\mathcal{J} \Rightarrow -\mathcal{J}$

$$P(\nu_\beta \to \nu_\alpha) = \delta_{\alpha\beta} - 4\sum_{i<j}^{3} \mathrm{Re}\left[V_{\alpha i}V_{\beta j}V_{\alpha j}^{*}V_{\beta i}^{*}\right]\sin^2\frac{\Delta m_{ji}^2 L}{4E}$$

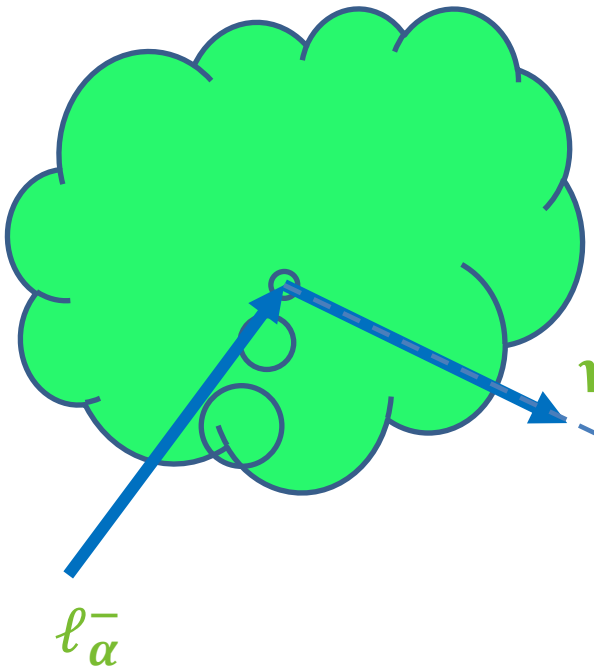
$$-8\mathcal{J}\sum_{\gamma}\varepsilon_{\alpha\beta\gamma}\sin\frac{\Delta m_{21}^2 L}{4E}\sin\frac{\Delta m_{32}^2 L}{4E}\sin\frac{\Delta m_{31}^2 L}{4E}$$

CPT theorem: $P(\nu_\alpha \to \nu_\beta) = P(\bar{\nu}_\beta \to \bar{\nu}_\alpha)$ **Invalid if matter effects exist**

Neutrino-antineutrino Oscillations

- Neutrino-antineutrino oscillations can take place only if neutrinos are Majorana particles, as the lepton number is violated in this case.

Neutrino Production



ν_α

ℓ_α^-

Quantization of neutrino fields: $\nu^C(x) = \nu(x)$

$$\nu(x) = \int \frac{d^3p}{(2\pi)^3 2E} \sum_{h=\pm} \left[a_h(p) u_h(p) e^{-ipx} + a_h^\dagger(p) v_h(p) e^{ipx} \right]$$

$$|\nu(p)\rangle = |\nu(p,-)\rangle$$
$$|\bar{\nu}(p)\rangle = |\nu(p,+)\rangle$$

creation operator for both ν and $\bar{\nu}$

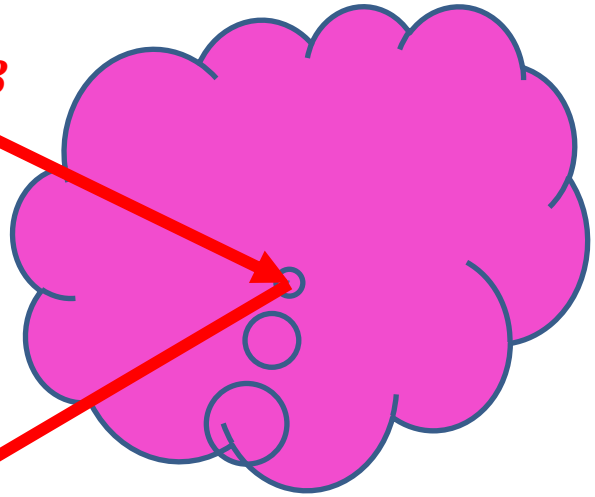
Neutrino Propagation

$\bar{\nu}_\beta$

$$A(\nu_\alpha \to \bar{\nu}_\beta) = \sum_i V_{\alpha i}^* V_{\beta i}^* \frac{m_i}{E_i} e^{-iE_i t}$$

$$A(\bar{\nu}_\alpha \to \nu_\beta) = \sum_i V_{\alpha i} V_{\beta i} \frac{m_i}{E_i} e^{+iE_i t}$$

ℓ_β^+



Highly suppressed by a factor of m_i/E

Neutrino Detection

Neutrino-antineutrino Oscillations

Oscillation probabilities

Pontecorvo, 57, 67; Bahcall & Primakoff, 78; Schechter & Valle, 81; …

$$P(\nu_\alpha \to \bar{\nu}_\beta) = \frac{1}{E^2}\left[|\langle m \rangle_{\alpha\beta}|^2 - 4\sum_{i<j} m_i m_j \mathrm{Re}\left[V_{\alpha i} V_{\beta i} V_{\alpha j}^* V_{\beta j}^*\right] \sin^2 \frac{\Delta m_{ji}^2 L}{4E} + 2\sum_{i<j} m_i m_j \mathrm{Im}\left[V_{\alpha i} V_{\beta i} V_{\alpha j}^* V_{\beta j}^*\right] \sin \frac{\Delta m_{ji}^2 L}{2E}\right]$$

$$P(\bar{\nu}_\alpha \to \nu_\beta) = \frac{1}{E^2}\left[|\langle m \rangle_{\alpha\beta}|^2 - 4\sum_{i<j} m_i m_j \mathrm{Re}\left[V_{\alpha i} V_{\beta i} V_{\alpha j}^* V_{\beta j}^*\right] \sin^2 \frac{\Delta m_{ji}^2 L}{4E} - 2\sum_{i<j} m_i m_j \mathrm{Im}\left[V_{\alpha i} V_{\beta i} V_{\alpha j}^* V_{\beta j}^*\right] \sin \frac{\Delta m_{ji}^2 L}{2E}\right]$$

$$10^{-12} \left(\frac{m_i}{\mathrm{eV}}\right)^2 \left(\frac{\mathrm{MeV}}{E}\right)^2$$

$$\langle m \rangle_{\alpha\beta} \equiv \sum_{i=1}^{3} V_{\alpha i} V_{\alpha i} m_i$$

Dependence on all three CP-violating phases

The oscillation probabilities are extremely small, but enhanced for $\bar{\nu}_e$ from ^{3}H decays.

Parametrization of PMNS Matrix

Dirac neutrinos

$c_{ij} \equiv \cos\theta_{ij}\,, \qquad s_{ij} \equiv \sin\theta_{ij}$

$$V = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix} \cdot \begin{pmatrix} c_{13} & 0 & s_{13}e^{-i\delta} \\ 0 & 1 & 0 \\ -s_{13}e^{i\delta} & 0 & c_{13} \end{pmatrix} \cdot \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Majorana neutrinos

$$V = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix} \cdot \begin{pmatrix} c_{13} & 0 & s_{13}e^{-i\delta} \\ 0 & 1 & 0 \\ -s_{13}e^{i\delta} & 0 & c_{13} \end{pmatrix} \cdot \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} e^{i\rho} & 0 & 0 \\ 0 & e^{i\sigma} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Exercise: prove that a general 2 x 2 unitary matrix contains four real parameters, and can be written as follows

$$V = \begin{pmatrix} e^{i\alpha} & 0 \\ 0 & e^{i\beta} \end{pmatrix} \cdot \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \cdot \begin{pmatrix} e^{i\gamma} & 0 \\ 0 & 1 \end{pmatrix}$$

then all the phases can be removed by redefining the charged-lepton and neutrino fields in the Dirac case. So, no CP violation if there are 2 generations of fermions. Consider also the Majorana case.

