Neutrino Masses and Lepton Flavor Mixing

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Neutrino Oscillations

Neutrino oscillation experiments can precisely measure two independent neutrino mass-squared differences

$$\Delta m^2_{21} \equiv m^2_2 - m^2_1 \approx 8.1 \times 10^{-5}\ \mathrm{eV}^2\ , \tag{2.1}$$

$$|\Delta m^2_{31}| \equiv |m^2_3 - m^2_1| \approx 2.5 \times 10^{-3}\ \mathrm{eV}^2\ , \tag{2.2}$$

but not absolute neutrino masses. We are left with either normal mass ordering (NO): $\Delta m^2_{31} > 0$ or inverted mass ordering (IO): $\Delta m^2_{31} < 0$.

Example

Let us assume that the lightest neutrino is massless, namely, $m_1 = 0$. Then we have

$$m_2 = \sqrt{\Delta m^2_{21}} \approx 9\ \mathrm{meV}\ , \quad m_3 = \sqrt{\Delta m^2_{31}} \approx 50\ \mathrm{meV}\ . \tag{2.3}$$

Tritium Beta (β) Decays

The KATRIN experiment has reported its first result of the upper bound on the effective neutrino mass m_β for tritium β-decays $^3\mathrm{H} \to {}^3\mathrm{He} + e^- + \overline{\nu}_e$

$$m_\beta \equiv \sqrt{m_1^2 |V_{e1}|^2 + m_2^2 |V_{e2}|^2 + m_3^2 |V_{e3}|^2} \lesssim 0.45 \text{ eV} , \tag{2.4}$$

at the 90% confidence level, where V_{ei} (for $i = 1, 2, 3$) denote the lepton flavor mixing matrix elements.

Example

Now we assume that the neutrino mass spectrum is nearly degenerate, i.e., $m_1 \approx m_2 \approx m_3 \approx m_i$. Then the effective mass m_β can be estimated as

$$m_\beta \approx \sqrt{m_i^2 \left(|V_{e1}|^2 + |V_{e2}|^2 + |V_{e3}|^2\right)} = m_i \lesssim 0.45 \text{ eV} . \tag{2.5}$$

Neutrinoless Double-Beta ($0\nu\beta\beta$) Decays

The most restrictive bound on the effective neutrino mass $m_{\beta\beta}$ for $0\nu\beta\beta$ decays (e.g., ${}^{136}_{54}\mathrm{Xe} \to {}^{136}_{56}\mathrm{Ba} + 2e^-$) is obtained by KamLAND-Zen

$$m_{\beta\beta} \equiv \left| m_1 |V_{e1}|^2 e^{\mathrm{i}\rho} + m_2 |V_{e2}|^2 + m_3 |V_{e3}|^2 e^{\mathrm{i}\sigma} \right| \lesssim (28 \cdots 122)\ \mathrm{meV}\ ,$$

at the 90% confidence level, where ρ and σ are two Majorana CP phases.

Example

For IO with $m_3 = 0$, we get $m_1 \approx 50\ \mathrm{meV}$ and $m_2 \approx 51\ \mathrm{meV}$. In this case, together with $|V_{e1}|^2 \approx 0.67$ and $|V_{e2}|^2 \approx 0.31$, one obtains

$$(m_1 |V_{e1}|^2 - m_2 |V_{e2}|^2) \lesssim m_{\beta\beta} \lesssim (m_1 |V_{e1}|^2 + m_2 |V_{e2}|^2)\ , \qquad (2.6)$$

leading to $m_{\beta\beta} \in [18, 50]\ \mathrm{meV}$. A promising scenario for $0\nu\beta\beta$ decays!

Cosmological Bounds

Massive neutrinos have played an important role in the evolution of our Universe, affecting the power spectrum of cosmic microwave background (CMB) and the formation of large-scale structures (LSS). The Planck collaboration (CMB + BAO) place an upper bound on neutrino masses

$$\Sigma = m_1 + m_2 + m_3 \lesssim 0.12 \text{ eV} \tag{2.7}$$

at the 95% confidence level. We expect that $m_i < 0.1$ eV for $i = 1, 2, 3$.

Example

One can figure out that there exists a lower bound on Σ, i.e.,

$$\begin{cases} \Sigma \gtrsim m_2 + m_3 \approx 0.059 \text{ eV}, & \text{NO}; \\ \Sigma \gtrsim m_1 + m_2 \approx 0.101 \text{ eV}, & \text{IO}. \end{cases} \tag{2.8}$$

Right-handed Neutrinos

Similar to charged fermions, namely, quarks and charged leptons, we can simply introduce three right-handed neutrinos into the SM. The lepton fields under the $\mathrm{SU(2)_L} \times \mathrm{U(1)_Y}$ gauge symmetry transform as

$$\begin{aligned}
\text{lepton doublets} \quad & L_\alpha \; : \quad \begin{pmatrix} \nu_{e\mathrm{L}} \\ e_{\mathrm{L}} \end{pmatrix} , \quad \begin{pmatrix} \nu_{\mu\mathrm{L}} \\ \mu_{\mathrm{L}} \end{pmatrix} , \quad \begin{pmatrix} \nu_{\tau\mathrm{L}} \\ \tau_{\mathrm{L}} \end{pmatrix} ; \\
\text{charged} - \text{lepton singlets} \quad & R_\alpha \; : \quad e_{\mathrm{R}} , \quad \mu_{\mathrm{R}} , \quad \tau_{\mathrm{R}} ; \\
\text{neutrino singlets} \quad & N_\alpha \; : \quad \nu_{e\mathrm{R}} , \quad \nu_{\mu\mathrm{R}} , \quad \nu_{\tau\mathrm{R}} ,
\end{aligned}$$

where right-handed neutrino singlets are completely neutral: $I_3(N_\alpha) = 0$ and $Y(N_\alpha) = 0$ with $\alpha = e, \mu, \tau$.

Right-handed neutrinos never interact with any other particles in the SM.

Recall the Higgs doublet in the SM and its vev

$$H = \begin{pmatrix} \varphi^+ \\ \varphi^0 \end{pmatrix} , \quad \widetilde{H} = \mathrm{i}\sigma^2 H^* = \begin{pmatrix} \varphi^{0*} \\ -\varphi^- \end{pmatrix} , \quad \langle H \rangle = \frac{v}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \end{pmatrix} . \tag{2.9}$$

Lepton Yukawa Interactions

The gauge-invariant Yukawa interactions of leptons can be written as

$$\mathcal{L}_{\rm Y} = -\left(Y_l\right)_{\alpha\beta} \overline{L_\alpha} H R_\beta - \left(Y_\nu\right)_{\alpha\beta} \overline{L_\alpha} \widetilde{H} N_\beta + \text{h.c.} , \tag{2.10}$$

where $\alpha, \beta = e, \mu, \tau$. After the SSB, the lepton mass terms are given by

$$\mathcal{L}_{\rm m} = -\left(M_l\right)_{\alpha\beta} \overline{\alpha_{\rm L}} \beta_{\rm R} - \left(M_\nu\right)_{\alpha\beta} \overline{\nu_{\alpha{\rm L}}} \nu_{\beta{\rm R}} + \text{h.c.} , \tag{2.11}$$

where $M_l = Y_l v/\sqrt{2}$ and $M_\nu = Y_\nu v/\sqrt{2}$ are charged-lepton and Dirac neutrino mass matrices, respectively.

Leptonic CC Interaction

As we have shown, the leptonic CC interaction reads

$$\mathcal{L}^{\rm l}_{\rm CC} = \frac{g}{\sqrt{2}} \overline{\alpha_{\rm L}} \gamma^\mu \nu_{\alpha {\rm L}} W^-_\mu + {\rm h.c.} \; . \tag{2.12}$$

Leptonic Mass Terms

Recall the leptonic mass terms from the previous slide

$$\mathcal{L}_{\rm m} = - \left(M_l\right)_{\alpha\beta} \overline{\alpha_{\rm L}} \beta_{\rm R} - \left(M_\nu\right)_{\alpha\beta} \overline{\nu_{\alpha {\rm L}}} \nu_{\beta {\rm R}} + {\rm h.c.} \; . \tag{2.13}$$

Leptonic Flavor Mixing

Convert into the mass basis via

$$\alpha_{\rm L} = V_{l{\rm L}} \widehat{\alpha}_{\rm L} \; , \quad \alpha_{\rm R} = V_{l{\rm R}} \widehat{\alpha}_{\rm R} \; , \quad \nu_{\rm L} = V_{\nu{\rm L}} \widehat{\nu}_{\rm L} \; , \quad \nu_{\rm R} = V_{\nu{\rm R}} \widehat{\nu}_{\rm R} \; . \tag{2.14}$$

Leptonic Flavor Mixing

In the mass basis, we have

$$\mathcal{L}^{\rm l}_{\rm CC} = \frac{g}{\sqrt{2}} \overline{\widehat{\alpha}_{\rm L}} \gamma^\mu \left(V^\dagger_{l{\rm L}} V_{\nu{\rm L}} \right)_{\alpha i} \widehat{\nu}_{i{\rm L}} W^-_\mu + {\rm h.c.} \; , \tag{2.15}$$

where $V^\dagger_{l{\rm L}} V_{\nu{\rm L}} \equiv V$ is a 3×3 unitary matrix. This flavor mixing matrix is called Pontecorvo-Maki-Nakagawa-Sakata (PMNS) matrix.

Leptonic Mass Terms

$$\begin{aligned} \mathcal{L}_{\rm m} &= -\left(M_l\right)_{\alpha\beta} \overline{\alpha_{\rm L}} \beta_{\rm R} - \left(M_\nu\right)_{\alpha\beta} \overline{\nu_{\alpha{\rm L}}} \nu_{\beta{\rm R}} + {\rm h.c.} \; , \\ &= -\left(V^\dagger_{l{\rm L}} M_l V_{l{\rm R}} \right)_{\alpha\beta} \overline{\widehat{\alpha}_{\rm L}} \widehat{\beta}_{\rm R} - \left(V^\dagger_{\nu{\rm L}} M_\nu V_{\nu{\rm R}} \right)_{ij} \overline{\widehat{\nu}_{i{\rm L}}} \widehat{\nu}_{j{\rm R}} + {\rm h.c.} \; , \\ &= -m_\alpha \overline{\widehat{\alpha}_{\rm L}} \widehat{\alpha}_{\rm R} - m_i \overline{\widehat{\nu}_{i{\rm L}}} \widehat{\nu}_{i{\rm R}} + {\rm h.c.} \; . \end{aligned} \tag{2.16}$$

Lepton Masses

$$V_{l\mathrm{L}}^{\dagger} M_l V_{l\mathrm{R}} = \widehat{M}_l = \begin{pmatrix} m_e & 0 & 0 \\ 0 & m_\mu & 0 \\ 0 & 0 & m_\tau \end{pmatrix} , \quad (2.17)$$

$$V_{\nu\mathrm{L}}^{\dagger} M_\nu V_{\nu\mathrm{R}} = \widehat{M}_\nu = \begin{pmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{pmatrix} . \quad (2.18)$$

where $m_e = 0.511\ \mathrm{MeV}$, $m_\mu = 105\ \mathrm{MeV}$ and $m_\tau = 1.78\ \mathrm{GeV}$ are the charged-lepton masses.

