Further Discussions on Massive Neutrinos

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A Simple Derivation

Equal Momenta

As we have shown before, the neutrino oscillation probabilities can be computed as

$$P(\nu_\alpha \to \nu_\beta; t) = \left| \sum_i V^*_{\alpha i} e^{-\mathrm{i} E_i t} V_{\beta i} \right|^2 . \tag{1.1}$$

Then we take the following approximation ($E_i \approx p \gg m_i$)

$$E_i = \sqrt{p^2 + m_i^2} = p\sqrt{1 + m_i^2/p^2} \approx p + m_i^2/2p \tag{1.2}$$

and sometimes $E_i \approx p \approx E$ as well. The energy difference reads

$$E_i - E_j \approx (m_i^2 - m_j^2)/(2p) \equiv \Delta m^2_{ij}/(2p) \, , \tag{1.3}$$

and finally the time-to-distance conversion $t \approx L$ is performed.

A Simple Derivation

Equal Energies

We assume that the energies of neutrino mass eigenstates are the same, and they are propagating in space, leading to the oscillation probabilities

$$P(\nu_\alpha \to \nu_\beta; x) = \left| \sum_i V^*_{\alpha i} e^{+\mathrm{i} p_i x} V_{\beta i} \right|^2 \; . \tag{1.4}$$

Then we take the following approximation ($p_i \approx E \gg m_i$)

$$p_i = \sqrt{E^2 - m_i^2} = E \sqrt{1 - m_i^2/E^2} \approx E - m_i^2/2E \tag{1.5}$$

and sometimes $p_i \approx p \approx E$ as well. The energy difference reads

$$p_i - p_j \approx -(m_i^2 - m_j^2)/(2E) \equiv -\Delta m^2_{ij}/(2E) \; , \tag{1.6}$$

and no time-to-distance conversion is needed.

Equal Energies or Momenta

Which is Correct?

The answer is that both are wrong. Consider the production of neutrino mass eigenstates from $\pi^+ \to \mu^+ + \nu_i$. If the energies and momenta of π^+ and μ^+ are given, then we have

$$E_{\pi^+} = E_{\mu^+} + E_i \; , \quad \mathbf{p}_{\pi^+} = \mathbf{p}_{\mu^+} + \mathbf{p}_i \; , \tag{1.7}$$

where the energies and momenta of ν_i (for $i = 1, 2, 3$) fulfill the relation

$$E_i = \sqrt{\mathbf{p}_i^2 + m_i^2} \; . \tag{1.8}$$

Hence both energies and momenta of different neutrino mass eigenstates are different. In the rest frame of π^+, we have $E_{\pi^+} = m_\pi$ and $\mathbf{p}_{\pi^+} = 0$:

$$|\mathbf{p}_i| = \sqrt{\left[(m_\pi - m_\mu)^2 - m_i^2\right]\left[(m_\pi + m_\mu)^2 - m_i^2\right]}/(2m_\pi) \; . \tag{1.9}$$

Wave-packet Formalism

A Realistic Description

The correct description should be the wave-packet formalism:

$$|\nu_\alpha(t,\mathbf{x})\rangle = \sum_i V^*_{\alpha i}\psi_i(t,\mathbf{x})|\nu_i\rangle \; , \tag{1.10}$$

where the wave packet for a free neutrino mass eigenstate is

$$\psi_i(t,\mathbf{x}) = \int \frac{\mathrm{d}^3 p}{(2\pi)^{3/2}} f_i^{\mathrm{S}}(\mathbf{p}-\mathbf{p}_i) e^{\mathrm{i}[\mathbf{p}\cdot\mathbf{x} - E_i(p)t]} \; . \tag{1.11}$$

Some comments are helpful:

- $f_i^{\mathrm{S}}(\mathbf{p}-\mathbf{p}_i)$ is the momentum distribution sharply peaked at $\mathbf{p}_i$ for the neutrino eigenstates at the source with the width $\sigma_p \ll p_i$.
- $E_i(p) = \sqrt{\mathbf{p}^2 + m_i^2}$ is the energy corresponding to the momentum $\mathbf{p}$.

A Realistic Description

Expand the energy $E_i(p)$ around the mean momentum p_i

$$\begin{aligned} E_i(p) &= E_i(p_i) + \left.\frac{\partial E_i(p)}{\partial p^j}\right|_{\mathbf{p}_i} (p - p_i)^j \\ &\quad + \left.\frac{1}{2}\frac{\partial^2 E_i(p)}{\partial p^j \partial p^k}\right|_{\mathbf{p}_i} (p - p_i)^j (p - p_i)^k + \cdots \; . \end{aligned} \tag{1.12}$$

We can rewrite $\psi_i(t, \mathbf{x})$ as below

$$\begin{aligned} \psi_i(t, \mathbf{x}) &\approx \int \frac{\mathrm{d}^3 p}{(2\pi)^{3/2}} f_i^{\mathrm{S}}(\mathbf{p} - \mathbf{p}_i) e^{\mathrm{i}\left\{\mathbf{p}\cdot\mathbf{x} - \left[E_i(p_i) + \left.\frac{\partial E_i(p)}{\partial p^j}\right|_{\mathbf{p}_i} (p - p_i)^j\right] t\right\}} \\ &= e^{\mathrm{i}[\mathbf{p}_i \cdot \mathbf{x} - E_i(p_i) t]} \cdot g_i^{\mathrm{S}}(\mathbf{x} - \mathbf{v}_{i\mathrm{g}} t) \; . \end{aligned} \tag{1.13}$$