Parametrization of PMNS Matrix

Standard Parametrization

$$V = \begin{pmatrix} c_{13}c_{12} & c_{13}s_{12} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & +c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & c_{13}s_{23} \\ +s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{13}c_{23} \end{pmatrix} \begin{pmatrix} e^{i\rho} & 0 & 0 \\ 0 & e^{i\sigma} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Extraction of mixing angles and CP-violating phases from the PMNS matrix

$$\sin^2\theta_{13} = |V_{e3}|^2$$

$$\sin^2\theta_{12} = \frac{|V_{e2}|^2}{1-|V_{e3}|^2}$$

$$\sin^2\theta_{23} = \frac{|V_{\mu 3}|^2}{1-|V_{e3}|^2}$$

$$\begin{aligned} \mathcal{J} &= \mathrm{Im}\left[V_{e2}V_{\mu 3}V_{e3}^*V_{\mu 2}^*\right] \\ &= \mathrm{Im}\left[(c_{13}s_{12})\cdot(c_{13}s_{23})\cdot(s_{13}e^{i\delta})\cdot(+c_{12}c_{23})\right] \\ &= s_{12}c_{12}s_{23}c_{23}s_{13}c_{13}^2\sin\delta \end{aligned}$$

$$\sin\delta = \frac{\mathrm{Im}\left[V_{e2}V_{\mu 3}V_{e3}^*V_{\mu 2}^*\right]}{s_{12}c_{12}s_{23}c_{23}s_{13}c_{13}^2}$$

three mixing angles are restricted to the first quadrat

Exercise: prove that a general 3 x 3 unitary matrix contains nine real parameters, and there will be one non-factorizable phase.

Disappearance Channels

Most oscillation experiments are measuring the disappearance of neutrinos

$$P(\nu_\alpha \to \nu_\beta) = \delta_{\alpha\beta} - 4\sum_{i<j}^{3} \mathrm{Re}\left[V_{\alpha i}V_{\beta j}V_{\alpha j}^{*}V_{\beta i}^{*}\right]\sin^2\frac{\Delta m_{ji}^2 L}{4E} + 2\sum_{i<j}^{3} \mathrm{Im}\left[V_{\alpha i}V_{\beta j}V_{\alpha j}^{*}V_{\beta i}^{*}\right]\sin\frac{\Delta m_{ji}^2 L}{2E}$$

$$\mathrm{Re}\left[V_{\alpha i}V_{\alpha j}V_{\alpha j}^{*}V_{\alpha i}^{*}\right] = |V_{\alpha i}|^2|V_{\alpha j}|^2$$

$$\mathrm{Im}\left[V_{\alpha i}V_{\alpha j}V_{\alpha j}^{*}V_{\alpha i}^{*}\right] = 0$$

$$P(\nu_\alpha \to \nu_\alpha) = 1 - 4\sum_{i<j}^{3} |V_{\alpha i}|^2|V_{\alpha j}|^2 \sin^2\frac{\Delta m_{ji}^2 L}{4E}$$

$$= 1 - 4|V_{\alpha 1}|^2|V_{\alpha 2}|^2 \sin^2\frac{\Delta m_{21}^2 L}{4E} - 4|V_{\alpha 1}|^2|V_{\alpha 3}|^2 \sin^2\frac{\Delta m_{31}^2 L}{4E} - 4|V_{\alpha 2}|^2|V_{\alpha 3}|^2 \sin^2\frac{\Delta m_{32}^2 L}{4E}$$

Comments:

- **No CP asymmetry, so valid for both ν and $\bar{\nu}$ oscillations**
- **In a specific experiment with L and E fixed, so sensitive to a particular range of Δm_{ji}^2**
- **2-flavor oscillations if θ_{13} is small and $|\Delta m_{21}^2| \ll |\Delta m_{31}^2| \simeq |\Delta m_{32}^2|$**

One can determine the mixing angles and neutrino mass-squared differences

Appearance Channels

$$P(\nu_\mu \to \nu_e) = \sin^2\theta_{23}\sin^2 2\theta_{13}\sin^2\frac{\Delta m^2_{31}L}{4E}$$

$$+ \textbf{CPV} + \textbf{Matter Effects}$$

For accelerator neutrinos (T2K)

$E \simeq 0.6\ \text{GeV}$ $L \simeq 295\ \text{km}$

$\Downarrow$

$\frac{L}{E} \simeq 500\ \text{m/MeV}$

$\Downarrow$

$\Delta m^2 \simeq 2.5 \times 10^{-3}\ \text{eV}^2$

Comments:

- **Sensitive to θ_{13}, depending on other mixing parameters**

3-favor results without matter effects

$$P(\nu_\mu \to \nu_e) = (\sin^2\theta_{23}\sin^2 2\theta_{13} - 8\mathcal{J}\sin\frac{\Delta m^2_{21}L}{2E})\sin^2\frac{\Delta m^2_{31}L}{4E}$$

leading terms

$$+ c^2_{13}\left[\sin^2 2\theta_{12}(c^2_{23} - s^2_{23}s^2_{13}) + \frac{1}{4}\sin 4\theta_{12}\sin 2\theta_{23}\cos\delta\right]\sin^2\frac{\Delta m^2_{21}L}{4E}$$

sub-leading terms

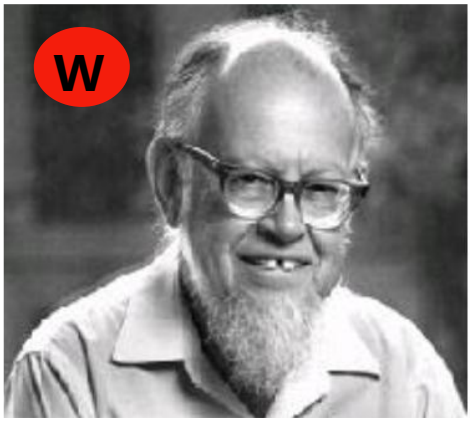
Main features of LBL accelerator neutrino experiments: sensitive to δ, octant of θ_{23}, and also sign of Δm^2_{31} when including matter effects

PHYSICAL REVIEW D VOLUME 17, NUMBER 9 1 MAY 1978

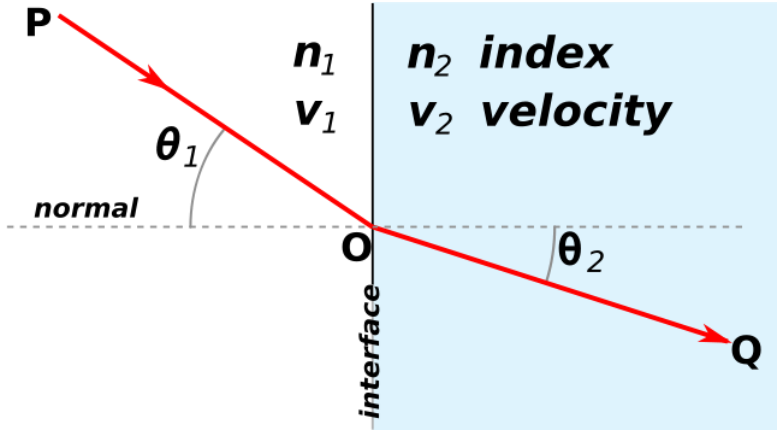
Neutrino oscillations in matter

L. Wolfenstein
Carnegie-Mellon University, Pittsburgh, Pennsylvania 15213
(Received 6 October 1977; revised manuscript received 5 December 1977)

The effect of coherent forward scattering must be taken into account when considering the oscillations of neutrinos traveling through matter. In particular, for the case of massless neutrinos for which vacuum oscillations cannot occur, oscillations can occur in matter if the neutral current has an off-diagonal piece connecting different neutrino types. Applications discussed are solar neutrinos and a proposed experiment involving transmission of neutrinos through 1000 km of rock.