Fermion Mass Hierarchy:

$$m_e = 0.511\ \mathrm{MeV} \ll m_\mu \ll m_\tau , \quad m_u \ll m_c \ll m_t = 171\ \mathrm{GeV} . \quad (2.19)$$

Fermion Mass Hierarchy

$$\frac{m_u}{m_c} \approx \frac{m_c}{m_t} \approx \lambda^4 \,, \quad \frac{m_d}{m_s} \approx \frac{m_s}{m_b} \approx \lambda^2 \,, \tag{2.20}$$

where $\lambda \approx \sin\theta_{\rm C} = 0.22$ with $\theta_{\rm C} \approx 13^\circ$ being the Cabibbo angle. For charged-lepton masses, we have $m_e/m_\mu \approx \lambda^4$ and $m_\mu/m_\tau \approx \lambda^2$.

Yukawa Couplings

$$y_t = \frac{\sqrt{2} m_t}{v} \approx 0.983 \,, \quad y_1 = \frac{\sqrt{2} m_1}{v} < 2.3 \times 10^{-12} \,, \tag{2.21}$$

where $v \approx 246\ {\rm GeV}$ is the Higgs vev, $m_t \approx 171\ {\rm GeV}$ the top-quark mass, and $m_1 < 0.4\ {\rm eV}$ the neutrino mass. Why twelve orders of magnitude?

Lepton Number Conservation

	L_e	L_μ	L_τ	L
$(\nu_{e\mathrm{L}}, e_{\mathrm{L}})$, e_{R}, $\nu_{e\mathrm{R}}$	$+1$	0	0	$+1$
$(\nu_{\mu\mathrm{L}}, \mu_{\mathrm{L}})$, μ_{R}, $\nu_{\mu\mathrm{R}}$	0	$+1$	0	$+1$
$(\nu_{\tau\mathrm{L}}, \tau_{\mathrm{L}})$, τ_{R}, $\nu_{\tau\mathrm{R}}$	0	0	$+1$	$+1$

Global $\mathrm{U}(1)$ Symmetry

$$\begin{aligned} \mathcal{L}_{\mathrm{k}} &= \overline{L_\alpha}\mathrm{i}\gamma^\mu D_\mu L_\alpha + \overline{R_\alpha}\mathrm{i}\gamma^\mu \partial'_\mu R_\alpha + \overline{N_\alpha}\mathrm{i}\gamma^\mu \partial_\mu N_\alpha \; , \\ \mathcal{L}_{\mathrm{Y}} &= -\overline{L_\alpha}\left(Y_l\right)_{\alpha\beta} H R_\beta - \overline{L_\alpha}\left(Y_\nu\right)_{\alpha\beta} \widetilde{H} N_\beta + \mathrm{h.c.} \; , \end{aligned} \tag{2.22}$$

are invariant under the global $\mathrm{U}(1)$ transformations: $L_\alpha \to L_\alpha e^{-\mathrm{i}\theta}$, $R_\alpha \to R_\alpha e^{-\mathrm{i}\theta}$ and $N_\alpha \to N_\alpha e^{-\mathrm{i}\theta}$. The total lepton number is conserved.

Lepton Number Conservation

The individual lepton number (L_e, L_μ and L_τ) and the total lepton number $L = L_e + L_\mu + L_\tau$ are conserved accidentally in the SM.

Violation of the global $\mathrm{U}(1)$ symmetry:

$$\mathcal{L}_{\rm Y} = -\overline{L_\alpha}\,(Y_l)_{\alpha\beta}\,H R_\beta - \overline{L_\alpha}\,(Y_\nu)_{\alpha\beta}\,\widetilde{H} N_\beta - \frac{1}{2}\overline{N_\alpha^{\rm C}}\,(M_N)_{\alpha\beta}\,N_\beta + {\rm h.c.}\;,$$

where the last term is NOT invariant under the $\mathrm{U}(1)$ transformations: $L_\alpha \to L_\alpha e^{-{\rm i}\theta}$, $R_\alpha \to R_\alpha e^{-{\rm i}\theta}$ and $N_\alpha \to N_\alpha e^{-{\rm i}\theta}$, i.e.,

$$N_\alpha^{\rm C} \equiv C\overline{N_\alpha}^{\rm T} = {\rm i}\gamma^2\gamma^0\left(N_\alpha^\dagger\gamma^0\right)^{\rm T} = {\rm i}\gamma^2 N_\alpha^* \Rightarrow N_\alpha^{\rm C} \to N_\alpha^{\rm C} e^{+{\rm i}\theta}\;; \quad (2.23)$$

and thus

$$\frac{1}{2}\overline{N_\alpha^{\rm C}}\,(M_N)_{\alpha\beta}\,N_\beta \to e^{-2{\rm i}\theta}\frac{1}{2}\overline{N_\alpha^{\rm C}}\,(M_N)_{\alpha\beta}\,N_\beta\;. \quad (2.24)$$

Difficulties for Dirac Neutrino Masses

- Worsen the problem of strong fermion mass hierarchy:

$$y_t \sim 1 \gg y_e \sim 10^{-6} \gg y_i \sim 10^{-12} \; ; \tag{2.25}$$

- Require the lepton number conservation to forbid Majorana neutrino mass terms

$$\frac{1}{2} \overline{N_\alpha^{\rm C}} \left(M_N\right)_{\alpha\beta} N_\beta \; , \tag{2.26}$$

  which are obviously allowed by the gauge symmetry of the SM. Unlike the electric charge, the lepton number conservation corresponds to a global $\mathrm{U}(1)$ symmetry, which is actually anomalous within the SM and expected to be violated by gravity or blackholes.

Dirac Spinor

Consider the electrons and positrons, which are described by the Dirac spinor $\psi(x)$ in QED. The Dirac equation reads

$$(\mathrm{i}\gamma^\mu \partial_\mu - m)\psi(x) = 0 \,, \tag{2.27}$$

where the Dirac matrices in the Weyl representation are given by

$$\gamma^0 = \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix} \,, \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} \,, \tag{2.28}$$

with Pauli matrices

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \,, \quad \sigma^2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} \,, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \,. \tag{2.29}$$

Dirac Spinor

As is well known, there are four degrees of freedom for a Dirac spinor, which can be identified as the left/right-handed electron/positron, i.e.,

$$e_{\rm L}\;, \quad e_{\rm R}\;, \quad \bar{e}_{\rm L}\;, \quad \bar{e}_{\rm R}\;, \tag{2.30}$$

where "L/R" refer to the helicities, e the electron, and $\bar{e}$ the positron.

Since the electron is massive, a left-handed electron $e_{\rm L}$ will be observed as a right-handed one $e_{\rm R}$ in a boosted reference frame. Due to the electric charge conservation, $e_{\rm L}$ can never be changed to $\bar{e}_{\rm R}$.

Dirac Neutrinos

Similarly, for massive Dirac neutrinos, we should also have four states:

$$\nu_{\rm L}\;, \quad \nu_{\rm R}\;, \quad \bar{\nu}_{\rm L}\;, \quad \bar{\nu}_{\rm R}\;. \tag{2.31}$$

Weyl Spinor

Introducing $\gamma_5 = \mathrm{i}\gamma^0\gamma^1\gamma^2\gamma^3$, which can be explicitly figured out as

$$\begin{aligned} \gamma_5 &= \mathrm{i} \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix} \begin{pmatrix} 0 & \sigma^1 \\ -\sigma^1 & 0 \end{pmatrix} \begin{pmatrix} 0 & \sigma^2 \\ -\sigma^2 & 0 \end{pmatrix} \begin{pmatrix} 0 & \sigma^3 \\ -\sigma^3 & 0 \end{pmatrix} \\ &= \begin{pmatrix} +\mathrm{i}\sigma^1\sigma^2\sigma^3 & 0 \\ 0 & -\mathrm{i}\sigma^1\sigma^2\sigma^3 \end{pmatrix} = \begin{pmatrix} -\mathbf{1} & 0 \\ 0 & +\mathbf{1} \end{pmatrix} , \end{aligned} \tag{2.32}$$

and the chiral projection

$$\psi_{\mathrm{L/R}}(x) \equiv P_{\mathrm{L/R}}\psi(x) = \frac{1 \mp \gamma_5}{2}\psi(x) , \tag{2.33}$$

one can obtain the Weyl equation for massless Dirac spinors

$$\mathrm{i}\gamma^\mu\partial_\mu\psi_{\mathrm{L}} = 0 , \quad \mathrm{i}\gamma^\mu\partial_\mu\psi_{\mathrm{R}} = 0 . \tag{2.34}$$

Weyl Spinor

Since massless fermions have to travel at the speed of light, there exists no boosted reference frame in which the helicity can be changed. Hence either $\nu_{\rm L}$ or $\bar{\nu}_{\rm R}$ can be used to construct a Lorentz-invariant theory, such as the SM.

Majorana Spinor

In 1937, Ettore Majorana tried to simplify the Dirac equation and retain only its positive-energy solution. This can be accomplished by imposing the so-called Majorana condition on the Dirac spinor, i.e.,

$$\psi^{\rm C} \equiv C\overline{\psi}^{\rm T} = \psi \; , \tag{2.35}$$

where $C = {\rm i}\gamma^2\gamma^0$ and $\overline{\psi} \equiv \psi^\dagger\gamma^0$ have been defined.

Majorana Neutrinos

Only electrically neutral particles can be Majorana particles.

