

Quantum fluctuation energies over a spatially inhomogeneous field background in a chiral soliton model

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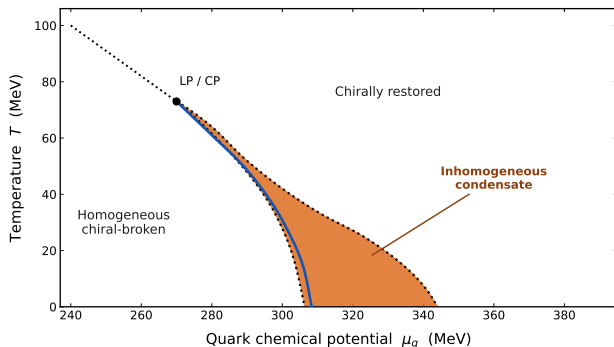
Phys. Rev. C **113** (2026) 4, 045202

Spicy Gluons, Wuhan, October 17–20, 2026



Inhomogeneous Condensates and the QCD Phase Diagram

Two-flavor NJL model: phase boundaries with and without spatially modulated condensates.



Evidence for QCD

QGP in heavy-ion collisions;
crossover near $\mu_B = 0$ from lattice QCD.

Model predictions

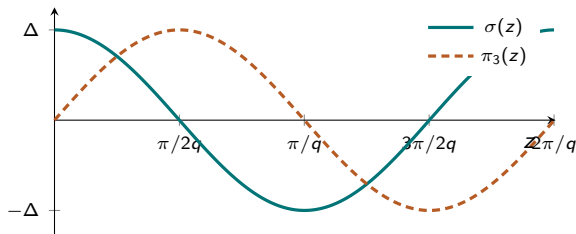
Critical point, high-density first-order transition and spatially inhomogeneous phases are *not experimentally established*.

Allowing a nonuniform condensate can change the candidate phase boundaries.

Shaded region: inhomogeneous NJL solution. Blue: homogeneous-only first-order boundary. Approximate vector reproduction of Carignano & Buballa, *Acta Phys. Pol. B Proc. Suppl.* **5**, 641 (2012), Fig. 1, based on Nickel, *PRD* **80**, 074025 (2009); axes use quark μ_q , not μ_B .

Spatially Inhomogeneous Chiral Backgrounds

A spatial modulation of the chiral condensate



Analytic CDW/spiral ansatz (illustrative model profile, not measured data).

$$\sigma(z) + i\pi_3(z) = \Delta e^{iqz}.$$

A periodic example

The scalar and pion fields vary with position.

Our calculation

A different configuration: a spatially localized chiral soliton.

Spatial inhomogeneity motivates the question; our numerical background is a localized soliton.

Refs: M. Buballa and S. Carignano, Prog. Part. Nucl. Phys. 81, 39 (2015).

The Chiral Soliton Model

Two-flavor linear σ model with quark Yukawa coupling

$$\mathcal{L} = \bar{\psi} [i\gamma^\mu \partial_\mu - g(\sigma + i\gamma_5 \boldsymbol{\tau} \cdot \boldsymbol{\pi})] \psi + \frac{1}{2}(\partial_\mu \sigma)^2 + \frac{1}{2}(\partial_\mu \boldsymbol{\pi})^2 - U,$$
$$U = \frac{\lambda}{4}(\sigma^2 + \boldsymbol{\pi}^2 - \nu^2)^2 - H\sigma + \text{const.}$$

- Vacuum: $\sigma = f_\pi$, $\boldsymbol{\pi} = 0$.
- Constituent quark mass: $m_q = g f_\pi$.
- Explicit chiral breaking: $H = f_\pi m_\pi^2$.

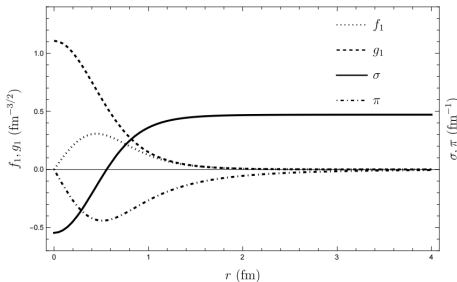
Chiral-model parameter inputs

$$f_\pi = 93 \text{ MeV}, \quad m_\pi = 138 \text{ MeV}$$
$$m_q = 491 \text{ MeV}, \quad m_\sigma = 1200 \text{ MeV}$$

Refs: M. C. Birse and M. K. Banerjee, PLB 136, 284 (1984); C. V. Christov et al., PPNP 37, 91 (1996).

The Mean-Field Hedgehog Soliton

Mean-field chiral background



$$\sigma(\mathbf{r}) = \sigma(r), \quad \pi(\mathbf{r}) = \hat{\mathbf{r}} \pi(r)$$

A tractable spectral problem

- Spatially localized, rather than a periodic bulk condensate.
- Dirac partial-wave channels.
- Bound levels and continuum phase shifts.
- Renormalized fermion one-loop correction.

This VPE calculation concerns a localized mean-field profile, not the phase diagram of bulk CDW matter.