A Realistic Description

The shape factor is defined as

$$g_i^{\rm S}(\mathbf{x}-\mathbf{v}_{i\rm g}t)=\int\frac{{\rm d}^3p}{(2\pi)^{3/2}}f_i^{\rm S}(\mathbf{p})e^{{\rm i}[\mathbf{p}\cdot(\mathbf{x}-\mathbf{v}_{i\rm g}t)]}\;, \tag{1.14}$$

where the group velocity of the wave packet is

$$\mathbf{v}_{i\rm g}=\left.\frac{\partial E_i(p)}{\partial\mathbf{p}}\right|_{\mathbf{p}_i}=\left.\frac{\mathbf{p}}{E_i(p)}\right|_{\mathbf{p}_i}\;. \tag{1.15}$$

- The shape factor $g_i^{\rm S}(\mathbf{x}-\mathbf{v}_{\rm g}t)$ stands for the distribution of the wave packet in the coordinate space.
- The wave packet is travelling at the speed of $\mathbf{v}_{i\rm g}$ without changing its shape. The width is $\sigma_x\gtrsim\sigma_p^{-1}$ according to the uncertainty principle.

Wave-packet Formalism

A Realistic Description

Next, we have to specify the neutrino state at the detector, which is taken to be another wave packet located at $\mathbf{x} = \mathbf{L}$, i.e.,

$$\begin{aligned} |\nu_\beta(\mathbf{x}-\mathbf{L})\rangle &= \sum_i V^*_{\beta i}\psi^{\rm D}_i(\mathbf{x}-\mathbf{L})|\nu_i\rangle \\ &= \sum_i V^*_{\beta i} e^{{\rm i}\mathbf{p}_i\cdot(\mathbf{x}-\mathbf{L})} g^{\rm D}_i(\mathbf{x}-\mathbf{L})|\nu_i\rangle \end{aligned} \tag{1.16}$$

where $g^{\rm D}_i$ stands for the shape of the wave packet of $|\nu_i\rangle$ at the detector.

Oscillation Amplitudes

The amplitudes are given by

$$\mathcal{A}_{\alpha\beta}(t,\mathbf{L}) = \int {\rm d}^3x \langle \nu_\beta(\mathbf{x}-{\rm L})|\nu_\alpha(t,\mathbf{x})\rangle \; . \tag{1.17}$$

Oscillation Amplitudes

The amplitudes are given by

$$\begin{aligned} \mathcal{A}_{\alpha\beta}(t, \mathbf{L}) &= \int \mathrm{d}^3 x \langle \nu_\beta(\mathbf{x} - \mathrm{L}) | \nu_\alpha(t, \mathbf{x}) \rangle \\ &= \sum_i V^*_{\alpha i} V_{\beta i} G(\mathbf{L} - \mathbf{v}_{i\mathrm{g}} t) e^{-\mathrm{i} E_i(p_i) t + \mathrm{i} \mathbf{p}_i \cdot \mathbf{L}} \; , \end{aligned} \tag{1.18}$$

where

$$G(\mathbf{L} - \mathbf{v}_{i\mathrm{g}} t) \equiv \int \mathrm{d}^3 x \; g_i^{\mathrm{S}}(\mathbf{x} - \mathbf{v}_{i\mathrm{g}} t) \cdot g_i^{\mathrm{D}*}(\mathbf{x} - \mathbf{L}) \; . \tag{1.19}$$

Note that $g_i^{\mathrm{S}}(\mathbf{x} - \mathbf{v}_{i\mathrm{g}} t)$ decreases rapidly when $\mathbf{x}$ deviates far from $\mathbf{v}_{i\mathrm{g}} t$, so does $g_i^{\mathrm{D}}(\mathbf{x} - \mathbf{L})$ for $\mathbf{x}$ from $\mathbf{L}$. Hence $G(\mathbf{L} - \mathbf{v}_{i\mathrm{g}} t)$ peaks at $\mathbf{L} = \mathbf{v}_{i\mathrm{g}} t$, i.e., the amplitudes are not small only when the wave packets overlap.

Density Matrix Formalism: General Comments

Pure States

Given a pure quantum state $|\psi(t)\rangle$, one can construct the density matrix operator via

$$\widehat{\rho}(t) \equiv |\psi(t)\rangle\langle\psi(t)| \; , \tag{2.1}$$

where $\langle\psi(t)| \equiv |\psi(t)\rangle^\dagger$ has been defined such that the density matrix operator is Hermitian $\widehat{\rho}^\dagger = \widehat{\rho}$.