- Dirac neutrinos can be massive, however, only $\nu_{\rm L}$ and $\bar{\nu}_{\rm R}$ take part in weak interactions. Neutrinos and antineutrinos are discriminated by their lepton number.
- Weyl neutrinos should be massless, so $\nu_{\rm L}$ and $\bar{\nu}_{\rm R}$ are independent components and no other degrees of freedom are needed. Neutrinos and antineutrinos differ by their lepton number or helicities.

Massive Majorana Spinor

If $\nu^{\rm C} = \nu$ is satisfied, the lepton number is NOT a good quantum number. Though neutrinos are massive, only two degrees of freedom are needed:

$$\nu_{\rm L} \; , \quad \nu_{\rm R} = \bar{\nu}_{\rm R} \; . \tag{2.36}$$

Type-I Seesaw Model

Extend the SM with three right-handed neutrinos but without lepton number conservation. The gauge-invariant Lagrangian is

$$\mathcal{L}_{\rm SS} = \mathcal{L}_{\rm SM} + \overline{N_\alpha} {\rm i}\gamma^\mu \partial_\mu N_\alpha - \left[\overline{L_\alpha} \left(Y_\nu\right)_{\alpha\beta} \widetilde{H} N_\beta + \frac{1}{2} \overline{N_\alpha^{\rm C}} \left(M_N\right)_{\alpha\beta} N_\beta + {\rm h.c.}\right] \; ,$$

where M_N is the Majorana mass matrix of right-handed neutrinos.

Theorem

The Majorana mass matrix M_N should be symmetric, i.e., $M_N^{\rm T} = M_N$.

$$\begin{aligned} \left[\overline{N_\alpha^{\rm C}} \left(M_N\right)_{\alpha\beta} N_\beta\right]^{\rm T} &= -N_\beta^{\rm T} \left(M_N\right)_{\alpha\beta} \overline{N_\alpha^{\rm C}}^{\rm T} \\ &= -N_\beta^{\rm T} C^{-1} \left(M_N\right)_{\alpha\beta} C \overline{N_\alpha^{\rm C}}^{\rm T} = \overline{N_\beta^{\rm C}} \left(M_N\right)_{\alpha\beta} N_\alpha \; . \end{aligned}$$

Properties of the C Matrix

The charge-conjugation matrix $C = \mathrm{i}\gamma^2\gamma^0$ has the following properties:

$$C^{-1} = -\mathrm{i}(\gamma^0)^{-1}(\gamma^2)^{-1} = \mathrm{i}\gamma^0\gamma^2 = -C = C^{\mathrm{T}} \; , \tag{2.37}$$

together with $C\gamma^\mu C^{-1} = -(\gamma^\mu)^{\mathrm{T}}$ and $C\gamma_5 = \gamma_5 C$.

Properties of the Mass Terms

The Dirac and Majorana mass terms are respectively written as

$$\overline{\psi_{\mathrm{L}}} m \psi_{\mathrm{R}} + \overline{\psi_{\mathrm{R}}} m \psi_{\mathrm{L}} = \overline{\psi} m \psi \; , \tag{2.38}$$

$$\frac{1}{2}\overline{\psi^{\mathrm{C}}_{\mathrm{R}}} m \psi_{\mathrm{R}} + \frac{1}{2}\overline{\psi_{\mathrm{R}}} m \psi^{\mathrm{C}}_{\mathrm{R}} = \frac{1}{2}\overline{\psi} m \psi \; , \tag{2.39}$$

where $\psi = \psi_{\mathrm{L}} + \psi_{\mathrm{R}}$ (Dirac) and $\psi = \psi^{\mathrm{C}}_{\mathrm{R}} + \psi_{\mathrm{R}} = \psi^{\mathrm{C}}$ (Majorana).

Why 1/2?

For a free Majorana fermion, the Lagrangian is

$$\begin{aligned}\mathcal{L}_{\rm M} &= \overline{\nu_{\rm R}}{\rm i}\gamma^\mu\partial_\mu\nu_{\rm R} - \left[\frac{1}{2}\overline{\nu_{\rm R}^{\rm C}}m\nu_{\rm R} + {\rm h.c.}\right] \\ &= \frac{1}{2}\overline{\nu_{\rm R}}{\rm i}\gamma^\mu\partial_\mu\nu_{\rm R} + \frac{1}{2}\overline{\nu_{\rm R}^{\rm C}}{\rm i}\gamma^\mu\partial_\mu\nu_{\rm R}^{\rm C} - \frac{1}{2}\left[\overline{\nu_{\rm R}^{\rm C}}m\nu_{\rm R} + \overline{\nu_{\rm R}}m\nu_{\rm R}^{\rm C}\right] \\ &= \frac{1}{2}\overline{\nu}{\rm i}\gamma^\mu\partial_\mu\nu - \frac{1}{2}m\overline{\nu}\nu \; , \end{aligned} \tag{2.40}$$

where $\nu \equiv \nu_{\rm R}^{\rm C} + \nu_{\rm R} = \nu^{\rm C}$ is a Majorana spinor.

Hints:

$$\nu_{\rm R}^{\rm C} \equiv (\nu_{\rm R})^{\rm C} = C\overline{\nu_{\rm R}}^{\rm T} = C\left[(P_{\rm R}\nu)^\dagger\gamma^0\right]^{\rm T} = P_{\rm L}\nu^{\rm C} \; , \tag{2.41}$$

where $\gamma^0 P_{\rm L/R} = P_{\rm R/L}\gamma^0$ and $CP_{\rm L/R} = P_{\rm L/R}C$.

Why 1/2?

Notice that

$$\overline{(\nu_{\rm R}+\nu_{\rm R}^{\rm C})}{\rm i}\gamma^\mu\partial_\mu(\nu_{\rm R}+\nu_{\rm R}^{\rm C}) = \overline{\nu_{\rm R}}{\rm i}\gamma^\mu\partial_\mu\nu_{\rm R}+\overline{\nu_{\rm R}^{\rm C}}{\rm i}\gamma^\mu\partial_\mu\nu_{\rm R}^{\rm C}\,, \quad (2.42)$$

$$\overline{(\nu_{\rm R}+\nu_{\rm R}^{\rm C})}(\nu_{\rm R}+\nu_{\rm R}^{\rm C}) = \overline{\nu_{\rm R}^{\rm C}}\nu_{\rm R}+\overline{\nu_{\rm R}}\nu_{\rm R}^{\rm C}\,, \quad (2.43)$$

where $P_{\rm L}P_{\rm R}=P_{\rm R}P_{\rm L}=0$, $P_{\rm L}^2=P_{\rm L}$ and $P_{\rm R}^2=P_{\rm R}$ have been used; and

$$\begin{aligned}\left[\overline{\nu_{\rm R}}{\rm i}\gamma^\mu\partial_\mu\nu_{\rm R}\right]^{\rm T} &= -\left(\partial_\mu\nu_{\rm R}\right)^{\rm T}(\gamma^\mu)^{\rm T}\overline{\nu_{\rm R}}^{\rm T}\\ &= -\left(\partial_\mu\nu_{\rm R}\right)^{\rm T}C^{-1}C(\gamma^\mu)^{\rm T}C^{-1}C\overline{\nu_{\rm R}}^{\rm T}\\ &= -{\rm i}\left(\partial_\mu\overline{\nu_{\rm R}^{\rm C}}\right)\gamma^\mu\nu_{\rm R}^{\rm C}\\ &= \overline{\nu_{\rm R}^{\rm C}}{\rm i}\gamma^\mu\partial_\mu\nu_{\rm R}^{\rm C}-{\rm i}\partial_\mu\left(\overline{\nu_{\rm R}^{\rm C}}\gamma^\mu\nu_{\rm R}^{\rm C}\right)\,. \quad (2.44)\end{aligned}$$

Why 1/2?

Derive Dirac equation from the Lagrangian for a free Dirac fermion, i.e.,

$$\mathcal{L}_{\rm D} = \overline{\nu} {\rm i} \gamma^\mu \partial_\mu \nu - m \overline{\nu} \nu = -{\rm i} (\partial_\mu \overline{\nu}) \gamma^\mu \nu - m \overline{\nu} \nu \; . \tag{2.45}$$

From the Euler-Lagrange equation, we have

$$\partial_\mu \left[\frac{\delta \mathcal{L}_{\rm D}}{\delta (\partial_\mu \overline{\nu})} \right] - \frac{\delta \mathcal{L}_{\rm D}}{\delta \overline{\nu}} = - \left({\rm i} \gamma^\mu \partial_\mu - m \right) \nu = 0 \; . \tag{2.46}$$

Now do the same thing for the Lagrangian of a free Majorana fermion

$$\begin{aligned} \mathcal{L}_{\rm M} &= \frac{1}{2} \overline{\nu} {\rm i} \gamma^\mu \partial_\mu \nu - \frac{1}{2} m \overline{\nu} \nu = \frac{1}{2} \overline{\nu^{\rm C}} {\rm i} \gamma^\mu \partial_\mu \nu - \frac{1}{2} m \overline{\nu^{\rm C}} \nu \\ &= \frac{1}{2} \overline{\nu^{\rm C}} {\rm i} \gamma^\mu \partial_\mu \nu - \frac{1}{2} m \overline{\nu^{\rm C}} \nu = \frac{1}{2} \nu^{\rm T} C {\rm i} \gamma^\mu \partial_\mu \nu - \frac{1}{2} m \nu^{\rm T} C \nu \; . \end{aligned} \tag{2.47}$$

Majorana Representation

In the Majorana representation, the Dirac matrices turn out to be

$$\gamma^0 = \begin{pmatrix} 0 & \sigma^2 \\ \sigma^2 & 0 \end{pmatrix} , \qquad \gamma^1 = \begin{pmatrix} \mathrm{i}\sigma^3 & 0 \\ 0 & \mathrm{i}\sigma^3 \end{pmatrix} , \tag{2.48}$$

$$\gamma^2 = \begin{pmatrix} 0 & -\sigma^2 \\ \sigma^2 & 0 \end{pmatrix} , \qquad \gamma^3 = \begin{pmatrix} -\mathrm{i}\sigma^1 & 0 \\ 0 & -\mathrm{i}\sigma^1 \end{pmatrix} , \tag{2.49}$$

which are purely imaginary. The definition of charge-conjugation matrix becomes $C = -\gamma^0$, and thus we have

$$\psi^{\mathrm{C}} \equiv C\overline{\psi}^{\mathrm{T}} = -\gamma^0(\gamma^0)^{\mathrm{T}}\psi^* = \psi^* , \tag{2.50}$$

implying that the Majorana field in the Majorana representation is real.