Lincoln Wolfenstein (1923-2015)



Refraction of light in media

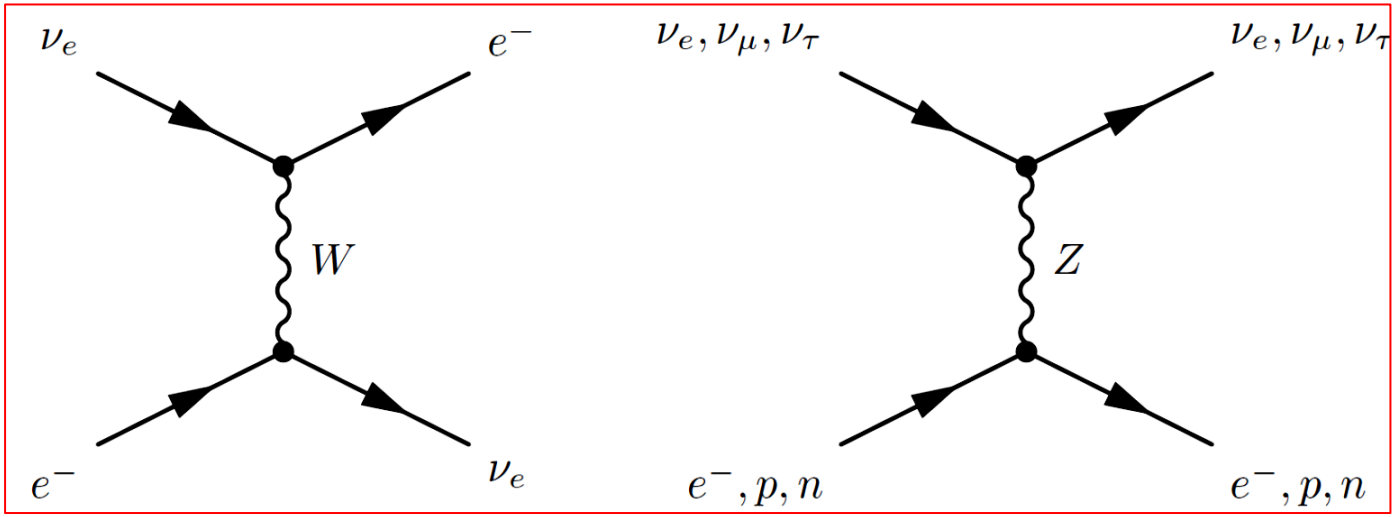
$$\nu_e : \exp[ipx(n_{nc} + n_{cc} - 1)]$$
$$\nu_\mu : \exp[ipx(n_{nc} - 1)]$$
$$\nu_\tau : \exp[ipx(n_{nc} - 1)]$$

Refraction of neutrinos in media, where both CC and NC interactions contribute to refractive indices (not far from 1)

When neutrinos are traveling in matter, the effect of coherent forward scattering with background particles leads to a modification of their energies. Such a modification can be described by an potential energy. The difference between the potentials of distinct neutrino flavors is relevant for neutrino oscillations.

Matter Potentials for Neutrinos

Ordinary matter contains only electrons, neutrons and protons:



Incoherent scattering

$$\ell_{\rm matter} \sim \frac{10^{14}\,{\rm cm}}{(E/{\rm GeV})}$$

We take the number density of particles in normal matter to be about 10^{24} cm^{-3}.

First of all, we look at the Hamiltonian of free neutrinos in vacuum

in flavor basis

$$\widehat{\mathcal{H}}\begin{pmatrix}|\nu_1\rangle\\|\nu_2\rangle\end{pmatrix} = \begin{pmatrix}E_1 & 0\\0 & E_2\end{pmatrix}\begin{pmatrix}|\nu_1\rangle\\|\nu_2\rangle\end{pmatrix}$$
$$= \left[E\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix} + \frac{1}{2E}\begin{pmatrix}m_1^2 & 0\\0 & m_2^2\end{pmatrix}\right]\begin{pmatrix}|\nu_1\rangle\\|\nu_2\rangle\end{pmatrix}$$

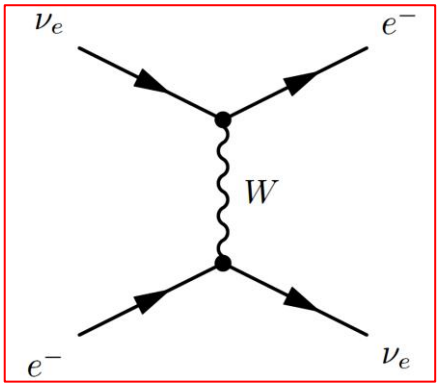
in mass basis

$$\begin{pmatrix}|\nu_e\rangle\\|\nu_\mu\rangle\end{pmatrix} = \begin{pmatrix}c_\theta & s_\theta\\-s_\theta & c_\theta\end{pmatrix}\begin{pmatrix}|\nu_1\rangle\\|\nu_2\rangle\end{pmatrix} \equiv U\begin{pmatrix}|\nu_1\rangle\\|\nu_2\rangle\end{pmatrix}$$

$$\widehat{\mathcal{H}}\begin{pmatrix}c_\theta & -s_\theta\\s_\theta & c_\theta\end{pmatrix}\begin{pmatrix}|\nu_e\rangle\\|\nu_\mu\rangle\end{pmatrix} = \frac{1}{2E}\begin{pmatrix}m_1^2 & 0\\0 & m_2^2\end{pmatrix}\begin{pmatrix}c_\theta & -s_\theta\\s_\theta & c_\theta\end{pmatrix}\begin{pmatrix}|\nu_e\rangle\\|\nu_\mu\rangle\end{pmatrix} \Rightarrow \boldsymbol{\mathcal{H} = \frac{1}{2E}U\begin{pmatrix}m_1^2 & 0\\0 & m_2^2\end{pmatrix}U^\dagger}$$

Matter Potentials for Neutrinos

Effective Hamiltonian density for weak interactions



$$\mathcal{H}_{\rm cc}(x) = \frac{G_{\rm F}}{\sqrt{2}}[\bar{\nu}_e(x)\gamma^\mu(1-\gamma_5)e(x)]\left[\bar{e}(x)\gamma_\mu(1-\gamma_5)\nu_e(x)\right]$$

Fierz transformation

$$\mathcal{H}_{\rm cc}(x) = \frac{G_{\rm F}}{\sqrt{2}}[\bar{\nu}_e(x)\gamma^\mu(1-\gamma_5)\nu_e(x)\,]\left[\bar{e}(x)\gamma_\mu(1-\gamma_5)e(x)\right]$$

Averaged Hamiltonian density over the electron background

$$\bar{\mathcal{H}}_{\rm cc}(x) = \frac{G_{\rm F}}{\sqrt{2}}\bar{\nu}_e(x)\gamma^\mu(1-\gamma_5)\nu_e(x)\int d^3p_e \boldsymbol{f(p_e)}$$ Distribution function of electrons

Averaged over electron helicities $\times \frac{1}{2}\sum_{h_e=\pm}\langle e(p_e,h_e)|\bar{e}(x)\gamma_\mu(1-\gamma_5)e(x)|e(p_e,h_e)\rangle$

$$|e(p_e,h_e)\rangle = \frac{1}{2EV}a_e^{(h_e)\dagger}|0\rangle$$

$$\int d^3p_e f(p_e) = N_e V$$

$$\bar{\mathcal{H}}_{\rm cc}(x) = \frac{G_{\rm F}}{\sqrt{2}}\int d^3p_e f(p_e)\,\bar{\nu}_e(x)\frac{\boldsymbol{p_{e\mu}}}{\boldsymbol{E_e V}}\gamma^\mu(1-\gamma_5)\nu_e(x)$$