Matrix Representation

Expanding the state $|\psi(t)\rangle$ in terms of a complete set of orthonormal eigenstates $|\phi_i\rangle$ (for $i = 1, 2, \cdots, n$) of the Hamiltonian, we have

$$|\psi(t)\rangle = \sum_{i=1}^{n} a_i(t)|\phi_i\rangle \; . \tag{2.2}$$

Density Matrix Formalism: General Comments

Matrix Representation

Then, the density matrix can be written as

$$\widehat{\rho}(t) = |\psi(t)\rangle\langle\psi(t)| = \sum_{i,j=1}^{n} a_i(t) a_j^*(t) |\phi_i\rangle\langle\phi_j| \; , \tag{2.3}$$

and its element is given by

$$\widehat{\rho}_{kl}(t) \equiv \langle\phi_k|\widehat{\rho}(t)|\phi_l\rangle = \sum_{i,j=1}^{n} a_i(t) a_j^*(t) \langle\phi_k|\phi_i\rangle\langle\phi_j|\phi_l\rangle = a_k(t) a_l^*(t) \; . \tag{2.4}$$

The trace of the density matrix for a pure state is

$$\mathrm{Tr}\left[\widehat{\rho}(t)\right] = \sum_{i=1}^{n} |a_i(t)|^2 = 1 \; , \tag{2.5}$$

where the state $|\psi(t)\rangle$ is normalized to one.

Density Matrix Formalism: General Comments

Liouville-von Neumann Equation

Starting with the Schrödinger equation

$$\mathrm{i}\frac{\partial}{\partial t}|\psi(t)\rangle = \widehat{H}|\psi(t)\rangle \; , \tag{2.6}$$

and its Hermitian conjugate

$$-\mathrm{i}\frac{\partial}{\partial t}\langle\psi(t)| = \langle\psi(t)|\widehat{H} \; , \tag{2.7}$$

we can find the time evolution of the density matrix

$$\mathrm{i}\frac{\partial\widehat{\rho}(t)}{\partial t} = \mathrm{i}\frac{\partial|\psi(t)\rangle}{\partial t}\langle\psi(t)| + \mathrm{i}|\psi(t)\rangle\frac{\partial\langle\psi(t)|}{\partial t} = [\widehat{H},\widehat{\rho}] \; . \tag{2.8}$$

Here $\widehat{H}$ is the Hamiltonian operator. One can figure out the matrix representations of any operators in the chosen basis of state vectors.

Density Matrix Formalism: Neutrino Oscillations

Neutrino Oscillations in Vacuum

We apply the density matrix formalism to two-flavor neutrino oscillations in vacuum, for which the Hamiltonian is given by

$$\begin{aligned} \widehat{H} &= \frac{1}{2E}\begin{pmatrix} m_1^2 c_\theta^2 + m_2^2 s_\theta^2 & \Delta m_{21}^2 s_\theta c_\theta \\ \Delta m_{21}^2 s_\theta c_\theta & m_1^2 s_\theta^2 + m_2^2 c_\theta^2 \end{pmatrix} \\ &= \frac{m_1^2 + m_2^2}{4E} + \frac{\Delta m_{21}^2}{4E}\begin{pmatrix} -c_{2\theta} & s_{2\theta} \\ s_{2\theta} & +c_{2\theta} \end{pmatrix} \\ &\equiv \omega_0 \sigma^0 + \frac{1}{2}\omega \mathbf{B}\cdot\boldsymbol{\sigma} \; , \end{aligned} \tag{2.9}$$

where $\omega \equiv \Delta m_{21}^2/(2E)$ is the oscillation frequency and

$$\sigma^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} , \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} , \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} . \tag{2.10}$$

Density Matrix Formalism: Neutrino Oscillations

Neutrino Oscillations in Vacuum

The term $\omega_0 \sigma^0$ with $\omega_0 = (m_1^2 + m_2^2)/(4E)$ contributes a common phase factor and is irrelevant. The "magnetic field" vector can be identified as

$$\mathbf{B} = (B_1, B_2, B_3) \equiv (s_{2\theta}, 0, -c_{2\theta}) \; . \tag{2.11}$$

On the other hand, we can expand the density matrix in terms of Pauli matrices and obtain the polarization vector $\mathbf{P}(t)$:

$$\widehat{\rho}(t) = P_0 \sigma^0 + \frac{1}{2} \mathbf{P} \cdot \boldsymbol{\sigma} \; . \tag{2.12}$$

Notice that the Hamiltonian $\widehat{H}$ has been given in the flavor basis, for which the basis vectors are $|\nu_e\rangle$ and $|\nu_\mu\rangle$, namely,

$$\widehat{H} = H_{\alpha\beta} |\nu_\alpha\rangle \langle \nu_\beta| \; , \tag{2.13}$$

for $\alpha, \beta = e, \mu$. What are $\widehat{\rho}(0)$ and $\widehat{\rho}(t)$ in the same basis?

Density Matrix Formalism: Neutrino Oscillations

Neutrino Oscillations in Vacuum

For a general state $|\psi(t)\rangle = a_e(t)|\nu_e\rangle + a_\mu(t)|\nu_\mu\rangle$ in the flavor basis, we can get the expression of the density matrix

$$\widehat{\rho}(t) = \begin{pmatrix} |a_e(t)|^2 & a_e(t)a_\mu^*(t) \\ a_e^*(t)a_\mu(t) & |a_\mu(t)|^2 \end{pmatrix} , \tag{2.14}$$

and its value at $t = 0$

$$\widehat{\rho}(0) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} , \tag{2.15}$$

where $a_e(0) = 1$ and $a_\mu(0) = 0$ imply that the initial state is $|\psi(0)\rangle = |\nu_e\rangle$.

