

Entropy Currents and the H-Theorem for the Berry-Dressed Matrix Wigner Function

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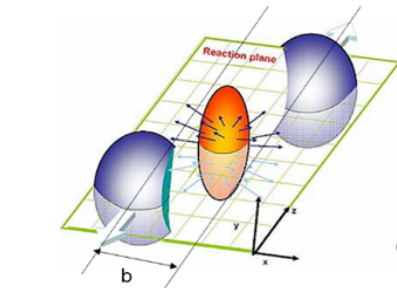
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- motivation and application
- The matrix-valued Wigner function and the total Berry
- transition amplitude decomposition
- H-theorem at leading order
- fixed equation of motion and entropy production
- summary

Motivation

- In spin kinetic theory, **nonlocal collisions** convert orbital into spin angular momentum and polarize the fluid along the vorticity.
- The side jump is encoded by a matrix-valued shift of order the Compton wavelength (Berry curvature).
- Missing piece: a **positive, frame-independent entropy production** for the matrix-valued theory.
- Goal: entropy current + H-theorem, with *fixed* equation of motion.



Weickgenannt, Speranza, Sheng, Wang, Rischke (2020, 2021) Wagner, Weickgenannt, Rischke (2022)
Sheng, Wang, Rischke (2022)

The Collision Operator

Binary elastic collisions, molecular chaos: $\hat{\rho} = \hat{W}_1 \otimes \hat{W}_2$, $\hat{\rho} \rightarrow S\hat{\rho}S^\dagger$, $S = 1 - it$:

$$\Delta\hat{\rho} = t\hat{\rho}t^\dagger - \frac{1}{2}\{t^\dagger t, \hat{\rho}\} - \frac{i}{2}[t + t^\dagger, \hat{\rho}]$$

- Three terms: **gain, loss, coherent spin precession**.
- Unitarity $\Rightarrow$ operator optical theorem: $i(t - t^\dagger) = t^\dagger t$.
- Trace over the unobserved partner and integrate over phase space:

$$\hat{C}[\hat{W}_1] = \int d\Gamma_2 d\Gamma'_1 d\Gamma'_2 \delta^{(4)}(P) \text{Tr}_2[\Delta\hat{\rho}]$$

The Matrix-Valued Wigner Function

The one-particle spin density matrix in phase space is a 2×2 Hermitian matrix:

$$\hat{W}(x, p) = \frac{1}{2} \left[F(x, p) \mathbb{1} + A_\nu(x, p) n^\nu(p) \right]$$

- F : scalar distribution; A_ν : spin polarization (axial vector).
- Positivity of the eigenvalues of $\hat{W}$ is what makes $\ln \hat{W}$ (and the entropy) well defined.
- On-shell measure: $dP = d^4 p \delta(p^2 - m^2)$ ($\theta(p^0)$ understood).

Transition Amplitude Decomposition and Spin-Flip Model

General two-particle transition operator in the 16-element Pauli-product basis:

$$t = a_1 \mathbb{1} \otimes \mathbb{1} + \vec{a}_2 \cdot (\vec{\sigma} \otimes \mathbb{1}) + \vec{a}_3 \cdot (\mathbb{1} \otimes \vec{\sigma}) + a_4^{ij} \sigma_i \otimes \sigma_j$$

- **Explicit two-leg spin-flip model** along collision axis $\hat{n}$ ($a_1 = A$, $\vec{a}_2 = \vec{a}_3 = 0$, $a_4^{ij} = d n_i n_j$):

$$t = A I + d \Sigma, \quad I \equiv \mathbb{1}_1 \otimes \mathbb{1}_2, \quad \Sigma \equiv (\hat{n} \cdot \vec{\sigma}_1) \otimes (\hat{n} \cdot \vec{\sigma}_2)$$

- Since $\Sigma^2 = I$, $\{I, \Sigma\}$ closes under the optical theorem $i(t - t^\dagger) = t^\dagger t$:

Quadratic Rate Operator and Cross Sections

$$t^\dagger t = \sigma_0 I + \sigma_\times \Sigma$$

$$\sigma_0 \equiv |A|^2 + |d|^2 = -2 \operatorname{Im} A, \quad \sigma_\times \equiv 2 \operatorname{Re}(A^* d) = -2 \operatorname{Im} d$$

Gain, Loss, and Coherent Sectors After Tr_2

With $\hat{W}_j = \frac{1}{2}(F_j \mathbb{1} + \vec{A}_j \cdot \vec{\sigma})$, $\vec{A}_j = A_j^\parallel \hat{n} + \vec{A}_j^\perp$, and $\text{Tr}_2(A \otimes B) = A \text{Tr}(B)$:

- **Gain sector** $\text{Tr}_2[t \hat{\rho}' t^\dagger]$ (with $\hat{\rho}' = \hat{W}_3 \otimes \hat{W}_4$):

$$\begin{aligned} & \frac{1}{2}(\sigma_0 F_3 F_4 + \sigma_\times A_3^\parallel A_4^\parallel) \mathbb{1} + \frac{1}{2}(\sigma_0 F_4 A_3^\parallel + \sigma_\times F_3 A_4^\parallel) \hat{n} \cdot \vec{\sigma} \\ & + \frac{1}{2} \left[(|A|^2 - |d|^2) F_4 \vec{A}_3^\perp + 2 \text{Im}(A^* d) A_4^\parallel (\hat{n} \times \vec{A}_3) \right] \cdot \vec{\sigma} \end{aligned}$$

- **Loss sector** $\text{Tr}_2[-\frac{1}{2}\{t^\dagger t, \hat{\rho}\}]$ (with $\hat{\rho} = \hat{W}_1 \otimes \hat{W}_2$):

$$-\frac{1}{2}(\sigma_0 F_1 F_2 + \sigma_\times A_1^\parallel A_2^\parallel) \mathbb{1} - \frac{1}{2} \left[(\sigma_0 F_2 A_1^\parallel + \sigma_\times F_1 A_2^\parallel) \hat{n} + \sigma_0 F_2 \vec{A}_1^\perp \right] \cdot \vec{\sigma}$$

- **Coherent sector** $\text{Tr}_2[-\frac{i}{2}[t + t^\dagger, \hat{\rho}]]$ (scalar A drops out via $[I, \hat{\rho}] = 0$):

$$+ \text{Re}(d) A_2^\parallel (\hat{n} \times \vec{A}_1) \cdot \vec{\sigma}$$

Final Result: Decomposed Collision Kernel

Decomposing the collision operator as $\hat{C}[\hat{W}_1] = C_F[\hat{W}] \mathbb{1} + \vec{C}_A[\hat{W}] \cdot \vec{\sigma}$:

Scalar (Particle Number) Channel $C_F[\hat{W}]$

$$C_F[\hat{W}] = \frac{1}{2} \int d\Gamma_{234} \left[\sigma_0 (F_3 F_4 - F_1 F_2) + \sigma_{\times} (A_3^{\parallel} A_4^{\parallel} - A_1^{\parallel} A_2^{\parallel}) \right]$$

Spin-Polarization Channel $\vec{C}_A[\hat{W}]$

$$\begin{aligned} \vec{C}_A[\hat{W}] = \frac{1}{2} \int d\Gamma_{234} \left\{ \left[\sigma_0 (F_4 A_3^{\parallel} - F_2 A_1^{\parallel}) + \sigma_{\times} (F_3 A_4^{\parallel} - F_1 A_2^{\parallel}) \right] \hat{n} \right. \\ \left. + (|A|^2 - |d|^2) F_4 \vec{A}_3^{\perp} - \sigma_0 F_2 \vec{A}_1^{\perp} \right. \\ \left. + 2 \operatorname{Im}(A^* d) A_4^{\parallel} (\hat{n} \times \vec{A}_3) + 2 \operatorname{Re}(d) A_2^{\parallel} (\hat{n} \times \vec{A}_1) \right\} \end{aligned}$$

The Total Shift Δ_η^μ

Two pieces, built on the same Hermitian matrix structure $\epsilon^{\mu\nu\alpha\beta} u_\nu p_\alpha n_\beta(p)$ with real prefactors:

$$\begin{aligned}\Delta_\eta^\mu &= \Delta_{\eta_0, \eta}^\mu + \chi_{\eta_0}^\mu \\ \Delta_{\eta_0, \eta}^\mu &= \Delta_{\eta_0}^\mu - \Delta_\eta^\mu\end{aligned}$$

$\chi_{\eta_0}^\mu$: the second term of the $\hat{W} \rightarrow F_{\eta_0}$ Berry redefinition.