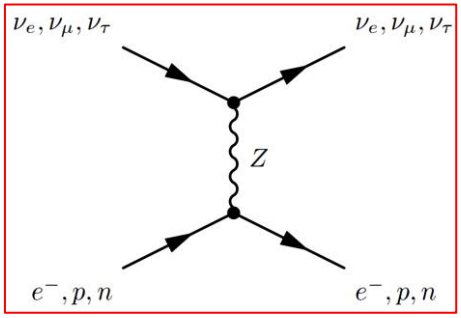
Matter Potentials for Neutrinos

$$\bar{\mathcal{H}}_{\rm cc}(x) = \frac{G_{\rm F}}{\sqrt{2}} \int d^3p_e f(p_e)\, \bar{\nu}_e(x) \frac{p_{e\mu}}{E_e V} \gamma^\mu (1-\gamma_5) \nu_e(x)$$

$$= \frac{G_{\rm F}}{\sqrt{2}V} \bar{\nu}_e(x) \int d^3p_e f(p_e) \left(\gamma^0 - \frac{\vec{p}_e \cdot \vec{\gamma}}{E_e} \right) (1-\gamma_5) \nu_e(x)$$

$$= \frac{G_{\rm F}}{\sqrt{2}V} \bar{\nu}_e(x) \gamma^0 (1-\gamma_5) \nu_e(x) N_e V$$

$$= \sqrt{2} G_{\rm F} N_e \nu_{e\rm L}^\dagger(x) \nu_{e\rm L}(x)$$



NC interactions contribute equally to all flavors, which is irrelevant for oscillations

$$V_e = \sqrt{2} G_{\rm F} N_e$$

Matter Potential (minus sign for $\bar{\nu}$)

Then, we obtain the Hamiltonian for neutrinos travelling in matter

$$A = 2\sqrt{2} G_{\rm F} N_e E$$

$$\mathcal{H}_{\rm m} = \frac{1}{2E} U \begin{pmatrix} m_1^2 & 0 \\ 0 & m_2^2 \end{pmatrix} U^\dagger + \begin{pmatrix} V_e & 0 \\ 0 & 0 \end{pmatrix} = \frac{1}{2E} \left[U \begin{pmatrix} m_1^2 & 0 \\ 0 & m_2^2 \end{pmatrix} U^\dagger + \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} \right]$$

Exercise: evaluate the electron number density in Earth matter of a mass density $\rho = 4\ \mathrm{g/cm^3}$. Note that the normal matter is electrically neutral, so the number fraction of electrons is $Y_e = 0.5$.

Effective Mixing Parameters

The Hamiltonian in a more compact form:

$$\mathcal{H}_{\rm m} = \frac{1}{4E}\left[U\begin{pmatrix} -\Delta m^2_{21} & 0 \\ 0 & +\Delta m^2_{21} \end{pmatrix} U^\dagger + \begin{pmatrix} A & 0 \\ 0 & -A \end{pmatrix}\right] + \frac{m_1^2 + m_2^2 + A}{4E}$$

$$\mathcal{H}_{\rm m} = \frac{1}{4E}\begin{pmatrix} A - \Delta m^2_{21} c_{2\theta} & \Delta m^2_{21} s_{2\theta} \\ \Delta m^2_{21} s_{2\theta} & \Delta m^2_{21} c_{2\theta} - A \end{pmatrix}$$

in flavor basis

Converted into the mass basis

$$\mathcal{H}_{\rm m} = \frac{1}{4E}\widetilde{U}\begin{pmatrix} -\Delta \widetilde{m}^2_{21} & 0 \\ 0 & +\Delta \widetilde{m}^2_{21} \end{pmatrix} \widetilde{U}^\dagger$$

Mixing matrix & mass states in matter

$$\begin{pmatrix} |\nu_e\rangle \\ |\nu_\mu\rangle \end{pmatrix} = \begin{pmatrix} c_{\widetilde{\theta}} & s_{\widetilde{\theta}} \\ -s_{\widetilde{\theta}} & c_{\widetilde{\theta}} \end{pmatrix} \begin{pmatrix} |\widetilde{\nu}_1\rangle \\ |\widetilde{\nu}_2\rangle \end{pmatrix} \equiv \widetilde{U} \begin{pmatrix} |\widetilde{\nu}_1\rangle \\ |\widetilde{\nu}_2\rangle \end{pmatrix}$$

$$\Delta \widetilde{m}^2_{21} = \sqrt{\left(\Delta m^2_{21} c_{2\theta} - A\right)^2 + \left(\Delta m^2_{21} s_{2\theta}\right)^2} \qquad \tan 2\widetilde{\theta} = \frac{\Delta m^2_{21} s_{2\theta}}{\Delta m^2_{21} c_{2\theta} - A}$$

Relationship between the mixing angle (mass difference) in vacuum and that in matter

Some Discussions

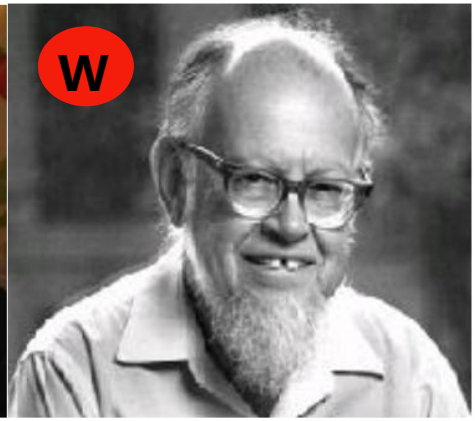
Mixing angle in matter

$$\tan 2\tilde{\theta} = \frac{\Delta m^2_{21} s_{2\theta}}{\Delta m^2_{21} c_{2\theta} - A}$$

MSW resonance $\tilde{\theta} = 45^\circ$

Resonance condition:
$\Delta m^2_{21} c_{2\theta} = 2\sqrt{2} G_F N_e E$

Stanislav Mikheyev (1940-2011) Alexei Smirnov (1951-)



Lincoln Wolfenstein (1923-2015)

Exercise: assume that the energy of solar ^{8}B neutrinos is $E = 10$ MeV, and take $N_e = 100\ N_A/\text{cm}^3$ for $\rho = 150\ \text{g/cm}^3$ in the solar center. The density decreases from the center to the surface. Check if the MSW resonance can be reached, given $\Delta m^2_{21} = 7.5 \times 10^{-5}\ \text{eV}^2$.

Solution:

$$A = 2\sqrt{2} G_F N_e E \approx 2\sqrt{2} \cdot \left(1.17 \times 10^{-5}\ \text{GeV}^{-2}\right) \cdot 10\ \text{MeV} \cdot \left(6 \times 10^{25}\ \text{cm}^{-3}\right)$$

$$\approx 1.5 \times 10^{-4}\ \text{eV}^2$$

$$V_e = \sqrt{2} G_F N_e \approx 7.5 \times 10^{-6}\ \text{eV}^2/\text{MeV}$$

A useful relation: $197\ \text{MeV} \cdot 1\ \text{fm} = 1$

$$P(\nu_e \to \nu_\mu) = \sin^2 2\theta \sin^2 \frac{\Delta m^2 L}{4E}$$

$$\tilde{P}(\nu_e \to \nu_\mu) = \sin^2 2\tilde{\theta} \sin^2 \frac{\Delta \tilde{m}^2 L}{4E}$$

Varying Matter Density

The matter density $\rho(r)$ is varying in astrophysical environments, like the Sun and SNe

$$\tan 2\widetilde{\theta}(r) = \frac{\Delta m^2_{21} s_{2\theta}}{\Delta m^2_{21} c_{2\theta} - 2\sqrt{2} G_{\rm F} N_e(r) E} \qquad \mathcal{H}_{\rm m} = \frac{1}{4E} \widetilde{U} \begin{pmatrix} -\Delta \widetilde{m}^2_{21} & 0 \\ 0 & +\Delta \widetilde{m}^2_{21} \end{pmatrix} \widetilde{U}^\dagger$$