One can observe the physical meaning of diagonal elements of $\widehat{\rho}(t)$, i.e., $\widehat{\rho}_{ee}(t) = |a_e(t)|^2$ for probability in the state $|\nu_e\rangle$, and $\widehat{\rho}_{\mu\mu}(t) = |a_\mu(t)|^2$ for probability in the state $|\nu_\mu\rangle$.

Density Matrix Formalism: Neutrino Oscillations

Neutrino Oscillations in Vacuum

It is now straightforward to find out the polarization vector:

$$\begin{aligned}\widehat{\rho}(t) &= \frac{|a_e(t)|^2+|a_\mu(t)|^2}{2}+\frac{1}{2}\begin{pmatrix}|a_e(t)|^2-|a_\mu(t)|^2 & 2a_e(t)a_\mu^*(t)\\ 2a_e^*(t)a_\mu(t) & |a_\mu(t)|^2-|a_e(t)|^2\end{pmatrix}\\ &\equiv P_0\sigma^0+\frac{1}{2}\mathbf{P}(t)\cdot\boldsymbol{\sigma}\;,\end{aligned}$$

where $|a_e(t)|^2+|a_\mu(t)|^2=1$ holds, i.e., $P_0=1/2$ and

$$P_1=2\mathrm{Re}\left[a_e^*(t)a_\mu(t)\right]\;,P_2=2\mathrm{Im}\left[a_e^*(t)a_\mu(t)\right]\;,P_3=|a_e(t)|^2-|a_\mu(t)|^2\;.$$

Note that the probability to be in $|\nu_e\rangle$ is given by $|a_e(t)|^2$, or equivalently

$$|a_e(t)|^2=\frac{1}{2}\left[1+P_3(t)\right]\;. \tag{2.16}$$

Density Matrix Formalism: Neutrino Oscillations

Neutrino Oscillations in Vacuum

The initial value of the polarization vector can be extracted from $\widehat{\rho}(0)$, namely,

$$P_1(0) = 0 \,, \quad P_2(0) = 0 \,, \quad P_3(0) = 1 \,. \tag{2.17}$$

Equation of Motion (EoM)

The Liouville-von Neumann equation can be recast into

$$\begin{aligned} \mathrm{i}\frac{\partial}{\partial t}\left[P_0\sigma^0 + \frac{1}{2}\mathbf{P}(t)\cdot\boldsymbol{\sigma}\right] &= \left[\omega_0\sigma^0 + \frac{1}{2}\omega\mathbf{B}\cdot\boldsymbol{\sigma}, P_0\sigma^0 + \frac{1}{2}\mathbf{P}(t)\cdot\boldsymbol{\sigma}\right] \\ \Rightarrow \mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{P}(t)\cdot\boldsymbol{\sigma} &= \frac{1}{2}\omega B_i P_j(t)\left[\sigma^i, \sigma^j\right] = \mathrm{i}\epsilon^{ijk}\omega B_i P_j \sigma^k \\ \Rightarrow \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{P}(t) &= \omega\mathbf{B}\times\mathbf{P}(t) \,. \end{aligned} \tag{2.18}$$

The EoM should be solved with the initial conditions in Eq. (2.17).