Stacking identity

$$F_{\eta_0} + \Delta_{\eta_0, \eta} \cdot \partial F_{\eta_0} = \hat{W} + (\Delta_{\eta_0, \eta} + \chi_{\eta_0}) \cdot \partial \hat{W} = \hat{W}_\Delta$$

- Δ_η is Hermitian (both pieces: real prefactors on the same n -structure).
- The total Δ_η is the **phase space displacement** induced by berry curvature: it dresses $\hat{W}_\Delta$, drives $\int \Delta \cdot \partial c$ in the equation of motion, fills the entropy slot, and displaces the collision legs — one object, four appearances.

H-Theorem at Leading Order: Setup

Shiftless Entropy current and its divergence:

$$S^\mu(x) = - \int dP \operatorname{Tr}[p^\mu H(\hat{W})], \quad H(\hat{W}) \equiv \hat{W}(\ln \hat{W} - 1)$$
$$\begin{aligned} \partial_\mu S^\mu &= - \int dP \operatorname{Tr}[(p \cdot \partial \hat{W}) \ln \hat{W}] \\ &= - \int dP \operatorname{Tr}[C[\hat{W}] \ln \hat{W}], \quad \text{using } p \cdot \partial \hat{W} = C[\hat{W}] \end{aligned}$$

- Matrix chain rule: $\delta \operatorname{Tr}(\hat{W} \ln \hat{W} - \hat{W}) = \operatorname{Tr}[\ln \hat{W} \delta \hat{W}]$.
- Sign convention: production $= -\operatorname{Tr}[C \ln \hat{W}]$; the gain–loss structure of C makes this the positive side.

$$\partial_\mu S^\mu = \frac{1}{2} \sum_{I,J} \left[K_{J \leftarrow I} p_I \ln p_I - K_{I \leftarrow J} p_J \ln p_I \right]$$

Swap $I \leftrightarrow J$ in the first term and add the zero sum:

Theorem 1 (leading-order H-theorem)

$$\partial_\mu S^\mu = \frac{1}{2} \sum_{I,J} K_{I \leftarrow J} \left[p_J \ln \frac{p_J}{p_I} - p_J + p_I \right] \geq 0$$

by the scalar inequality $x \ln(x/y) - x + y \geq 0$ for $x, y > 0$, term by term. **Hypotheses:**
S-matrix unitarity and Hermiticity .

Furst, Mendl, Spohn, Wang (2018)

When Does the Production Vanish?

Every term is non-negative, so — *after* the theorem:

Remark (vanishing condition — derived, not assumed)

$\partial_\mu S^\mu = 0$ iff $p_I = p_J$ for every pair with $K_{I \leftarrow J} \neq 0$, i.e. the eigenvalues are constant along collision-connected configurations.

- The theorem itself assumes *only* unitarity + Hermiticity; the sum rule and the change of variables do the work.
- the condition is satisfied in equilibrium such that $C[\hat{W}] = 0$.

Two-particle probabilities ρ_I ; master equation for the two-particle entropy:

$$\partial S = \frac{1}{2} \sum_{I,J} \left[K_{J \leftarrow I} \rho_I \ln \rho_I - K_{I \leftarrow J} \rho_J \ln \rho_J \right], \quad \hat{\rho} = \hat{W}_1 \otimes \hat{W}_2$$

- Demand: a *closed one-particle* equation must reproduce this identically.
- The naive equation $p \cdot \partial \hat{W} = C[\hat{W}(x + \Delta)]$ **fails**: expanding the shifted kernel, the gradient bookkeeping on the loss legs does not close (mismatch $\propto \int \Delta \cdot \partial c$).
- The consistent equation carries the compensating nonlocal term.