Recall how to calculate neutrino oscillation probabilities in matter of a constant density

$$i \frac{\rm d}{{\rm d}r} \begin{pmatrix} |\nu_e(r)\rangle \\ |\nu_\mu(r)\rangle \end{pmatrix} = \frac{1}{4E} \widetilde{U} \begin{pmatrix} -\Delta \widetilde{m}^2_{21} & 0 \\ 0 & +\Delta \widetilde{m}^2_{21} \end{pmatrix} \widetilde{U}^\dagger \begin{pmatrix} |\nu_e(r)\rangle \\ |\nu_\mu(r)\rangle \end{pmatrix}$$

$$\begin{pmatrix} |\nu_e\rangle \\ |\nu_\mu\rangle \end{pmatrix} = \widetilde{U} \begin{pmatrix} |\widetilde{\nu}_1\rangle \\ |\widetilde{\nu}_2\rangle \end{pmatrix}$$

$$i \frac{\rm d}{{\rm d}r} \left[\widetilde{U} \begin{pmatrix} |\widetilde{\nu}_1(r)\rangle \\ |\widetilde{\nu}_2(r)\rangle \end{pmatrix} \right] = \frac{1}{4E} \widetilde{U} \begin{pmatrix} -\Delta \widetilde{m}^2_{21} & 0 \\ 0 & +\Delta \widetilde{m}^2_{21} \end{pmatrix} \begin{pmatrix} |\widetilde{\nu}_1(r)\rangle \\ |\widetilde{\nu}_2(r)\rangle \end{pmatrix}$$

$$i \frac{\rm d}{{\rm d}r} \begin{pmatrix} |\widetilde{\nu}_1(r)\rangle \\ |\widetilde{\nu}_2(r)\rangle \end{pmatrix} = \left[\frac{1}{4E} \begin{pmatrix} -\Delta \widetilde{m}^2_{21} & 0 \\ 0 & +\Delta \widetilde{m}^2_{21} \end{pmatrix} - i \widetilde{U}^\dagger \dot{\widetilde{U}} \right] \begin{pmatrix} |\widetilde{\nu}_1(r)\rangle \\ |\widetilde{\nu}_2(r)\rangle \end{pmatrix}$$

$$\widetilde{U}^\dagger \dot{\widetilde{U}} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \frac{{\rm d}\widetilde{\theta}}{{\rm d}r}$$

$$i \frac{\rm d}{{\rm d}r} \begin{pmatrix} |\widetilde{\nu}_1(r)\rangle \\ |\widetilde{\nu}_2(r)\rangle \end{pmatrix} = \frac{1}{4E} \begin{pmatrix} -\Delta \widetilde{m}^2_{21} & -4iE {\rm d}\widetilde{\theta}/{\rm d}r \\ 4iE {\rm d}\widetilde{\theta}/{\rm d}r & +\Delta \widetilde{m}^2_{21} \end{pmatrix} \begin{pmatrix} |\widetilde{\nu}_1(r)\rangle \\ |\widetilde{\nu}_2(r)\rangle \end{pmatrix}$$

If $\rho(r)$ or $\widetilde{\theta}(r)$ changes slowly, no transition between mass states

Adiabatic Evolution

Evolution of mass states in matter

$$i\frac{\mathrm{d}}{\mathrm{d}r}\begin{pmatrix}|\tilde{\nu}_1(r)\rangle\\|\tilde{\nu}_2(r)\rangle\end{pmatrix}=\frac{1}{4E}\begin{pmatrix}-\Delta\tilde{m}^2_{21} & -4iE\mathrm{d}\tilde{\theta}/\mathrm{d}r\\ 4iE\mathrm{d}\tilde{\theta}/\mathrm{d}r & +\Delta\tilde{m}^2_{21}\end{pmatrix}\begin{pmatrix}|\tilde{\nu}_1(r)\rangle\\|\tilde{\nu}_2(r)\rangle\end{pmatrix}$$

Exercise: calculate $\frac{\mathrm{d}\tilde{\theta}}{\mathrm{d}r}$

Adiabaticity parameter: the ratio between the diagonal and off-diagonal elements

$$\gamma=\frac{\Delta\tilde{m}^2_{21}}{4E|\mathrm{d}\tilde{\theta}/\mathrm{d}r|}=\frac{\left(\Delta\tilde{m}^2_{21}\right)^2}{2E\sin 2\tilde{\theta}\,|\mathrm{d}A/\mathrm{d}r|}$$

Mass ordering: sign of Δm^2

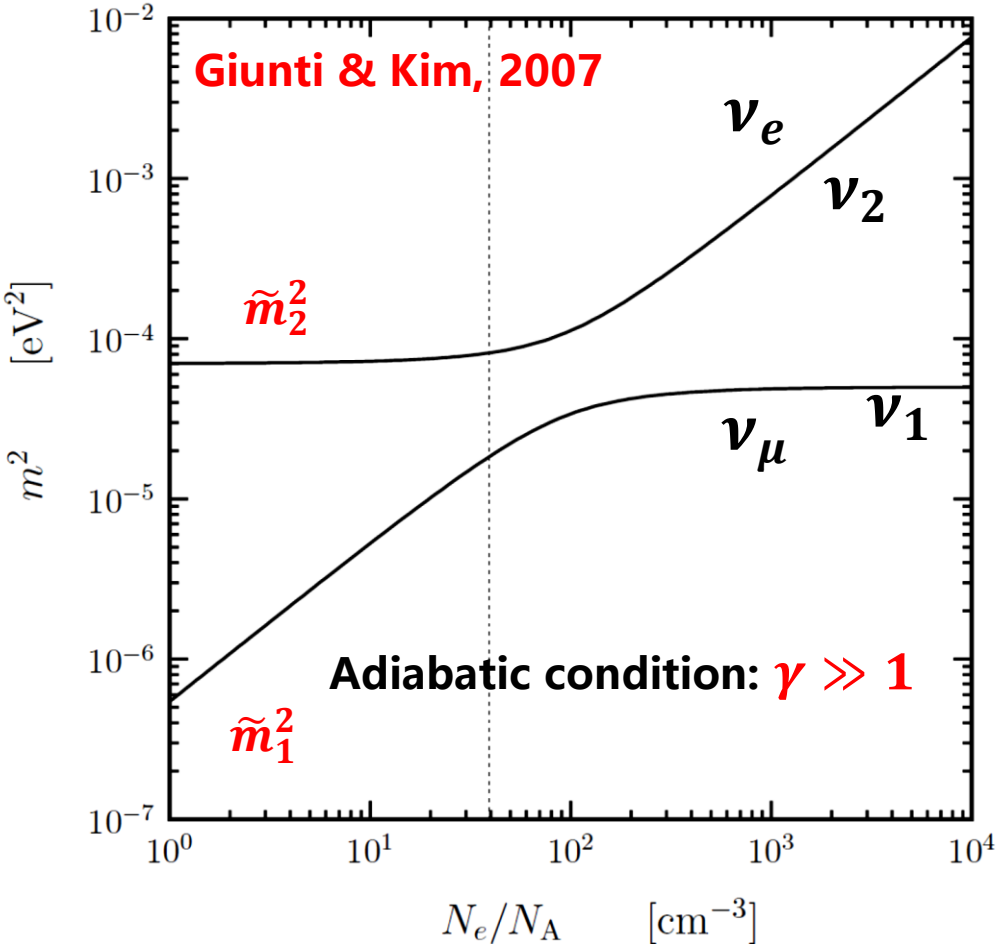
$$\mathcal{H}=U(\theta)\begin{pmatrix}m_1^2 & 0\\ 0 & m_2^2\end{pmatrix}U^{\dagger}(\theta)$$

invariant under

$$\theta\Rightarrow\frac{\pi}{2}-\theta \qquad m_1^2\Leftrightarrow m_2^2$$

$$|\nu_e\rangle\Rightarrow+|\nu_e\rangle \qquad |\nu_\mu\rangle\Rightarrow-|\nu_\mu\rangle$$