Neutrino Masses

Let us come back to the type-I seesaw model. After the SSB, we obtain the neutrino mass terms

$$\mathcal{L}^{\nu}_{\rm mass} = -\overline{\nu_{\alpha \rm L}} \left(M_{\rm D}\right)_{\alpha\beta} \nu_{\beta \rm R} - \frac{1}{2} \overline{\nu^{\rm C}_{\alpha \rm R}} \left(M_N\right)_{\alpha\beta} \nu_{\beta \rm R} + {\rm h.c.} \; , \tag{2.51}$$

where the Dirac neutrino mass matrix is $M_{\rm D} = Y_\nu v/\sqrt{2}$. One can prove that the mass terms can be collectively written as

$$\begin{aligned} \mathcal{L}^{\nu}_{\rm mass} &= -\frac{1}{2} \overline{\left(\nu_{\rm L} \quad \nu^{\rm C}_{\rm R}\right)} \begin{pmatrix} 0 & M_{\rm D} \\ M^{\rm T}_{\rm D} & M_N \end{pmatrix} \begin{pmatrix} \nu^{\rm C}_{\rm L} \\ \nu_{\rm R} \end{pmatrix} + {\rm h.c.} \\ &= -\frac{1}{2} \overline{\chi_{\rm L}} \mathcal{M} \chi^{\rm C}_{\rm L} + {\rm h.c.} \; , \end{aligned} \tag{2.52}$$

where $\chi_{\rm L} \equiv \left(\nu_{\rm L}, \nu^{\rm C}_{\rm R}\right)^{\rm T}$ and $\mathcal{M}^{\rm T} = \mathcal{M}$ can be easily identified.

Neutrino Masses

Hints:

$$\begin{aligned} \overline{\nu_{\rm L}} M_{\rm D} \nu_{\rm R} &= \left[\overline{\nu_{\rm L}} M_{\rm D} \nu_{\rm R}\right]^{\rm T} \\ &= -\nu_{\rm R}^{\rm T} M_{\rm D}^{\rm T} \overline{\nu_{\rm L}}^{\rm T} \\ &= -\nu_{\rm R}^{\rm T} C^{-1} M_{\rm D}^{\rm T} C \overline{\nu_{\rm L}}^{\rm T} \\ &= \overline{\nu_{\rm R}^{\rm C}} M_{\rm D}^{\rm T} \nu_{\rm L}^{\rm C} \; . \end{aligned} \tag{2.53}$$

In order to get neutrino masses, we have to diagonalize the overall 6×6 complex and symmetric matrix $\mathcal{M}$ via

$$\mathcal{U}^\dagger \mathcal{M} \mathcal{U}^* = \begin{pmatrix} V & R \\ S & U \end{pmatrix}^\dagger \begin{pmatrix} 0 & M_{\rm D} \\ M_{\rm D}^{\rm T} & M_N \end{pmatrix} \begin{pmatrix} V & R \\ S & U \end{pmatrix}^* = \begin{pmatrix} \widehat{M}_\nu & 0 \\ 0 & \widehat{M}_N \end{pmatrix} \; . \tag{2.54}$$

Neutrino Masses

In the limit of $\mathcal{O}(M_N) \gg \mathcal{O}(M_{\rm D})$, the effective Majorana mass matrix of three light neutrinos is given by the so-called seesaw formula

$$M_\nu = V \widehat{M}_\nu V^{\rm T} \approx -M_{\rm D} M_N^{-1} M_{\rm D}^{\rm T} \; , \tag{2.55}$$

and neutrino masses are $\widehat{M}_\nu = {\rm Diag}\{m_1, m_2, m_3\}$. The effective mass term of light Majorana neutrinos is

$$\mathcal{L}^\nu_{\rm mass} = -\frac{1}{2} \overline{\nu_{\rm L}} M_\nu \nu^{\rm C}_{\rm L} + {\rm h.c.} \; . \tag{2.56}$$

As an rough estimation, the heavy Majorana neutrino masses are

$$\mathcal{O}(M_N) \sim \frac{\mathcal{O}(M_{\rm D}^2)}{\mathcal{O}(M_\nu)} > \frac{(200 \text{ GeV})^2}{(0.4 \text{ eV})} \sim 10^{14} \text{ GeV} \; . \tag{2.57}$$

Example

Consider the one-generation light and heavy neutrinos. The overall mass matrix is a 2×2 complex and symmetric, i.e.,

$$\mathcal{M} = \begin{pmatrix} 0 & m_{\rm D} \\ m_{\rm D} & m_N \end{pmatrix} \; , \tag{2.58}$$

where $m_{\rm D}$ and m_N are further assumed to be real and $m_N \gg m_{\rm D}$ holds. It is straightforward to diagonalize this real and symmetric matrix via

$$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}^{\rm T} \begin{pmatrix} 0 & m_{\rm D} \\ m_{\rm D} & m_N \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} -m_1 & 0 \\ 0 & m_2 \end{pmatrix} \; , \tag{2.59}$$

with $\tan 2\theta = 2m_{\rm D}/m_N$ (or $\theta \approx m_{\rm D}/m_N$), $m_1 \approx m_{\rm D}^2/m_N$ and $m_2 \approx m_N$.

Example

Look at the mass terms again

$$\begin{aligned}
& \frac{1}{2}\overline{\begin{pmatrix}\nu_{\rm L} & \nu_{\rm R}^{\rm C}\end{pmatrix}}\begin{pmatrix}0 & m_{\rm D}\\ m_{\rm D} & m_N\end{pmatrix}\begin{pmatrix}\nu_{\rm L}^{\rm C}\\ \nu_{\rm R}\end{pmatrix} \\
= & \frac{1}{2}\overline{\begin{pmatrix}\nu_{\rm L} & \nu_{\rm R}^{\rm C}\end{pmatrix}}\begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}-m_1 & 0\\ 0 & m_2\end{pmatrix}\begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}^{\rm T}\begin{pmatrix}\nu_{\rm L}^{\rm C}\\ \nu_{\rm R}\end{pmatrix} \\
= & \frac{1}{2}\overline{\begin{pmatrix}\widehat{\nu}_{\rm L} & \widehat{\nu}_{\rm R}^{\rm C}\end{pmatrix}}\begin{pmatrix}m_1 & 0\\ 0 & m_2\end{pmatrix}\begin{pmatrix}\widehat{\nu}_{\rm L}^{\rm C}\\ \widehat{\nu}_{\rm R}\end{pmatrix} \, ,
\end{aligned}$$

where

$$\begin{pmatrix}\widehat{\nu}_{\rm L}^{\rm C}\\ \widehat{\nu}_{\rm R}\end{pmatrix} \equiv \begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}i & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}\nu_{\rm L}^{\rm C}\\ \nu_{\rm R}\end{pmatrix} \, . \qquad (2.60)$$

Example

The total neutrino mass term reads

$$\mathcal{L}^{\nu}_{\rm mass} = -\frac{1}{2}\overline{\widehat{\nu}_1} m_1 \widehat{\nu}_1 - \frac{1}{2}\overline{\widehat{\nu}_2} m_2 \widehat{\nu}_2 \tag{2.61}$$

where

$$\widehat{\nu}_1 \equiv \widehat{\nu}_{\rm L} + \widehat{\nu}^{\rm C}_{\rm L} \; , \quad \widehat{\nu}_2 \equiv \widehat{\nu}^{\rm C}_{\rm R} + \widehat{\nu}_{\rm R} \; . \tag{2.62}$$

The leptonic CC interaction in the mass basis becomes

$$\begin{aligned} \mathcal{L}^{\rm l}_{\rm CC} &= \frac{g}{\sqrt{2}} \overline{\alpha_{\rm L}} \gamma^{\mu} \nu_{\rm L} W^{-}_{\mu} + {\rm h.c.} \\ &= \frac{g}{\sqrt{2}} \overline{\alpha_{\rm L}} \gamma^{\mu} \left[\widehat{\nu}_{1{\rm L}} \cos\theta + \widehat{\nu}_{2{\rm L}} \sin\theta \right] W^{-}_{\mu} + {\rm h.c.} \; . \end{aligned} \tag{2.63}$$

SM as an Effective Theory

Retain the particle content and gauge symmetry of the SM, but give up the renormalizability, which is not necessary for an effective theory below a cutoff scale Λ. The effective Lagrangian can be written as

$$\mathcal{L} = \mathcal{L}_{\rm SM} + \sum_i \frac{c_5^i}{\Lambda} \mathcal{O}_5^i + \sum_i \frac{c_6^i}{\Lambda^2} \mathcal{O}_6^i + \cdots \; , \tag{2.64}$$

where $\mathcal{O}_5^i$ and $\mathcal{O}_6^i$ stand for the operators of mass dimension five and six, respectively, with the dimensionless couplings c_5^i and c_6^i. In 1979, Steven Weinberg showed that there exists a unique operator of dimension five, i.e.,

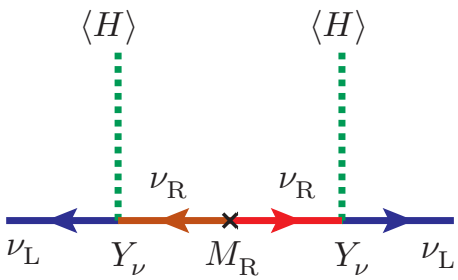
$$\mathcal{O}_5 = \left[\overline{L_\alpha} \widetilde{H}\right] \cdot \left[\widetilde{H}^{\rm T} L_\beta^{\rm C}\right] \; , \tag{2.65}$$

which gives rise to neutrino masses after the SSB.