Density Matrix Formalism: Neutrino Oscillations

Neutrino Oscillations in Vacuum: Geometrical Description

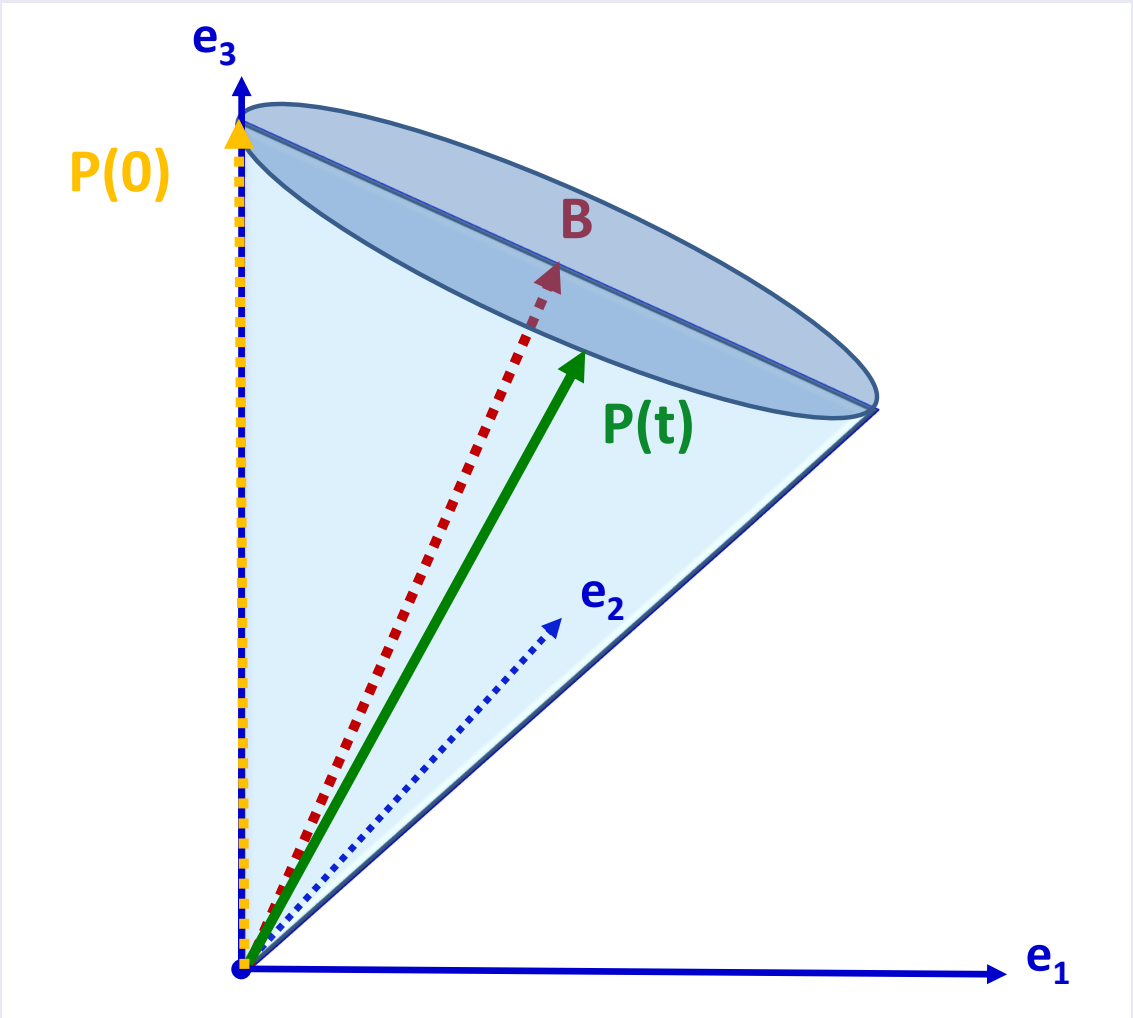

Figure: The polarization vector $\mathbf{P}(t)$ is revolving around the magnetic field $\mathbf{B}$.

Density Matrix Formalism: Neutrino Oscillations

Neutrino Oscillations in Vacuum: Geometrical Description

Recall that $\mathbf{B} = (s_{2\theta}, 0, -c_{2\theta})$ and write down the EoM of $\mathbf{P}(t)$ in its three components

$$\dot{P}_1 = \omega (B_2 P_3 - B_3 P_2) = \omega c_{2\theta} P_2 \; , \tag{2.19}$$

$$\dot{P}_2 = \omega (B_3 P_1 - B_1 P_3) = -\omega (c_{2\theta} P_1 + s_{2\theta} P_3) \; , \tag{2.20}$$

$$\dot{P}_3 = \omega (B_1 P_2 - B_2 P_1) = \omega s_{2\theta} P_2 \; . \tag{2.21}$$

The initial conditions are $\mathbf{P}(0) = (0, 0, 1)$ and $\dot{\mathbf{P}}(0) = (0, -\omega s_{2\theta}, 0)$. Then differentiate Eq. (2.20) and insert the other two equations into it. We get

$$\ddot{P}_2 = -\omega^2 P_2 \; , \tag{2.22}$$

whose general solution can be easily found $P_2(t) = A_0 \sin(\omega t + \varphi_0)$. The constants A_0 and φ_0 will be determined by the initial conditions.

Neutrino Oscillations in Vacuum: Geometrical Description

- For $P_2(0) = 0$, we have

$$P_2(0) = A_0 \sin\varphi_0 = 0 \; ; \tag{2.23}$$

- For $\dot{P}_2(0) = -\omega s_{2\theta}$, we obtain

$$\dot{P}_2(0) = \omega A_0 \cos\varphi_0 = -\omega s_{2\theta} \; . \tag{2.24}$$

Hence $A_0 = -s_{2\theta}$ and $\varphi_0 = 0$, leading to $P_2(t) = -s_{2\theta} \sin(\omega t)$, which can be inserted back into Eq. (2.22) to get

$$\dot{P}_3(t) = -\omega s^2_{2\theta} \sin(\omega t) \; , \tag{2.25}$$

with the initial condition $P_3(0) = 1$. The solution is

$$P_3(t) = s^2_{2\theta} \cos(\omega t) + c^2_{2\theta} \; . \tag{2.26}$$

Neutrino Oscillations in Vacuum: Geometrical Description

Finally, we arrive at the survival probability

$$\begin{aligned} P(\nu_e \to \nu_e) = |a_e(t)|^2 &= \frac{1}{2}\left[1 + P_3(t)\right] \\ &= \frac{1}{2}\left[1 + c_{2\theta}^2 + s_{2\theta}^2 \cos(\omega t)\right] \\ &= 1 - s_{2\theta}^2 \frac{1 - \cos(\omega t)}{2} \\ &= 1 - s_{2\theta}^2 \sin^2 \frac{\omega t}{2} = 1 - \sin^2 2\theta \sin^2 \frac{\Delta m_{21}^2 L}{4E} \; , \end{aligned}$$

where $\omega = \Delta m_{21}^2/(2E)$ and $t \approx L$ have been used. The ordinary oscillation probability is reproduced in the geometrical language.