$\theta\in[0,\frac{\pi}{4}]$, $\Delta m^2_{21}>0$ $\Leftrightarrow$ $\theta\in[\frac{\pi}{4},\frac{\pi}{2}]$, $\Delta m^2_{21}<0$



3-flavor Oscillations in Matter

The effective Hamiltonian in matter

$$\mathcal{H}_{\rm m} = \frac{1}{2E}\left[V \begin{pmatrix} m_1^2 & 0 & 0 \\ 0 & m_2^2 & 0 \\ 0 & 0 & m_3^2 \end{pmatrix} V^\dagger + \begin{pmatrix} A & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right]$$

Diagonalize the effective Hamiltonian in matter

$$\tilde{V}^\dagger \mathcal{H}_{\rm m} \tilde{V} = \frac{1}{2E} \begin{pmatrix} \widetilde{m}_1^2 & 0 & 0 \\ 0 & \widetilde{m}_2^2 & 0 \\ 0 & 0 & \widetilde{m}_3^2 \end{pmatrix}$$

Parametrize $\tilde{V}$ in the standard way by three mixing angles $\widetilde{\theta}_{ij}$ and one CP-violating phase $\widetilde{\delta}$ in matter

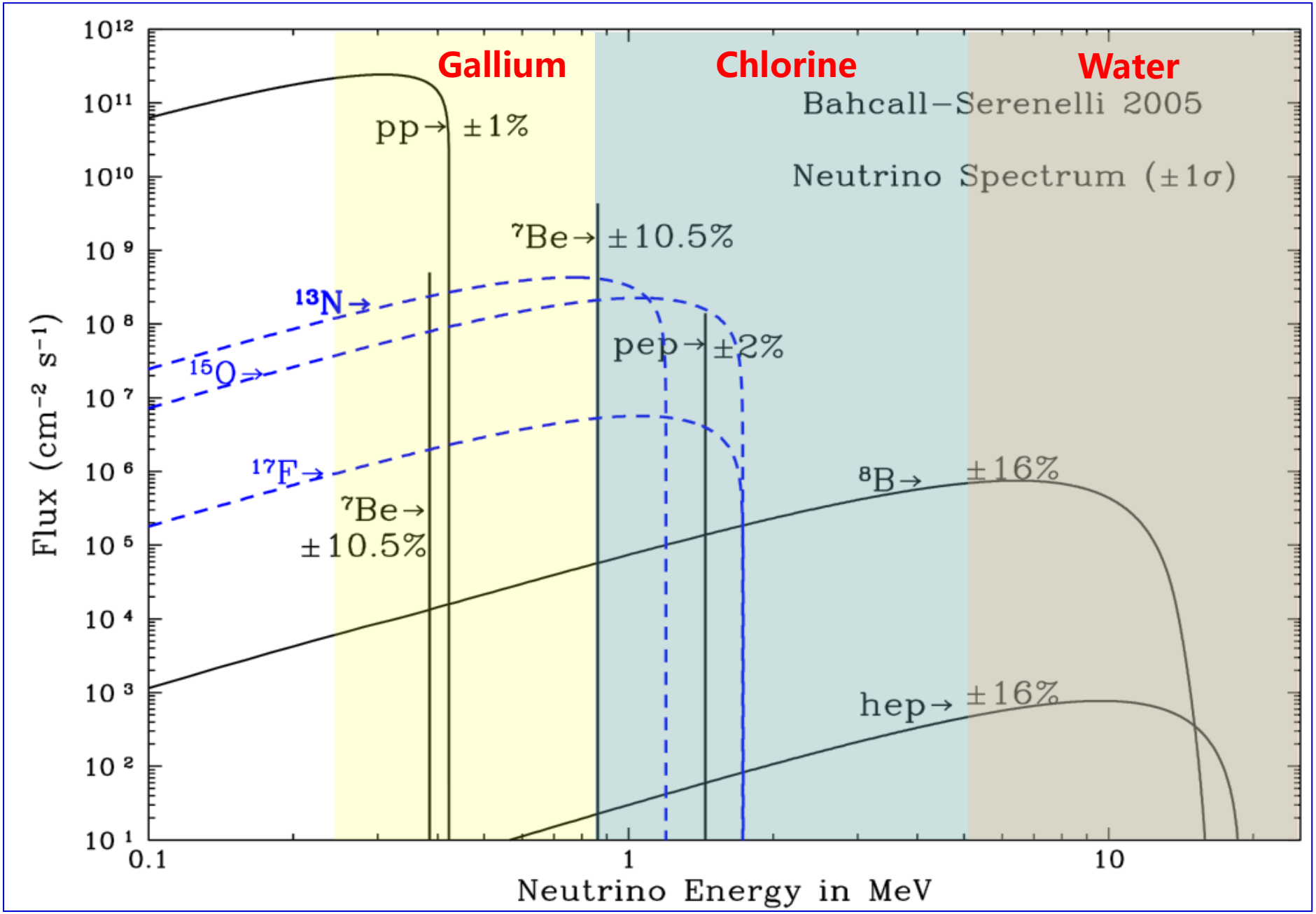
Oscillation probabilities in matter of a constant density

$$\tilde{P}(\nu_\alpha \to \nu_\beta) = \delta_{\alpha\beta} - 4 \sum_{i<j}^{3} \mathrm{Re}\left[\tilde{V}_{\alpha i} \tilde{V}_{\beta j} \tilde{V}_{\alpha j}^* \tilde{V}_{\beta i}^*\right] \sin^2 \frac{\Delta \widetilde{m}_{ji}^2 L}{4E}$$

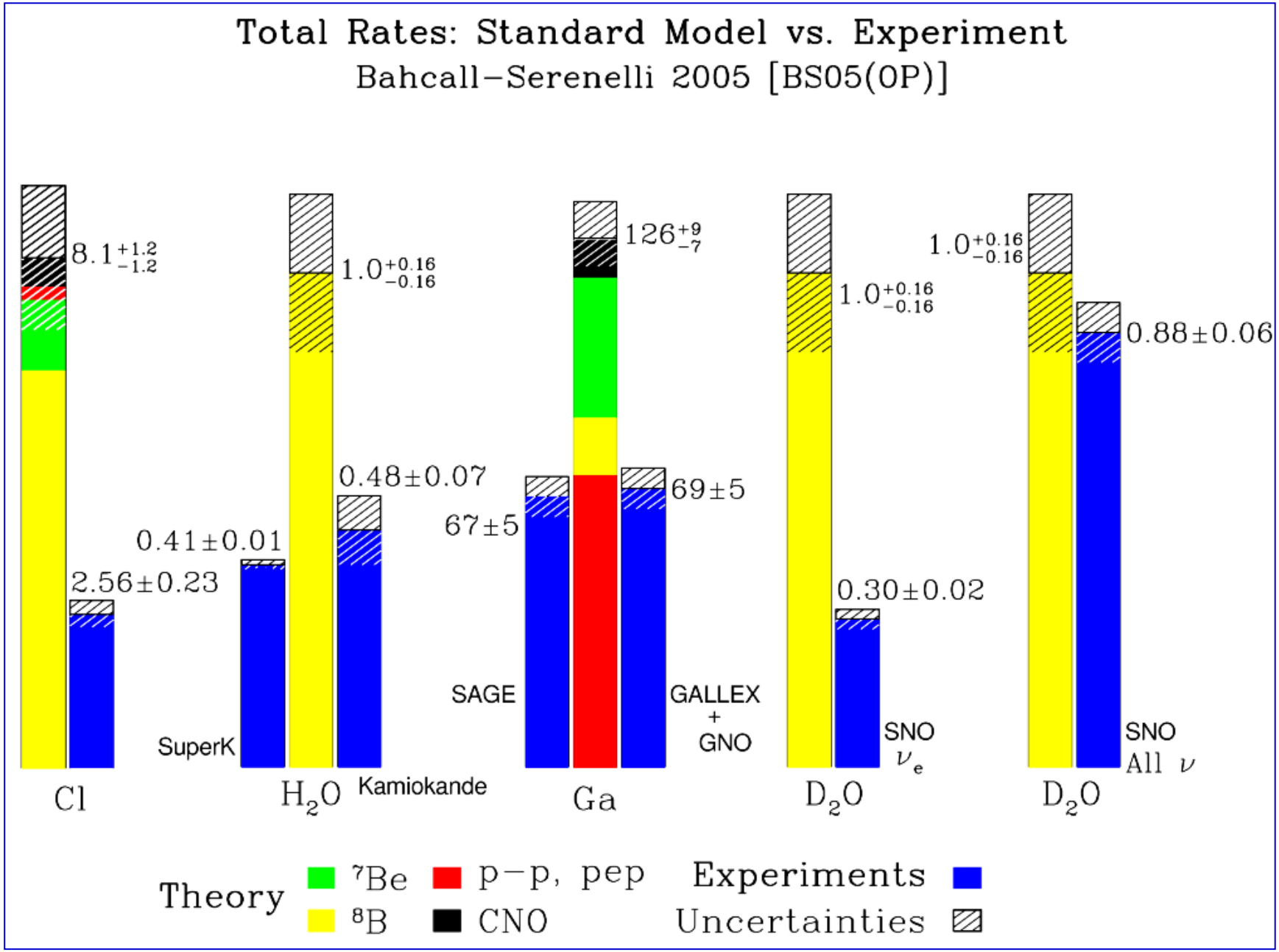
$$+ 8\tilde{\mathcal{J}} \sum_{\gamma} \varepsilon_{\alpha\beta\gamma} \sin \frac{\Delta \widetilde{m}_{21}^2 L}{4E} \sin \frac{\Delta \widetilde{m}_{32}^2 L}{4E} \sin \frac{\Delta \widetilde{m}_{31}^2 L}{4E}$$