Neutrino Mass Models

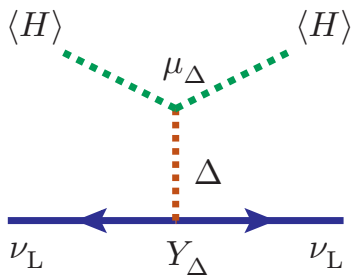
Canonical seesaw models

a



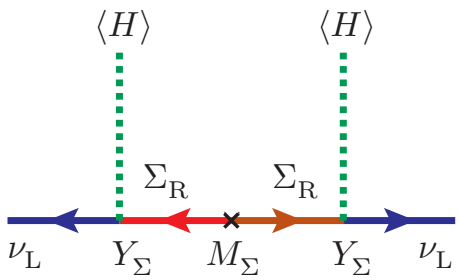
$$M_\nu = -\langle H\rangle^2 Y_\nu M_{\rm R}^{-1} Y_\nu^{\rm T}$$

b



$$M_\nu = \langle H\rangle^2 Y_\Delta \mu_\Delta / M_\Delta^2$$

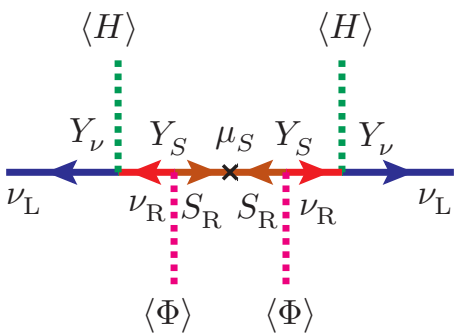
c



$$M_\nu = -\langle H\rangle^2 Y_\Sigma M_\Sigma^{-1} Y_\Sigma^{\rm T}$$

Inverse seesaw model

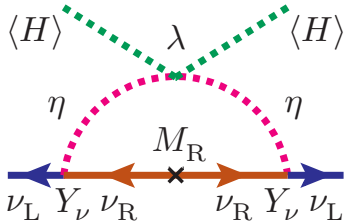
d



$$M_\nu = F \mu_S F^{\rm T}$$

The scotogenic model

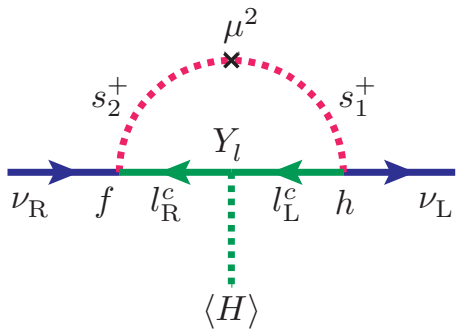
e



$$M_\nu = -\lambda \frac{\langle H\rangle^2}{16\pi^2} Y_\nu M_{\rm R}^{-1} Y_\nu^{\rm T}$$

Radiative Dirac model

f



$$M_\nu = \frac{h Y_l f}{16\pi^2} \langle H\rangle I(\mu^2, M_{s_1}^2, M_{s_2}^2)$$

Summary

- Current observational data from neutrino oscillations, tritium β decays, $0\nu\beta\beta$ decays and cosmology indicate that neutrino masses should be at the sub-eV level.
- Similar to charged fermions, neutrinos can acquire Dirac masses if three right-handed neutrino singlets are introduced into the SM.
- However, Dirac neutrino masses exaggerate the puzzle of strong fermion mass hierarchy, and require the global $\mathrm{U}(1)$ symmetry to forbid the Majorana mass term of right-handed neutrinos.
- Majorana neutrino masses appear in a more natural way, and can be tested in $0\nu\beta\beta$ decays and other types of experiments.

Quark Masses and Flavor Mixing

CC Interaction and Mass Terms

In terms of flavor or interaction eigenstates, we have

$$\mathcal{L}^{\rm q}_{\rm CC} = \frac{g}{\sqrt{2}} \overline{u_{i\rm L}} \gamma^\mu d_{i\rm L} W^+_\mu + {\rm h.c.} \; , \tag{4.1}$$

$$\mathcal{L}^{\rm q}_{\rm mass} = -\overline{u_{i\rm L}} \left(M_u\right)_{ij} u_{j\rm R} - \overline{d_{i\rm L}} \left(M_d\right)_{ij} d_{j\rm R} + {\rm h.c.} \; . \tag{4.2}$$

Now convert into the mass basis via the unitary transformation in the flavor basis:

$$M_u = V_{u\rm L} \widehat{M}_u V^\dagger_{u\rm R} \; , \quad M_d = V_{d\rm L} \widehat{M}_d V^\dagger_{d\rm R} \; , \tag{4.3}$$

and rewrite the mass terms

$$\begin{aligned} \mathcal{L}^{\rm q}_{\rm mass} &= -\overline{u_{i\rm L}} \left(V_{u\rm L} \widehat{M}_u V^\dagger_{u\rm R}\right)_{ij} u_{j\rm R} - \overline{d_{i\rm L}} \left(V_{d\rm L} \widehat{M}_d V^\dagger_{d\rm R}\right)_{ij} d_{j\rm R} + {\rm h.c.} \; , \\ &= -\overline{\widehat{u}_{i\rm L}} m_{u_i} \delta_{ij} \widehat{u}_{j\rm R} - \overline{\widehat{d}_{i\rm L}} m_{d_i} \delta_{ij} \widehat{d}_{j\rm R} + {\rm h.c.} \end{aligned} \tag{4.4}$$

CC Interaction and Mass Terms

Note that $\widehat{M}_u = \mathrm{Diag}\{m_u, m_c, m_t\}$ and $\widehat{M}_d = \mathrm{Diag}\{m_d, m_s, m_b\}$, and the mass eigenstates are defined as

$$\widehat{u}_{i\mathrm{L}} = (V_{u\mathrm{L}}^{\dagger})_{ij} u_{j\mathrm{L}} \; , \quad \widehat{u}_{i\mathrm{R}} = (V_{u\mathrm{R}}^{\dagger})_{ij} u_{j\mathrm{R}} \; , \quad \widehat{u}_i = \widehat{u}_{i\mathrm{L}} + \widehat{u}_{i\mathrm{R}} \; ; \tag{4.5}$$

$$\widehat{d}_{i\mathrm{L}} = (V_{d\mathrm{L}}^{\dagger})_{ij} d_{j\mathrm{L}} \; , \quad \widehat{d}_{i\mathrm{R}} = (V_{d\mathrm{R}}^{\dagger})_{ij} d_{j\mathrm{R}} \; , \quad \widehat{d}_i = \widehat{d}_{i\mathrm{L}} + \widehat{d}_{i\mathrm{R}} \; . \tag{4.6}$$

In the mass basis, the CC interaction appears to be

$$\begin{aligned} \mathcal{L}_{\mathrm{CC}}^{\mathrm{q}} &= \frac{g}{\sqrt{2}} \overline{u_{i\mathrm{L}}} \gamma^{\mu} d_{i\mathrm{L}} W_{\mu}^{+} + \mathrm{h.c.} \; , \\ &= \frac{g}{\sqrt{2}} \overline{\widehat{u}_{i\mathrm{L}}} \gamma^{\mu} U_{ij} \widehat{d}_{j\mathrm{L}} W_{\mu}^{+} + \mathrm{h.c.} \; , \end{aligned} \tag{4.7}$$

where the CKM matrix $U = V_{u\mathrm{L}}^{\dagger} V_{d\mathrm{L}}$ is a 3×3 unitary matrix.

CC Interaction and Mass Terms (Dirac)

If massive neutrinos are Dirac particles, we have

$$\mathcal{L}^{\rm l}_{\rm CC} = \frac{g}{\sqrt{2}} \overline{\alpha_{\rm L}} \gamma^\mu \nu_{\alpha {\rm L}} W^-_\mu + {\rm h.c.} \; , \tag{4.8}$$

$$\mathcal{L}^{\rm l}_{\rm mass} = -\overline{\alpha_{\rm L}} \left(M_l\right)_{\alpha\beta} \beta_{\rm R} - \overline{\nu_{\alpha {\rm L}}} \left(M_\nu\right)_{\alpha\beta} \nu_{\beta {\rm R}} + {\rm h.c.} \; . \tag{4.9}$$

Now convert into the mass basis via the unitary transformation in the flavor basis:

$$M_l = V_{l{\rm L}} \widehat{M}_l V^\dagger_{l{\rm R}} \; , \quad M_\nu = V_{\nu{\rm L}} \widehat{M}_\nu V^\dagger_{\nu{\rm R}} \; , \tag{4.10}$$

and rewrite the mass terms

$$\begin{aligned} \mathcal{L}^{\rm l}_{\rm mass} &= -\overline{\alpha_{\rm L}} \left(V_{l{\rm L}} \widehat{M}_l V^\dagger_{l{\rm R}} \right)_{\alpha\beta} \beta_{\rm R} - \overline{\nu_{\alpha {\rm L}}} \left(V_{\nu{\rm L}} \widehat{M}_\nu V^\dagger_{\nu{\rm R}} \right)_{\alpha\beta} \nu_{\beta {\rm R}} + {\rm h.c.} \; , \\ &= -\overline{\widehat{\alpha}_{\rm L}} m_\alpha \delta_{\alpha\beta} \widehat{\beta}_{\rm R} - \overline{\widehat{\nu}_{i{\rm L}}} m_i \delta_{ij} \widehat{\nu}_{j{\rm R}} + {\rm h.c.} \end{aligned} \tag{4.11}$$

CC Interaction and Mass Terms (Dirac)

Note that $\widehat{M}_l = \mathrm{Diag}\{m_e, m_\mu, m_\tau\}$ and $\widehat{M}_\nu = \mathrm{Diag}\{m_1, m_2, m_3\}$, and the mass eigenstates are defined as

$$\widehat{\alpha}_{\rm L} = (V_{l{\rm L}}^\dagger)_{\alpha\beta}\beta_{\rm L}\;, \quad \widehat{\alpha}_{\rm R} = (V_{l{\rm R}}^\dagger)_{\alpha\beta}\beta_{\rm R}\;, \quad \widehat{\alpha} = \widehat{\alpha}_{\rm L} + \widehat{\alpha}_{\rm R}\;; \tag{4.12}$$

$$\widehat{\nu}_{i{\rm L}} = (V_{\nu{\rm L}}^\dagger)_{i\beta}\nu_{\beta{\rm L}}\;, \quad \widehat{\nu}_{i{\rm R}} = (V_{\nu{\rm R}}^\dagger)_{i\beta}\nu_{\beta{\rm R}}\;, \quad \widehat{\nu}_i = \widehat{\nu}_{i{\rm L}} + \widehat{\nu}_{i{\rm R}}\;. \tag{4.13}$$

In the mass basis, the CC interaction appears to be

$$\begin{aligned} \mathcal{L}_{\rm CC}^{\rm l} &= \frac{g}{\sqrt{2}}\overline{\alpha_{\rm L}}\gamma^\mu \nu_{\alpha{\rm L}} W_\mu^- + {\rm h.c.}\;, \\ &= \frac{g}{\sqrt{2}}\overline{\widehat{\alpha}_{\rm L}}\gamma^\mu V_{\alpha i}\widehat{\nu}_{i{\rm L}} W_\mu^- + {\rm h.c.}\;, \end{aligned} \tag{4.14}$$

where the MNS matrix $V = V_{l{\rm L}}^\dagger V_{\nu{\rm L}}$ is a 3×3 unitary matrix.