Proposition 2 (EOM selection by two-particle consistency)

$$p \cdot \partial \hat{W}(x, p) = C[\hat{W}](x, p) + \int dP_2 dP_3 dP_4 \Delta^\mu \partial_\mu c[\hat{W}](x, p)$$

- Equivalently, with the dressed distribution $\hat{W}_\Delta \equiv \hat{W} + \Delta^\mu \partial_\mu \hat{W}$ and Δ independent of x : $p \cdot \partial \hat{W}_\Delta = C[\hat{W}_\Delta] + O(\hbar^2)$ — exact at $O(\hbar)$.
- All four legs shifted (finite-shift form) $\Leftrightarrow$ dressed streaming form: same statement, two coordinate choices.
- **Provenance:** $\Delta \equiv \Delta_\eta$ (the total of the earlier slides) — the displacement in $\int \Delta \cdot \partial c$ is by construction the same one that dresses $\hat{W}_\Delta$ and fills the entropy slot.

Entropy Current with the Shift Correction

The naive current paired with the dressed collision term leaves a mismatch $\propto \text{Tr}[\Delta \cdot \partial C H'(\hat{W})]$. The fix — a correction slot in the current:

$$S^\mu(x) = - \int dP \text{Tr}[p^\mu H(\hat{W}_\Delta)] - \int dP dP_2 dP_3 dP_4 \text{Tr}[\Delta^\mu c[\hat{W}] H'(\hat{W}_\Delta)] + \partial_\nu S^{\mu\nu},$$

Theorem 3 (production identity)

With the EOM of Prop. 2, exactly at $O(\hbar)$:

$$\partial_\mu S^\mu = - \int dP \text{Tr}[C[\hat{W}_\Delta] \ln \hat{W}_\Delta]$$

- Proof: chain rule on the first slot; the second slot supplies exactly the $\int \Delta \cdot \partial c$ bookkeeping; mismatch cancels term by term.
- $S^{\mu\nu} = -S^{\nu\mu}$: identically conserved (magnetization-type); carries no production.

Theorem 4: The Full H-Theorem

Theorem 4 (positivity of the dressed production)

$\partial_{\vec{k}} S^\mu \geq 0$, with equality iff the vanishing condition of the Remark holds on the eigenvalues of $\hat{W}_\Delta$.

- Expansion of the dressed production to $O(\hbar) =$ the leading-order production of Theorem 1 (on the shifted eigenbasis) + the Δ -correction.
- The correction is exactly the divergence slot already absorbed into the current $\Rightarrow$ the inequality reduces to Theorem 1.
- The full theorem is Theorem 1 *plus bookkeeping* — which is why the leading-order proof came first.

- **Spin hydrodynamics:** covariant entropy current with manifestly positive divergence — the input for moment methods and entropy-current analyses; transport coefficients from the linearized $C[\hat{W}_\Delta]$ near the stationary state.
- **Heavy-ion phenomenology:** Λ polarization; the vanishing condition as the freeze-out criterion for spin; equilibration time scales from the linearized operator.
- **Chiral media:** anomaly-driven corrections
- **quantum computation:** using a projective form of the transition amplitude allows for controlled spin switches.

- Leading-order H-theorem from semi-detailed balance and a change of variables alone — no equilibrium condition assumed.
- t decomposition and spin flipping.
- The total shift is $\Delta_\eta^\mu = \Delta_{\eta_0, \eta}^\mu + \chi_{\eta_0}^\mu$: the cocycle lives on the frame-difference piece only; the dressed $\hat{W}_\Delta$ transports by pure conjugation.
- Two-particle consistency *fixes* the equation of motion: the fully shift-dressed form; the naive equation fails.
- Entropy current with the $\Delta c H'$ slot: production positive definite and Lorentz invariant; the naive equation violates the second law at $O(\hbar)$.
- Applications (spin hydrodynamics, heavy-ion phenomenology, chiral media): in progress.