The oscillation probabilities for antineutrinos can be obtained by $V \Rightarrow V^*$ and $A \Rightarrow -A$

Solar Neutrino Experiments



Solar Neutrino Problem after SNO



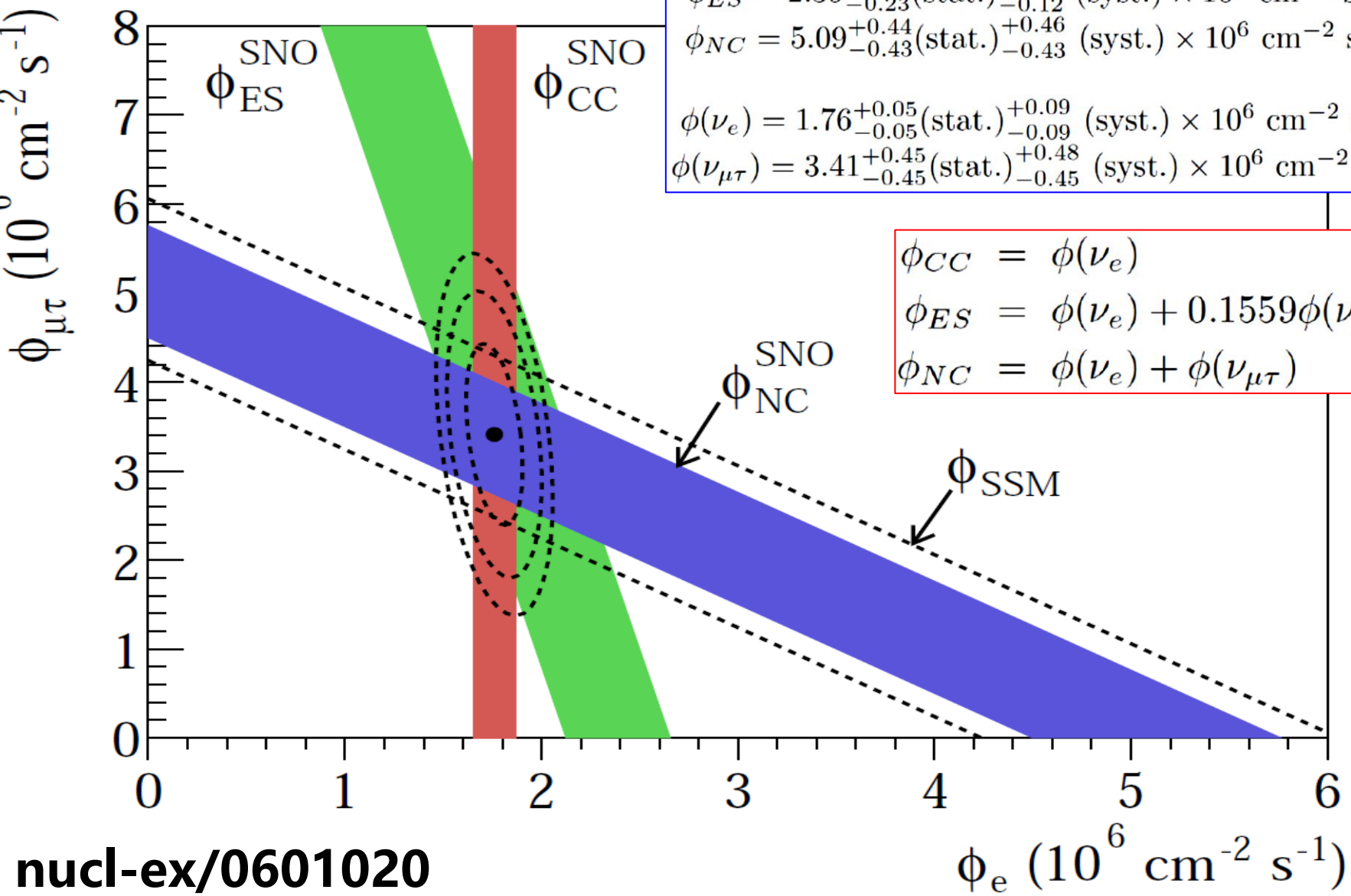
Results from SNO

Solar ν_e converted to ν_μ or ν_τ

$$\phi_{CC} = 1.76^{+0.06}_{-0.05}(\text{stat.})^{+0.09}_{-0.09}\ (\text{syst.}) \times 10^6\ \text{cm}^{-2}\ \text{s}^{-1}$$
$$\phi_{ES} = 2.39^{+0.24}_{-0.23}(\text{stat.})^{+0.12}_{-0.12}\ (\text{syst.}) \times 10^6\ \text{cm}^{-2}\ \text{s}^{-1}$$
$$\phi_{NC} = 5.09^{+0.44}_{-0.43}(\text{stat.})^{+0.46}_{-0.43}\ (\text{syst.}) \times 10^6\ \text{cm}^{-2}\ \text{s}^{-1}$$