CC Interaction and Mass Terms (Majorana)

If massive neutrinos are Majorana particles, we have

$$\mathcal{L}^{\rm l}_{\rm CC} = \frac{g}{\sqrt{2}} \overline{\alpha_{\rm L}} \gamma^\mu \nu_{\alpha {\rm L}} W^-_\mu + {\rm h.c.} \; , \tag{4.15}$$

$$\mathcal{L}^{\rm l}_{\rm mass} = -\overline{\alpha_{\rm L}} \left(M_l\right)_{\alpha\beta} \beta_{\rm R} - \frac{1}{2} \overline{\nu_{\alpha {\rm L}}} \left(M_\nu\right)_{\alpha\beta} \nu^{\rm C}_{\beta {\rm L}} + {\rm h.c.} \; . \tag{4.16}$$

Go to the mass basis via the unitary transformation in the flavor basis:

$$M_l = V_{l{\rm L}} \widehat{M}_l V^\dagger_{l{\rm R}} \; , \quad M_\nu = V_{\nu {\rm L}} \widehat{M}_\nu V^{\rm T}_{\nu {\rm L}} \; , \tag{4.17}$$

and rewrite the mass terms

$$\begin{aligned} \mathcal{L}^{\rm l}_{\rm mass} &= -\overline{\alpha_{\rm L}} \left(V_{l{\rm L}} \widehat{M}_l V^\dagger_{l{\rm R}} \right)_{\alpha\beta} \beta_{\rm R} - \frac{1}{2} \overline{\nu_{\alpha {\rm L}}} \left(V_{\nu {\rm L}} \widehat{M}_\nu V^{\rm T}_{\nu {\rm L}} \right)_{\alpha\beta} \nu^{\rm C}_{\beta {\rm L}} + {\rm h.c.} \; , \\ &= -\overline{\widehat{\alpha}_{\rm L}} m_\alpha \delta_{\alpha\beta} \widehat{\beta}_{\rm R} - \frac{1}{2} \overline{\widehat{\nu}_{i {\rm L}}} m_i \delta_{ij} \widehat{\nu}^{\rm C}_{j {\rm L}} + {\rm h.c.} \end{aligned} \tag{4.18}$$

CC Interaction and Mass Terms (Majorana)

Note that $\widehat{M}_l = \mathrm{Diag}\{m_e, m_\mu, m_\tau\}$ and $\widehat{M}_\nu = \mathrm{Diag}\{m_1, m_2, m_3\}$, and the mass eigenstates are defined as

$$\widehat{\alpha}_{\rm L} = (V_{l{\rm L}}^\dagger)_{\alpha\beta}\beta_{\rm L} \; , \quad \widehat{\alpha}_{\rm R} = (V_{l{\rm R}}^\dagger)_{\alpha\beta}\beta_{\rm R} \; , \quad \widehat{\alpha} = \widehat{\alpha}_{\rm L} + \widehat{\alpha}_{\rm R} \; ; \tag{4.19}$$

$$\widehat{\nu}_{i{\rm L}} = (V_{\nu{\rm L}}^\dagger)_{i\beta}\nu_{\beta{\rm L}} \; , \quad \widehat{\nu}_{i{\rm L}}^{\rm C} = (V_{\nu{\rm R}}^{\rm T})_{i\beta}\nu_{\beta{\rm L}}^{\rm C} \; , \quad \widehat{\nu}_i = \widehat{\nu}_{i{\rm L}} + \widehat{\nu}_{i{\rm L}}^{\rm C} \; . \tag{4.20}$$

In the mass basis, the CC interaction appears to be

$$\begin{aligned} \mathcal{L}_{\rm CC}^{\rm l} &= \frac{g}{\sqrt{2}} \overline{\alpha_{\rm L}} \gamma^\mu \nu_{\alpha{\rm L}} W_\mu^- + {\rm h.c.} \; , \\ &= \frac{g}{\sqrt{2}} \overline{\widehat{\alpha}_{\rm L}} \gamma^\mu V_{\alpha i} \widehat{\nu}_{i{\rm L}} W_\mu^- + {\rm h.c.} \; , \end{aligned} \tag{4.21}$$

where the PMNS matrix $V = V_{l{\rm L}}^\dagger V_{\nu{\rm L}}$ is a 3×3 unitary matrix.

Parametrization of Mixing Matrix

Two Generations

Assume that there are only two generations of quarks in the SM, and the quark matrices are given by

$$M_u = \begin{pmatrix} m_u & 0 \\ 0 & m_c \end{pmatrix} , \quad M_d = \begin{pmatrix} 0 & a \\ a^* & b \end{pmatrix} , \tag{4.22}$$

where $a = |a| e^{\mathrm{i}\varphi}$ is complex and b is real with $b \gg |a|$. To find out the CKM matrix, we need to diagonalize the Hermitian matrix M_d. More explicitly, we have

$$\begin{aligned} \widetilde{M}_d &\equiv \begin{pmatrix} e^{-\mathrm{i}\varphi} & 0 \\ 0 & 1 \end{pmatrix} M_d \begin{pmatrix} e^{\mathrm{i}\varphi} & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & |a| \\ |a| & b \end{pmatrix} \\ &\approx \begin{pmatrix} c_\theta & s_\theta \\ -s_\theta & c_\theta \end{pmatrix} \begin{pmatrix} |a|^2/b & 0 \\ 0 & b \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} c_\theta & -s_\theta \\ s_\theta & c_\theta \end{pmatrix} . \end{aligned} \tag{4.23}$$

Parametrization of Mixing Matrix

Two Generations

The mass eigenvalues are given by

$$m_d \approx |a|^2/b \,, \quad m_s \approx b \,, \tag{4.24}$$

while the rotation angle is $\theta \approx |a|/b$. We can identify $V_{u\mathrm{L}} = V_{u\mathrm{R}} = \mathbf{1}$ and

$$V_{d\mathrm{L}} = \begin{pmatrix} e^{\mathrm{i}\varphi} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} c_\theta & s_\theta \\ -s_\theta & c_\theta \end{pmatrix} \,, \tag{4.25}$$

$$V_{d\mathrm{R}} = \begin{pmatrix} e^{\mathrm{i}\varphi} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} c_\theta & s_\theta \\ -s_\theta & c_\theta \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \,, \tag{4.26}$$

leading to the CKM matrix

$$U = V_{u\mathrm{L}}^\dagger V_{d\mathrm{L}} = \begin{pmatrix} e^{\mathrm{i}\varphi} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} c_\theta & s_\theta \\ -s_\theta & c_\theta \end{pmatrix} \,. \tag{4.27}$$

Parametrization of Mixing Matrix

Two Generations

The mixing angle is just the Cabibbo angle, i.e.,

$$|U_{us}| = \sin\theta \approx \frac{|a|}{b} = \sqrt{\frac{|a|^2}{b^2}} = \sqrt{\frac{m_d}{m_s}} \; . \tag{4.28}$$

Input the running masses $m_d = 2.69\ \mathrm{MeV}$ and $m_s = 53.5\ \mathrm{MeV}$ at the Fermi scale $M_Z = 91.2\ \mathrm{GeV}$, then we get

$$\sin\theta = \sqrt{\frac{2.69}{53.5}} \approx 0.22423 \; , \tag{4.29}$$

which is perfectly consistent with the experimental value $\sin\theta_{\rm C} = 0.22453$.

Note that φ is the unphysical phase, which can be absorbed by redefining the phases of quark fields (e.g., $\widehat{u}_i \to e^{{\rm i}\varphi}\widehat{u}_i$).

Parametrization of Mixing Matrix

Two Generations

Another example for quark matrices is to take

$$M_u = \begin{pmatrix} 0 & a_u \\ a_u^* & b_u \end{pmatrix} , \quad M_d = \begin{pmatrix} 0 & a_d \\ a_d^* & b_d \end{pmatrix} , \tag{4.30}$$

where $a_{u,d} = |a_{u,d}| e^{\mathrm{i}\varphi_{u,d}}$ and $b_{u,d} \gg |a_{u,d}|$. The quark masses are

$$m_u = |a_u|^2 / b_u , \quad m_c = b_u , \quad m_d = |a_d|^2 / b_d , \quad m_s = b_d ; \tag{4.31}$$

and the unitary matrices are

$$V_{u,d\mathrm{L}} = \begin{pmatrix} e^{\mathrm{i}\varphi_{u,d}} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\theta_{u,d} & \sin\theta_{u,d} \\ -\sin\theta_{u,d} & \cos\theta_{u,d} \end{pmatrix} , \tag{4.32}$$

$$V_{u,d\mathrm{R}} = \begin{pmatrix} e^{\mathrm{i}\varphi_{u,d}} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\theta_{u,d} & \sin\theta_{u,d} \\ -\sin\theta_{u,d} & \cos\theta_{u,d} \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} . \tag{4.33}$$

Parametrization of Mixing Matrix

Two Generations

Notice that $\theta_{u,d} \approx |a_{u,d}|/b_{u,d}$ are related to the quark mass ratios:

$$\sin\theta_u \approx \frac{|a_u|}{b_u} = \sqrt{\frac{m_u}{m_c}}\,, \quad \sin\theta_d \approx \frac{|a_d|}{b_d} = \sqrt{\frac{m_d}{m_s}}\,. \tag{4.34}$$

The CKM matrix can be calculated immediately

$$U = V_{u\mathrm{L}}^{\dagger} V_{d\mathrm{L}} = \begin{pmatrix} \cos\theta_u & -\sin\theta_u \\ \sin\theta_u & \cos\theta_u \end{pmatrix} \begin{pmatrix} e^{\mathrm{i}(\varphi_d - \varphi_u)} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\theta_d & \sin\theta_d \\ -\sin\theta_d & \cos\theta_d \end{pmatrix}$$

and we get

$$|U_{us}| \approx \left|\sin\theta_d - e^{\mathrm{i}\phi}\sin\theta_u\right| = \left|\sqrt{\frac{m_d}{m_s}} - e^{\mathrm{i}\phi}\sqrt{\frac{m_u}{m_c}}\right|\,, \tag{4.35}$$

where $\phi \equiv \varphi_u - \varphi_d$ is the relevant phase difference.