Density Matrix Formalism: Neutrino Oscillations

Neutrino Oscillations in Matter

We turn to two-flavor neutrino oscillations in matter for which the effective Hamiltonian is given by

$$\begin{aligned} \widehat{H}_{\rm m} &= \frac{1}{2E}\left[\begin{pmatrix} m_1^2 c_\theta^2 + m_2^2 s_\theta^2 & \Delta m_{21}^2 s_\theta c_\theta \\ \Delta m_{21}^2 s_\theta c_\theta & m_1^2 s_\theta^2 + m_2^2 c_\theta^2 \end{pmatrix} + \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}\right] \\ &= \frac{1}{4E}\left[\begin{pmatrix} -\Delta m_{21}^2 c_{2\theta} & \Delta m_{21}^2 s_{2\theta} \\ \Delta m_{21}^2 s_{2\theta} & \Delta m_{21}^2 c_{2\theta} \end{pmatrix} + A \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right] \\ &\equiv \frac{1}{2}\omega \mathbf{B}\cdot\boldsymbol{\sigma} + \frac{1}{2}\mu\boldsymbol{\lambda}\cdot\boldsymbol{\sigma}\;, \end{aligned} \tag{2.27}$$

where $\omega \equiv \Delta m_{21}^2/(2E)$ is the oscillation frequency in vacuum, the matter potential is $\mu \equiv \sqrt{2} G_{\rm F} N_e = A/(2E)$, and $\boldsymbol{\lambda} = (0, 0, 1)$ contributes as an extra magnetic field.

Neutrino Oscillations in Matter

The effective "magnetic field" can be computed as

$$\begin{aligned} \widehat{H}_{\rm m} &= \frac{1}{2}\omega \mathbf{B}\cdot\boldsymbol{\sigma} + \frac{1}{2}\mu\boldsymbol{\lambda}\cdot\boldsymbol{\sigma} \\ &= \frac{1}{2}\omega_{\rm m}\mathbf{B}_{\rm m}\cdot\boldsymbol{\sigma}\;, \end{aligned} \tag{2.28}$$

where the effective mass-squared difference $\Delta\widetilde{m}^2_{21}$ and the effective mixing angle $\widetilde{\theta}$ can be identified as below

$$\omega\mathbf{B} + \mu\boldsymbol{\lambda} = (\omega s_{2\theta}, 0, \mu - \omega c_{2\theta}) = \frac{\Delta\widetilde{m}^2_{21}}{2E}(s_{2\widetilde{\theta}}, 0, c_{2\widetilde{\theta}}) \equiv \omega_{\rm m}\mathbf{B}_{\rm m} \tag{2.29}$$

with

$$\omega_{\rm m} = \frac{\Delta\widetilde{m}^2_{21}}{2E} = \sqrt{(\mu - \omega c_{2\theta})^2 + \omega^2 s^2_{2\theta}}\;. \tag{2.30}$$