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- frame-difference piece $\Delta_{\eta_0, \eta}$ *only*; χ_{η_0} is static
- $\hat{W}_\Delta' = D \hat{W}_\Delta D^\dagger + O(\hbar^2)$
- D boost two by two matrices

object	transform
$\hat{W}$	carries boost dressing
$\Delta_{\eta_0, \eta}^\mu$	cocycle; $D \Lambda_\nu^\mu(\cdot) D^\dagger$
$\chi_{\eta_0}^\mu$	static
Δ_η^μ	total (physical)
$\hat{W}_\Delta$	$D(\cdot) D^\dagger$
$C[\hat{W}_\Delta], H'(\hat{W}_\Delta)$	$D(\cdot) D^\dagger$
dP	invariant

H-Theorem at Leading Order: Proof I

Symmetrize under $1 \leftrightarrow 2$, using $\ln \hat{\rho} = \ln \hat{W}_1 \otimes \mathbb{1}_2 + \mathbb{1}_1 \otimes \ln \hat{W}_2$ and $d\mu \equiv d\Gamma_1 d\Gamma_2 d\Gamma'_1 d\Gamma'_2 \delta^{(4)}(P)$:

$$\partial_\mu S^\mu = \frac{1}{2} \int d\mu \operatorname{Tr}_{12} [t^\dagger t \hat{\rho} \ln \hat{\rho} - t \hat{\rho}' t^\dagger \ln \hat{\rho}]$$

- The precession term vanishes identically: $\operatorname{Tr}_{12} [[t + t^\dagger, \hat{\rho}] \ln \hat{\rho}] = 0$ (cyclicity + $[\hat{\rho}, \ln \hat{\rho}] = 0$).
- Spin frames precess during the collision: $[\hat{\rho}, \hat{\rho}'] \neq 0 \Rightarrow$ diagonalize in *distinct* eigenbases,

$$\hat{\rho} = \sum_I p_I |I\rangle \langle I|, \quad \hat{\rho}' = \sum_J p_J |J\rangle \langle J|, \quad W_{I \leftarrow J} = |\langle I | t | J \rangle|^2$$

- All quantum interference is absorbed into $W_{I \leftarrow J}$.

Theorem 5 (frame invariance)

$\partial_{\mu'} S^{\mu'}(x') = \partial_{\mu} S^{\mu}(x)$: the *sign* of the production is frame-independent.

Three steps, one display each:

- 1 $\hat{W}_{\Delta} \rightarrow D \hat{W}_{\Delta} D^{\dagger}$ (stacking lemma: cocycle on $\Delta_{\eta_0, \eta}$; χ_{η_0} static),
- 2 $C[\hat{W}_{\Delta}] \rightarrow D C[\hat{W}_{\Delta}] D^{\dagger}$ (the D 's can be pulled out of the kernel),
- 3 $H'(\hat{W}_{\Delta}) \rightarrow D H'(\hat{W}_{\Delta}) D^{\dagger}$,
- 4 cyclicity of Tr removes the rest.

Each directional component relative to $\hat{n}$ carries a distinct physical role:

- **Number & Longitudinal Spin ($\parallel \hat{n}$):** Both scalar ($|A|^2$) and spin-flip ($|d|^2$) scattering conserve longitudinal spin via σ_0 , while the interference cross section $\sigma_{\times} = 2 \operatorname{Re}(A^* d)$ transfers longitudinal polarization between collision partners.
- **Transverse Dissipative Spin ($\perp \hat{n}$):** Conjugation by Σ acts as a π -rotation about $\hat{n}$ that flips incoming transverse polarization ($\vec{A}_3^{\perp} \rightarrow -\vec{A}_3^{\perp}$), yielding $(|A|^2 - |d|^2) F_4 \vec{A}_3^{\perp}$ in the gain term against $-\sigma_0 F_2 \vec{A}_1^{\perp}$ in the loss term.
- **Reversible Spin Precession ($\hat{n} \times \vec{A}$):** The coherent commutator is purely traceless and independent of A , driving mean-field Larmor precession $\propto \operatorname{Re}(d) A_2^{\parallel} (\hat{n} \times \vec{A}_1)$ without producing entropy.