Parametrization of Mixing Matrix

Two Generations

Let us input running quark masses $m_u = 1.24\ \mathrm{MeV}$ and $m_c = 0.62\ \mathrm{GeV}$ at M_Z, and assume $\phi = \pm\pi/2$. Then we can obtain

$$|U_{us}| \approx \left| \sqrt{\frac{m_d}{m_s}} \pm \mathrm{i} \sqrt{\frac{m_u}{m_c}} \right| = \sqrt{\frac{m_d}{m_s} + \frac{m_u}{m_c}} \approx 0.22865 \; , \tag{4.36}$$

which is slightly larger than the experimental value $|U_{us}| = 0.22453$.

Theorem

1. An arbitrary complex matrix M can be decomposed into $M = H \cdot U$, where H is Hermitian and U is unitary (Polar Decomposition Theorem).
2. An arbitrary complex matrix M can be diagonalized by two unitary matrices $V_{\mathrm{L,R}}$ via $V_{\mathrm{L}}^{\dagger} M V_{\mathrm{R}} = \widehat{M}$.
3. An arbitrary complex and symmetric matrix M can be diagonalized by one unitary matrix $V_{\mathrm{L}}^{\dagger} M V_{\mathrm{L}}^{*} = \widehat{M}$.

Parametrization of Mixing Matrix

Example

For a 2×2 unitary matrix U, how many degrees of freedom in terms of real parameters? Take the most general form

$$U = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{4.37}$$

where a, b, c, d are complex numbers. Impose the unitarity conditions $U^\dagger U = U U^\dagger = 1$, namely,

$$|a|^2 + |b|^2 = 1 \; , \quad |c|^2 + |d|^2 = 1 \; , \quad a^* c + b^* d = 0 \; , \tag{4.38}$$

reducing the number of real parameters from eight to four. Now we adopt the following parametrization

$$a = e^{\mathrm{i}\varphi_a} \cos\theta \; , \qquad b = e^{\mathrm{i}\varphi_b} \sin\theta \; , \tag{4.39}$$

$$c = e^{\mathrm{i}\varphi_c} \cos\vartheta \; , \qquad d = e^{\mathrm{i}\varphi_d} \sin\vartheta \; . \tag{4.40}$$

Parametrization of Mixing Matrix

Example

The first two identities in Eq. (1.38) are fulfilled automatically, so we need to examine the last one:

$$\cos\theta\cos\vartheta + e^{\mathrm{i}(\varphi_d+\varphi_a-\varphi_b-\varphi_c)}\sin\theta\sin\vartheta = 0 \; ,$$

whose solutions can be easily found

$$\left\{ \begin{array}{ll} \theta = +\vartheta + (2m+1)\pi/2 \; , & \varphi_d + \varphi_a - \varphi_b - \varphi_c = 2n\pi \quad ; \\ \theta = -\vartheta + (2m+1)\pi/2 \; , & \varphi_d + \varphi_a - \varphi_b - \varphi_c = (2n+1)\pi \quad . \end{array} \right. \tag{4.41}$$

Choose the first solution $\theta = \vartheta - \pi/2$ and $\varphi_a - \varphi_b = \varphi_c - \varphi_d \equiv \alpha$, and rewrite the unitary matrix as

$$U = \begin{pmatrix} e^{\mathrm{i}\varphi_b} & 0 \\ 0 & e^{\mathrm{i}\varphi_d} \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} e^{\mathrm{i}\alpha} & 0 \\ 0 & 1 \end{pmatrix} \; . \tag{4.42}$$

Parametrization of Mixing Matrix

Standard Parametrization: two-flavor case

In the two-flavor case, we have the most general mixing matrix

$$U = \begin{pmatrix} e^{\mathrm{i}\varphi_b} & 0 \\ 0 & e^{\mathrm{i}\varphi_d} \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} e^{\mathrm{i}\alpha} & 0 \\ 0 & 1 \end{pmatrix} , \tag{4.43}$$

and the CC interaction

$$\mathcal{L}^{\mathrm{q}}_{\mathrm{CC}} = \frac{g}{\sqrt{2}} \overline{\begin{pmatrix} \widehat{u}_{\mathrm{L}} & \widehat{c}_{\mathrm{L}} \end{pmatrix}} \gamma^{\mu} \begin{pmatrix} U_{ud} & U_{us} \\ U_{cd} & U_{cs} \end{pmatrix} \begin{pmatrix} \widehat{d}_{\mathrm{L}} \\ \widehat{s}_{\mathrm{L}} \end{pmatrix} W^{+}_{\mu} + \mathrm{h.c.} . \tag{4.44}$$

Redefining the phases of quark fields $\widehat{u} \to \widehat{u} e^{\mathrm{i}\varphi_b}$, $\widehat{c} \to \widehat{c} e^{\mathrm{i}\varphi_d}$, $\widehat{d} \to \widehat{d} e^{-\mathrm{i}\alpha}$, we have

$$U = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} . \tag{4.45}$$

Parametrization of Mixing Matrix

Standard Parametrization: three-flavor case

The 3×3 orthogonal matrix by three Euler angles $\{\theta, \omega, \vartheta\}$, namely,

$$O = R_{23}(\theta) \cdot R_{13}(\omega) \cdot R_{12}(\vartheta) \; , \tag{4.46}$$

where

$$R_{23}(\theta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_\theta & s_\theta \\ 0 & -s_\theta & c_\theta \end{pmatrix} \; , \tag{4.47}$$

$$R_{13}(\omega) = \begin{pmatrix} c_\omega & 0 & s_\omega \\ 0 & 1 & 0 \\ -s_\omega & 0 & c_\omega \end{pmatrix} \; , \tag{4.48}$$

$$R_{12}(\vartheta) = \begin{pmatrix} c_\vartheta & s_\vartheta & 0 \\ -s_\vartheta & c_\vartheta & 0 \\ 0 & 0 & 1 \end{pmatrix} \; . \tag{4.49}$$

Parametrization of Mixing Matrix

Standard Parametrization: three-flavor case

For the 3×3 unitary matrix, we further introduce three phases, i.e.,

$$U = R_{23}(\theta, \alpha) \cdot R_{13}(\omega, \delta) \cdot R_{12}(\vartheta, \gamma) \; , \tag{4.50}$$

where

$$R_{23}(\theta, \alpha) = \begin{pmatrix} e^{-\mathrm{i}\alpha} & 0 & 0 \\ 0 & c_\theta & s_\theta \\ 0 & -s_\theta & c_\theta \end{pmatrix} \; , \tag{4.51}$$

$$R_{13}(\omega, \delta) = \begin{pmatrix} c_\omega & 0 & s_\omega \\ 0 & e^{-\mathrm{i}\delta} & 0 \\ -s_\omega & 0 & c_\omega \end{pmatrix} \; , \tag{4.52}$$

$$R_{12}(\vartheta, \gamma) = \begin{pmatrix} c_\vartheta & s_\vartheta & 0 \\ -s_\vartheta & c_\vartheta & 0 \\ 0 & 0 & e^{-\mathrm{i}\gamma} \end{pmatrix} \; . \tag{4.53}$$

Parametrization of Mixing Matrix

Standard Parametrization: three-flavor case

Calculate the unitary matrix explicitly

$$U = P \cdot V \cdot Q \tag{4.54}$$

where

$$P = \begin{pmatrix} e^{-\mathrm{i}\alpha} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} , \tag{4.55}$$

$$V = \begin{pmatrix} c_\omega c_\vartheta & c_\omega s_\vartheta & s_\omega \\ -s_\theta s_\omega c_\vartheta - c_\theta s_\vartheta e^{-\mathrm{i}\delta} & -s_\theta s_\omega s_\vartheta + c_\theta c_\vartheta e^{-\mathrm{i}\delta} & s_\theta c_\omega \\ -c_\theta s_\omega c_\vartheta + s_\theta s_\vartheta e^{-\mathrm{i}\delta} & -c_\theta s_\omega s_\vartheta - s_\theta c_\vartheta e^{-\mathrm{i}\delta} & c_\theta c_\omega \end{pmatrix} , \tag{4.56}$$

$$Q = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{-\mathrm{i}\gamma} \end{pmatrix} . \tag{4.57}$$

Parametrization of Mixing Matrix

Standard Parametrization: three-flavor case

Properly redefine the phases in each row and column

$$U = P' \cdot V' \cdot Q' \tag{4.58}$$

where

$$P' = \begin{pmatrix} e^{-\mathrm{i}\alpha} & 0 & 0 \\ 0 & e^{-\mathrm{i}\delta} & 0 \\ 0 & 0 & e^{-\mathrm{i}\delta} \end{pmatrix} , \tag{4.59}$$

$$V' = \begin{pmatrix} c_\omega c_\vartheta & c_\omega s_\vartheta & s_\omega e^{-\mathrm{i}\delta} \\ -c_\theta s_\vartheta - s_\theta s_\omega c_\vartheta e^{\mathrm{i}\delta} & +c_\theta c_\vartheta - s_\theta s_\omega s_\vartheta e^{\mathrm{i}\delta} & s_\theta c_\omega \\ +s_\theta s_\vartheta - c_\theta s_\omega c_\vartheta e^{\mathrm{i}\delta} & -s_\theta c_\vartheta - c_\theta s_\omega s_\vartheta e^{\mathrm{i}\delta} & c_\theta c_\omega \end{pmatrix} , \tag{4.60}$$

$$Q' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{\mathrm{i}(\delta-\gamma)} \end{pmatrix} . \tag{4.61}$$

Parametrization of Mixing Matrix

Standard Parametrization: three-flavor case

As we have shown, the diagonal phase matrices P' and Q' can be removed by redefining the phases of fermion fields. Therefore, we are left with

$$U = V' = \begin{pmatrix} c_\omega c_\vartheta & c_\omega s_\vartheta & s_\omega e^{-\mathrm{i}\delta} \\ -c_\theta s_\vartheta - s_\theta s_\omega c_\vartheta e^{\mathrm{i}\delta} & +c_\theta c_\vartheta - s_\theta s_\omega s_\vartheta e^{\mathrm{i}\delta} & s_\theta c_\omega \\ +s_\theta s_\vartheta - c_\theta s_\omega c_\vartheta e^{\mathrm{i}\delta} & -s_\theta c_\vartheta - c_\theta s_\omega s_\vartheta e^{\mathrm{i}\delta} & c_\theta c_\omega \end{pmatrix} , \quad (4.62)$$

where three mixing angles $\{\theta, \omega, \vartheta\}$ and one Dirac CP-violating phase δ are physical parameters.