Density Matrix Formalism: Neutrino Oscillations

Neutrino Oscillations in Matter: Geometrical Description

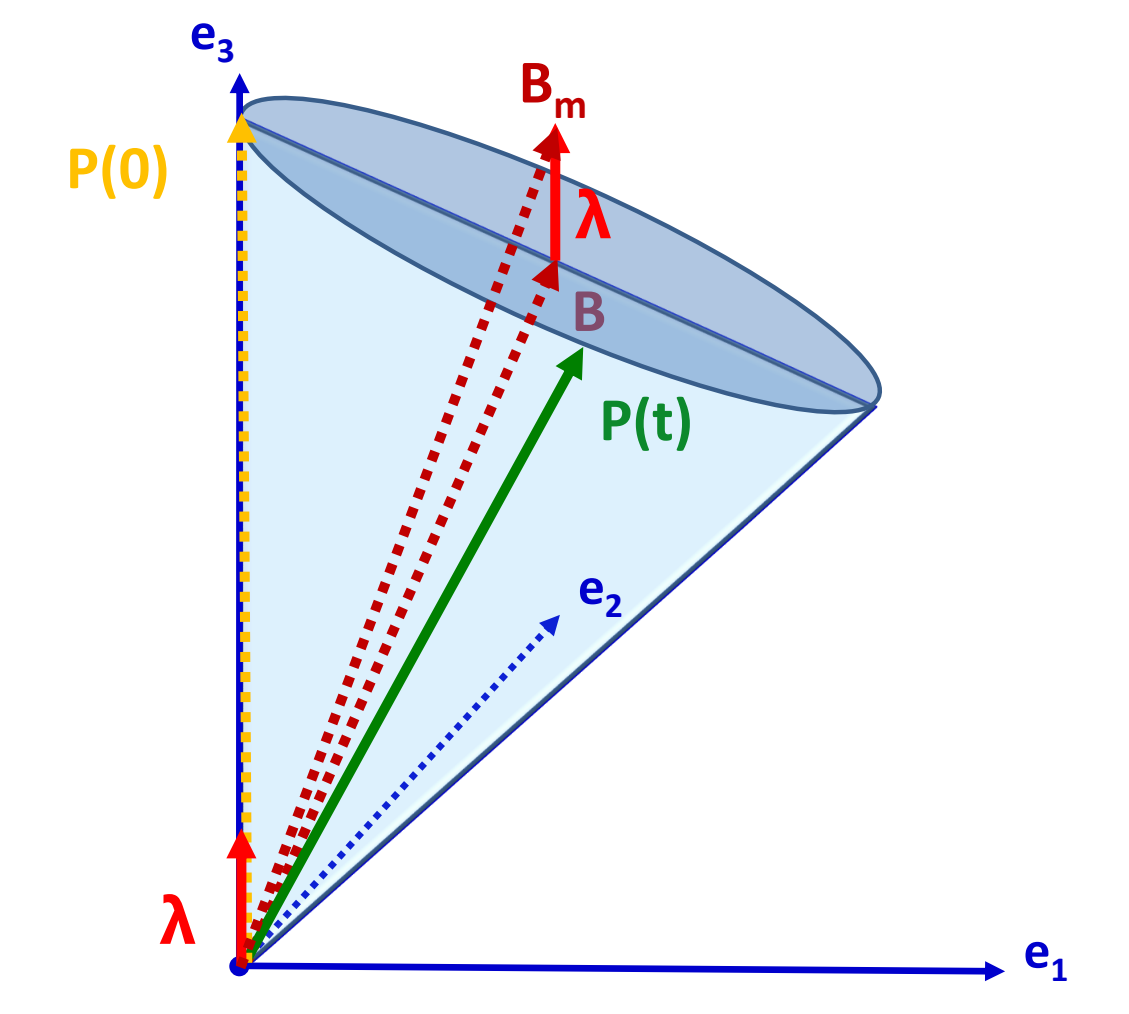

Figure: The polarization vector $\mathbf{P}(t)$ is revolving around the magnetic field $\mathbf{B}_{\rm m}$.

Fermion Masses

Why Yukawa interactions?

Main features of the standard model include (1) minimal particle content; (2) gauge symmetries & spontaneous breaking; (3) renormalizability.

Q: Why not mass terms? Take quarks for example. We have left-handed quark doublets and right-handed singlets

$$Q_i = \begin{pmatrix} u_{i\mathrm{L}} \\ d_{i\mathrm{L}} \end{pmatrix} \; , \quad u_{i\mathrm{R}} \; , \quad d_{i\mathrm{R}} \; . \tag{3.1}$$

The mass terms are not gauge invariant

$$(M_u)_{ij} \overline{u_{i\mathrm{L}}} u_{j\mathrm{R}} + (M_d)_{ij} \overline{d_{i\mathrm{L}}} d_{j\mathrm{R}} + \mathrm{h.c.} \; , \tag{3.2}$$

but Yukawa interactions are gauge invariant

$$(Y_u)_{ij} \overline{Q_i} \widetilde{H} u_{j\mathrm{R}} + (Y_d)_{ij} \overline{Q_i} H d_{j\mathrm{R}} + \mathrm{h.c.} \; . \tag{3.3}$$

Majorana Fermions

Why Majorana Condition?

The Dirac equation for a massive fermion reads

$$\mathrm{i}\gamma^\mu \partial_\mu \psi_{\rm L} = m\psi_{\rm R} \; , \tag{3.4}$$

$$\mathrm{i}\gamma^\mu \partial_\mu \psi_{\rm R} = m\psi_{\rm L} \; . \tag{3.5}$$

with $\psi = \psi_{\rm L} + \psi_{\rm R}$. Hence $\psi_{\rm L}$ is already related to $\psi_{\rm R}$. However, the Majorana equation is

$$\mathrm{i}\gamma^\mu \partial_\mu \psi_{\rm L} = m\psi^{\rm C}_{\rm L} \; , \tag{3.6}$$

$$\mathrm{i}\gamma^\mu \partial_\mu \psi^{\rm C}_{\rm L} = m\psi_{\rm L} \; . \tag{3.7}$$

with $\psi = \psi_{\rm L} + \psi^{\rm C}_{\rm L}$. The number of independent equations should be equal to the number of degrees, namely, two for Majorana fermions and four for Dirac fermions.

Right-handed Neutrinos

Dirac Neutrinos

For massive Dirac neutrinos, we have

$$\widehat{\nu} = \widehat{\nu}_{\rm L} + \widehat{\nu}_{\rm R} \; , \tag{3.8}$$

where there should be no confusion at all as massive Dirac fermions should have independent left- and right-handed chiral components.