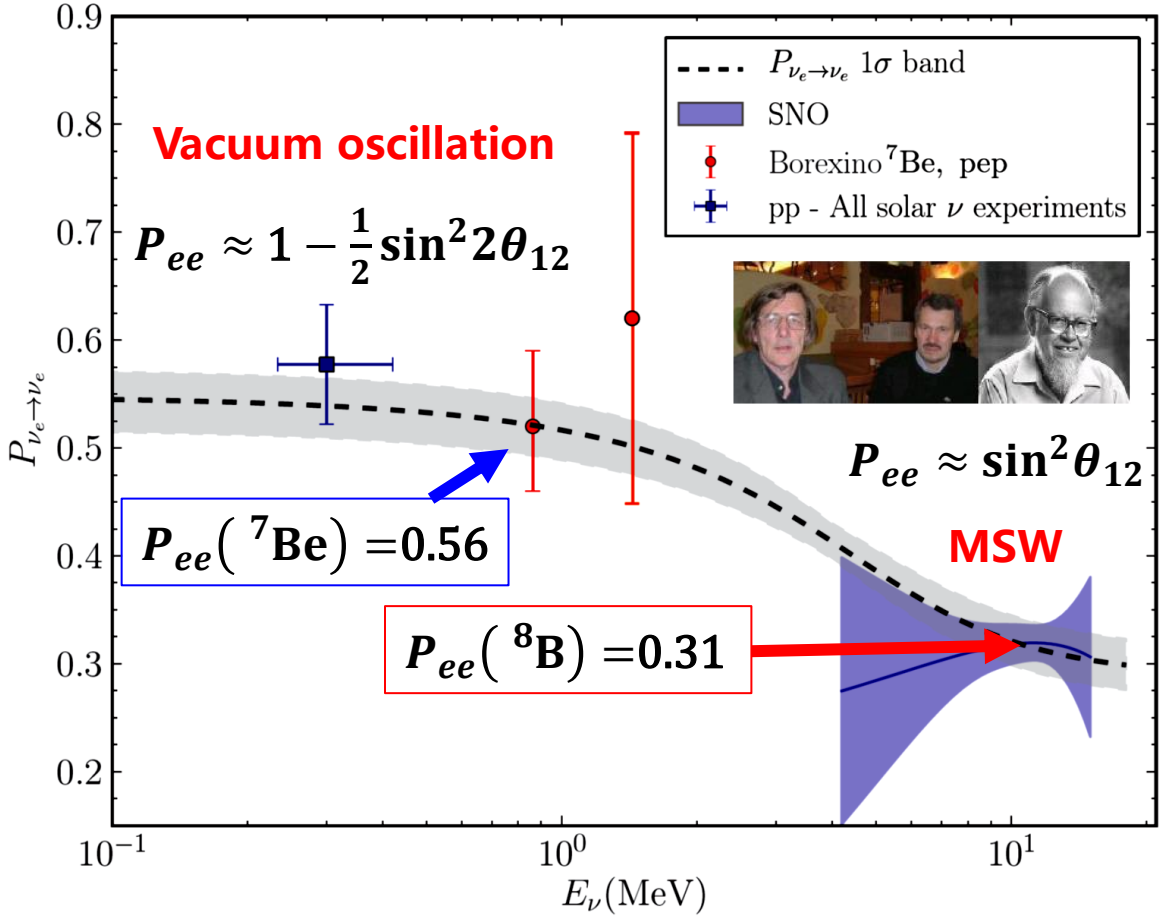
$$\phi(\nu_e) = 1.76^{+0.05}_{-0.05}(\text{stat.})^{+0.09}_{-0.09}\ (\text{syst.}) \times 10^6\ \text{cm}^{-2}\ \text{s}^{-1}$$
$$\phi(\nu_{\mu\tau}) = 3.41^{+0.45}_{-0.45}(\text{stat.})^{+0.48}_{-0.45}\ (\text{syst.}) \times 10^6\ \text{cm}^{-2}\ \text{s}^{-1}$$

$$\phi_{CC} = \phi(\nu_e)$$
$$\phi_{ES} = \phi(\nu_e) + 0.1559\phi(\nu_{\mu\tau})$$
$$\phi_{NC} = \phi(\nu_e) + \phi(\nu_{\mu\tau})$$



nucl-ex/0601020

The LMA-MSW Solution



For high-energy ^{8}B neutrinos

at production $r = 0$

$$\begin{pmatrix} |\tilde{\nu}_1(0)\rangle \\ |\tilde{\nu}_2(0)\rangle \end{pmatrix} = \begin{pmatrix} c_{\hat{\theta}} & -s_{\hat{\theta}} \\ s_{\hat{\theta}} & c_{\hat{\theta}} \end{pmatrix} \begin{pmatrix} |\nu_e(0)\rangle \\ |\nu_\mu(0)\rangle \end{pmatrix}$$

adiabatic evolution

$$\begin{pmatrix} |\tilde{\nu}_1(R)\rangle \\ |\tilde{\nu}_2(R)\rangle \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} |\tilde{\nu}_1(0)\rangle \\ |\tilde{\nu}_2(0)\rangle \end{pmatrix}$$

on the solar surface $r = R$

$$\begin{pmatrix} |\nu_e(R)\rangle \\ |\nu_\mu(R)\rangle \end{pmatrix} = \begin{pmatrix} c_\theta & s_\theta \\ -s_\theta & c_\theta \end{pmatrix} \begin{pmatrix} |\tilde{\nu}_1(R)\rangle \\ |\tilde{\nu}_2(R)\rangle \end{pmatrix}$$

survival probability

$$P_{ee} = c_{\hat{\theta}}^2 c_\theta^2 + s_{\hat{\theta}}^2 s_\theta^2 = \sin^2\theta$$

$\hat{\theta} \longrightarrow \pi/2$ as $A \gg \Delta m^2$

For the MSW resonance to happen

$\theta \in [0, \frac{\pi}{4}]$, $\Delta m^2_{21} > 0$ $\Longleftrightarrow$ $\theta \in [\frac{\pi}{4}, \frac{\pi}{2}]$, $\Delta m^2_{21} < 0$

$\theta_{12} = 34^\circ$

Normal neutrino mass ordering

For low-energy ^{7}Be neutrinos

$$P_{ee} \approx 1 - \frac{1}{2}\sin^2 2\theta_{12}$$

Oscillations in vacuum

Global-fit Analysis & Prospects

nu-fit.org

IC23 without SK atmospheric data

	Normal Ordering (best fit)		Inverted Ordering ($\Delta\chi^2 = 1.5$)	
	bfp $\pm 1\sigma$	3σ range	bfp $\pm 1\sigma$	3σ range
$\sin^2\theta_{12}$	$0.3088^{+0.0067}_{-0.0066}$	$0.2893 \to 0.3295$	$0.3088^{+0.0067}_{-0.0066}$	$0.2893 \to 0.3295$
$\theta_{12}/^\circ$	$33.76^{+0.42}_{-0.41}$	$32.54 \to 35.03$	$33.76^{+0.42}_{-0.41}$	$32.54 \to 35.03$
$\sin^2\theta_{23}$	$0.470^{+0.017}_{-0.014}$	$0.432 \to 0.587$	$0.555^{+0.013}_{-0.016}$	$0.437 \to 0.590$
$\theta_{23}/^\circ$	$43.27^{+1.0}_{-0.82}$	$41.11 \to 50.02$	$48.15^{+0.75}_{-0.92}$	$41.40 \to 50.21$
$\sin^2\theta_{13}$	$0.02249^{+0.00057}_{-0.00057}$	$0.02070 \to 0.02420$	$0.02261^{+0.00056}_{-0.00056}$	$0.02091 \to 0.02433$
$\theta_{13}/^\circ$	$8.62^{+0.11}_{-0.11}$	$8.27 \to 8.95$	$8.65^{+0.11}_{-0.11}$	$8.31 \to 8.97$
$\delta_{\rm CP}/^\circ$	207^{+23}_{-20}	$114 \to 405$	283^{+24}_{-28}	$202 \to 347$
$\frac{\Delta m^2_{21}}{10^{-5}\ \text{eV}^2}$	$7.537^{+0.094}_{-0.10}$	$7.236 \to 7.823$	$7.537^{+0.094}_{-0.10}$	$7.236 \to 7.822$
$\frac{\Delta m^2_{3\ell}}{10^{-3}\ \text{eV}^2}$	$+2.521^{+0.026}_{-0.018}$	$+2.454 \to +2.592$	$-2.500^{+0.024}_{-0.023}$	$-2.569 \to -2.430$

Neutrino Mass Ordering

- Reactor: JUNO
- LBL Acc.: T2K, NOvA, DUNE
- Atm: PINGU, ORCA, Hyper-K

Leptonic CP Violation

- LBL Acc.: T2HK,DUNE
- Super-B: ESSνSB, MOMENT
- NF & Beta-Beams

Absolute Masses: KATRIN, 0νββ (^{136}Xe, ^{76}Ge), cosmology, …