In 1973, Kobayashi and Maskawa generalized two generations of quarks to three generations, and found that there exists an unremovable phase δ in the mixing matrix, which will cause CP violation in the theory. Such a great discovery provides an explanation for the CP violation observed in the K^0-$\overline{K}^0$ system in 1964.

Parametrization of Mixing Matrix

Standard Parametrization: three-flavor case

For Dirac neutrinos, the parametrization of leptonic mixing matrix will be the same. However, for Majorana neutrinos, the rightmost phase matrix Q cannot be removed away.

$$V = \begin{pmatrix} c_\omega c_\vartheta & c_\omega s_\vartheta & s_\omega e^{-\mathrm{i}\delta} \\ -c_\theta s_\vartheta - s_\theta s_\omega c_\vartheta e^{\mathrm{i}\delta} & +c_\theta c_\vartheta - s_\theta s_\omega s_\vartheta e^{\mathrm{i}\delta} & s_\theta c_\omega \\ +s_\theta s_\vartheta - c_\theta s_\omega c_\vartheta e^{\mathrm{i}\delta} & -s_\theta c_\vartheta - c_\theta s_\omega s_\vartheta e^{\mathrm{i}\delta} & c_\theta c_\omega \end{pmatrix} \cdot Q \; , \quad (4.63)$$

where $Q = \mathrm{Diag}\{e^{\mathrm{i}\rho}, e^{\mathrm{i}\sigma}, 1\}$ with two extra CP-violating phases ρ and σ, i.e., Majorana CP phases.

For two-flavor Majorana neutrinos, it is straightforward to see

$$V = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} e^{\mathrm{i}\rho} & 0 \\ 0 & 1 \end{pmatrix} \; , \quad (4.64)$$

where one mixing angle θ and one Majorana CP phase ρ are involved.

C, P and CP Symmetry

Weak Interactions

Both QCD and QED possess C, P and CP symmetries. Parity is maximally violated in weak interactions, so is C symmetry. Take neutrinos in the SM for example, where we have left-handed neutrinos $\nu_{\rm L}$ and right-handed antineutrinos $\bar{\nu}_{\rm R}$, i.e.,

$$\hat{C}\nu_{\rm L} \to \bar{\nu}_{\rm L} \; , \quad \hat{P}\nu_{\rm L} \to \nu_{\rm R} \; , \quad \hat{C}\hat{P}\nu_{\rm L} \to \bar{\nu}_{\rm R} \; , \tag{5.1}$$

implying CP invariance in the lepton sector though both C and P are violated maximally.

CP Transformations

$$\hat{C}\hat{P}\psi = C\psi^* \; , \quad \hat{C}\hat{P}W^-_\mu \to (-1)^{\delta_{0\mu}} W^+_\mu \; , \tag{5.2}$$

which will be used to find out where CP symmetry could be violated.

C, P and CP Symmetry

Weak Interactions

In the mass basis, all the kinetic and mass terms are invariant under CP transformations. Consider the CC interaction of massive Dirac neutrinos

$$\mathcal{L}^{\rm l}_{\rm CC} = \frac{g}{\sqrt{2}} \left[\overline{\widehat{\alpha}_{\rm L}} \gamma^\mu V_{\alpha i} \widehat{\nu}_{i{\rm L}} W^-_\mu + \overline{\widehat{\nu}_{i{\rm L}}} \gamma^\mu V^*_{\alpha i} \widehat{\alpha}_{\rm L} W^+_\mu \right] \; , \tag{5.3}$$

and perform the CP transformations

$$\widehat{\alpha}_{\rm L} \to C \widehat{\alpha}^*_{\rm L} \; , \quad \widehat{\nu}_{i{\rm L}} \to C \widehat{\nu}^*_{i{\rm L}} \; , \quad W^-_\mu \to (-1)^{\delta_{0\mu}} W^+_\mu \; . \tag{5.4}$$

$$\begin{aligned} CP &= \left[\overline{(\gamma^0 \widehat{\alpha}_{\rm L})^{\rm C}} \gamma^\mu V_{\alpha i} (\gamma^0 \widehat{\nu}_{i{\rm L}})^{\rm C} (-1)^{\delta_{0\mu}} W^+_\mu \right]^{\rm T} \\ &= - \left[(\gamma^0 \widehat{\nu}_{i{\rm L}})^{\rm C} \right]^{\rm T} C^{-1} V_{\alpha i} C \gamma^{\mu {\rm T}} C^{-1} C \overline{(\gamma^0 \widehat{\alpha}_{\rm L})^{\rm C}}^{\rm T} (-1)^{\delta_{0\mu}} W^+_\mu \\ &= - \widehat{\nu}^\dagger_{i{\rm L}} V_{\alpha i} \gamma^\mu \gamma^0 \widehat{\alpha}_{\rm L} W^+_\mu (-1)^{\delta_{0\mu}} = \overline{\widehat{\nu}_{i{\rm L}}} \gamma^\mu V_{\alpha i} \widehat{\alpha}_{\rm L} W^+_\mu \; . \end{aligned} \tag{5.5}$$

C, P and CP Symmetry

Weak Interactions

Comparing between Eq. (2.3) and Eq. (2.5), we can draw the conclusion that CP conservation requires

$$V_{\alpha i}^{*} = V_{\alpha i} \; , \tag{5.6}$$

namely, the phases should vanish in the mixing matrix. The physical CP phases in V originate from the complex Yukawa couplings, which however are not constrained at all in the SM or its minimal extensions for massive neutrinos.

It is worth mentioning that CP violation serves as a necessary condition to dynamically generate matter-antimatter asymmetry in our Universe. The CP violation in the quark sector is not large enough for this purpose, and massive neutrinos may help us out.

General Discussions

Suppose that we can observe the decay or scattering process $i \to f$ with the amplitude $A = \langle f|i\rangle$, and the CP-conjugate channel $\bar{i} \to \bar{f}$ with the amplitude $\bar{A} = \langle \bar{f}|\bar{i}\rangle$. The CP asymmetry is defined as

$$\Delta_{\rm CP} = \frac{|A|^2 - |\bar{A}|^2}{|A|^2 + |\bar{A}|^2} \; . \tag{5.7}$$

For a nonzero CP asymmetry, we need both weak phase $\delta_{\rm w}$ and strong phase $\delta_{\rm s}$, the former of which changes sign under CP transformations but the latter not.

Let us write $A = a_0 + a_1 e^{{\rm i}(\delta_{\rm s} + \delta_{\rm w})}$ and $\bar{A} = a_0 + a_1 e^{{\rm i}(\delta_{\rm s} - \delta_{\rm w})}$, so we have

$$\begin{aligned} \Delta_{\rm CP} \; &\propto \; 2 a_0 a_1 \left[\cos(\delta_{\rm s} + \delta_{\rm w}) - \cos(\delta_{\rm s} - \delta_{\rm w})\right] \\ &= -4 a_0 a_1 \sin\delta_{\rm w} \sin\delta_{\rm s} \; . \end{aligned} \tag{5.8}$$

Jarlskog Invariant

In 1985, Jarlskog calculated the commutator of quark mass matrices $H_u \equiv M_u M_u^\dagger = V_{u\rm L} D_u V_{u\rm L}^\dagger$ and $H_d \equiv M_d M_d^\dagger = V_{d\rm L} D_d V_{d\rm L}^\dagger$, namely,

$$\begin{aligned} Z \equiv {\rm i}[H_u, H_d] &= {\rm i}[V_{u\rm L} D_u V_{u\rm L}^\dagger, V_{d\rm L} D_d V_{d\rm L}^\dagger] \\ &= {\rm i} V_{u\rm L} D_u V_{u\rm L}^\dagger V_{d\rm L} D_d V_{d\rm L}^\dagger - {\rm i} V_{d\rm L} D_d V_{d\rm L}^\dagger V_{u\rm L} D_u V_{u\rm L}^\dagger \\ &= {\rm i} V_{u\rm L} [D_u, V D_d V^\dagger] V_{u\rm L}^\dagger \; , \end{aligned} \tag{5.9}$$

with $D_u \equiv {\rm Diag}\{m_u^2, m_c^2, m_t^2\}$, $D_d \equiv {\rm Diag}\{m_d^2, m_s^2, m_b^2\}$ and $V = V_{u\rm L}^\dagger V_{d\rm L}$. Then we have

$${\rm Det} Z = {\rm iDet}[V D_d V^\dagger, D_u] = {\rm iDet} \begin{pmatrix} 0 & \Delta_{cu} X_{uc} & \Delta_{tu} X_{ut} \\ \Delta_{uc} X_{uc}^* & 0 & \Delta_{tc} X_{ct} \\ \Delta_{ut} X_{ut}^* & \Delta_{ct} X_{ct}^* & 0 \end{pmatrix}$$

where $\Delta_{ij} \equiv m_i^2 - m_j^2$ and $X_{ij} = V_{ik} V_{jk}^* m_k^2 = X_{ji}^*$ with $k = d, s, b$.

Jarlskog Invariant

After a direct calculation, we get

$$\mathrm{Det}Z = 2\mathcal{J}_{\rm q}\Delta_{uc}\Delta_{ct}\Delta_{tu}\Delta_{ds}\Delta_{sb}\Delta_{bd} \; , \tag{5.10}$$

where the Jarlskog invariant is given by

$$\mathcal{J}_{\rm q}\sum_{\gamma}\epsilon_{\alpha\beta\gamma}\sum_{k}\epsilon_{ijk} = \mathrm{Im}\left[V_{\alpha i}V^*_{\alpha j}V^*_{\beta i}V_{\beta j}\right] \; . \tag{5.11}$$

There is a unique Jarlskog invariant for three-flavor mixing, which can be expressed in the standard parametrization as below

$$\mathcal{J}_{\rm q} = s_\theta c_\theta s_\vartheta c_\vartheta s_\omega c^2_\omega s_\delta \; . \tag{5.12}$$

It measures the strength of CP violation, i.e., $\mathcal{J}_{\rm q} = 0 \Leftrightarrow \delta = 0$ or π.