Majorana Neutrinos

For massive Majorana neutrinos in seesaw models, we adopt the same notation $\nu_{\rm R}$. For clarity, one can use $N_{\rm R}$ instead of $\nu_{\rm R}$. In this case,

$$\widehat{\nu} = \widehat{\nu}_{\rm L} + \widehat{\nu}^{\rm C}_{\rm L} \tag{3.9}$$

for light Majorana neutrinos $\widehat{\nu}^{\rm C} = \widehat{\nu}$; but for heavy Majorana neutrinos

$$\widehat{N} = \widehat{N}^{\rm C}_{\rm R} + \widehat{N}_{\rm R} \tag{3.10}$$

Right-handed Neutrinos

Majorana Neutrinos

Let us recapitulate the one-generation version of seesaw model. Adding one right-handed neutrino $N_{\rm R}$ into the SM, we can obtain the neutrino mass terms

$$\mathcal{L}^{\nu}_{\rm mass} = -\frac{1}{2}\overline{\begin{pmatrix} \nu_{\rm L} & N^{\rm C}_{\rm R} \end{pmatrix}} \begin{pmatrix} 0 & m_{\rm D} \\ m_{\rm D} & m_N \end{pmatrix} \begin{pmatrix} \nu^{\rm C}_{\rm L} \\ N_{\rm R} \end{pmatrix} + {\rm h.c.} \; , \tag{3.11}$$

where $\nu^{\rm C}_{\rm L} \equiv C\overline{\nu_{\rm L}}^{\rm T}$ and $N^{\rm C}_{\rm R} \equiv C\overline{N_{\rm R}}^{\rm T}$, and $\{m_{\rm D}, m_N\}$ are positive numbers.

Diagonalization

The overall mass matrix

$$\mathcal{M} = \begin{pmatrix} 0 & m_{\rm D} \\ m_{\rm D} & m_N \end{pmatrix} \tag{3.12}$$

is real and symmetric, so it can be exactly diagonalized.

Right-handed Neutrinos

Diagonalization

First, let us diagonalize $\mathcal{M}$ by an orthogonal matrix $\mathcal{O}$, i.e.,

$$\mathcal{O} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} , \tag{3.13}$$

$$\mathcal{O}^{\rm T}\mathcal{M}\mathcal{O} = \begin{pmatrix} x_1 & 0 \\ 0 & x_2 \end{pmatrix} , \tag{3.14}$$

where the mixing angle and the eigenvalues can be exactly calculated:

$$x_1 = \frac{1}{2}\left[m_N - \sqrt{m_N^2 + 4m_{\rm D}^2}\right] , \tag{3.15}$$

$$x_2 = \frac{1}{2}\left[m_N + \sqrt{m_N^2 + 4m_{\rm D}^2}\right] , \tag{3.16}$$

$$\tan 2\theta = \frac{2m_{\rm D}}{m_N} . \tag{3.17}$$

Right-handed Neutrinos

Diagonalization

Assuming $m_N \gg m_{\rm D}$, we have the excellent approximations

$$x_1 = \frac{1}{2}\left[m_N - \sqrt{m_N^2 + 4m_{\rm D}^2}\right] \approx -m_{\rm D}^2/m_N \; , \tag{3.18}$$

$$x_2 = \frac{1}{2}\left[m_N + \sqrt{m_N^2 + 4m_{\rm D}^2}\right] \approx m_N \; , \tag{3.19}$$

$$\theta \approx m_{\rm D}/m_N \; . \tag{3.20}$$

Since x_1 is negative, it cannot be the mass of the light Majorana neutrino. We must perform another transformation

$$\begin{pmatrix} i & 0 \\ 0 & 1 \end{pmatrix}^{\rm T} \begin{pmatrix} -m_{\rm D}^2/m_N & 0 \\ 0 & m_N \end{pmatrix} \begin{pmatrix} i & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} m_{\rm L} & 0 \\ 0 & m_{\rm H} \end{pmatrix} \; , \tag{3.21}$$

where $m_{\rm L} = m_{\rm D}^2/m_N$ and $m_{\rm H} = m_N$ are light and heavy neutrino masses.

Right-handed Neutrinos

Mass Eigenstates

Since the overal mass matrix $\mathcal{M}$ is diagonalized via

$$V^\dagger \mathcal{M} V^* = \begin{pmatrix} m_{\rm L} & 0 \\ 0 & m_{\rm H} \end{pmatrix} , \quad V = \mathcal{O} \begin{pmatrix} -i & 0 \\ 0 & 1 \end{pmatrix} , \tag{3.22}$$

we rewrite the neutrino mass terms as

$$\mathcal{L}^\nu_{\rm mass} = -\frac{1}{2} \overline{\left(\nu_{\rm L} \quad N^{\rm C}_{\rm R}\right)} V \begin{pmatrix} m_{\rm L} & 0 \\ 0 & m_{\rm H} \end{pmatrix} V^{\rm T} \begin{pmatrix} \nu^{\rm C}_{\rm L} \\ N_{\rm R} \end{pmatrix} + {\rm h.c.} \, . \tag{3.23}$$

Therefore, the mass eigenstates of Majorana neutrinos can be found

$$\widehat{\nu}_{\rm L} = V^*_{11} \nu_{\rm L} + V^*_{21} N^{\rm C}_{\rm R} , \tag{3.24}$$

$$\widehat{N}_{\rm R} = V_{12} \nu^{\rm C}_{\rm L} + V_{22} N_{\rm R} , \tag{3.25}$$

where $V_{11} = {\rm i} c_\theta$, $V_{12} = s_\theta$, $V_{21} = {\rm i} s_\theta$ and $V_{22} = c_\theta$.