Refs: Model background: Xia-Shu-Li, arXiv:2503.04015v4, Fig. 1; Birse and Banerjee, PLB 136, 284 (1984).

Quark Vacuum Fluctuations around the Soliton

We evaluate quark vacuum fluctuations on the prescribed soliton profile. The reference is the homogeneous vacuum:

$$\sigma(r) = f_\pi + \tilde{\sigma}(r), \quad \boldsymbol{\pi}(r) = \hat{\mathbf{r}} \tilde{\pi}(r), \quad m_q = g f_\pi.$$

$$D_M[\tilde{\Phi}] = i\gamma^\mu \partial_\mu - m_q - g(\tilde{\sigma} + i\gamma_5 \boldsymbol{\tau} \cdot \hat{\mathbf{r}} \tilde{\pi}).$$

Integrating out the quark fields gives the relative one-loop effective action:

$$\Delta S_{M,F}^{\text{1loop}} = -i \text{Tr} \ln \left(D_M[\tilde{\Phi}] D_{M,0}^{-1} \right).$$

$D_{M,0}$ is the vacuum operator; tilded fields are fixed deviations. Meson loops are not included.

Refs: Xia–Shu–Li, arXiv:2503.04015v4, Sec. III.A; E. Farhi et al., NPB 630, 241 (2002).

Vacuum Energy: Effective Action and Spectral Sum

From the effective action.

For a time-independent background, the vacuum energy is minus the relative effective action per unit observation time:

$$E_{\text{VPE}}^{\text{formal}} = - \lim_{\mathcal{T} \rightarrow \infty} \frac{\Delta S_{M,F}^{\text{1loop}}}{\mathcal{T}}.$$

From the spectrum.

The same contribution can also be written as the difference between the background and homogeneous-vacuum level sums:

$$E_{\text{VPE}}^{\text{formal}} = -\frac{1}{2} \left[\sum_{\alpha} |E_{\alpha}| - \sum_k |E_q(k)| \right].$$

E_{α} includes bound and scattering states; $E_q(k)$ is the free reference. Both signed branches are counted. $\mathcal{T}$ is time, not temperature.

Local counterterms complete $E_{\text{vac}} = E_{\text{vac}}^{\psi} + E_{\text{ct}}$.

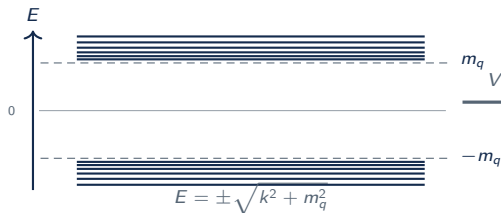
Refs: N. Graham, M. Quandt and H. Weigel, Spectral Methods in Quantum Field Theory, LNP 777 (2009).

From the Free Dirac Spectrum to Bound States

$$E_{\text{VPE}}^{\text{formal}} = -\frac{1}{2} \left[\sum_{\alpha} |E_{\alpha}| - \sum_k |E_q(k)| \right]$$

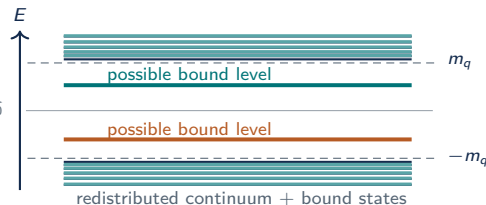
Homogeneous reference: H_0

$$H_0 \psi_{\alpha}^{(0)} = E_{\alpha}^{(0)} \psi_{\alpha}^{(0)}$$



Inhomogeneous profile: $H[\Phi]$

$$H[\Phi] \psi_{\alpha} = E_{\alpha} \psi_{\alpha}$$



A localized background can create gap bound levels and redistribute continuum states. Both changes contribute to the vacuum-energy difference.

Refs: Schematic spectrum, not numerical level data. Farhi et al., *Nucl. Phys. B* **630**, 241 (2002).

Vacuum Energy from the Density of States

Write $\omega_k = \sqrt{k^2 + m_q^2}$; all state multiplicities are included in b and ρ_s .

$$E_{\text{vac}}^{\text{hom}} = -\frac{1}{2} \sum_{s=\pm} \int_0^\infty dk \, \omega_k \rho_{0,s}(k),$$

$$E_{\text{vac}}^{\text{inhom}}[\Phi] = E_{\text{bound}}[\Phi] - \frac{1}{2} \sum_{s=\pm} \int_0^\infty dk \, \omega_k \rho_{\Phi,s}^{\text{cont}}(k).$$

$$E_{\text{bound}} = -\frac{1}{2} \sum_b |E_b|, \quad \Delta\rho_s = \rho_{\Phi,s}^{\text{cont}} - \rho_{0,s}.$$

$$\begin{aligned} E_{\text{VPE}}^{\text{formal}} &= E_{\text{vac}}^{\text{inhom}}[\Phi] - E_{\text{vac}}^{\text{hom}} \\ &= E_{\text{bound}} - \frac{1}{2} \sum_{s=\pm} \int_0^\infty dk \, \omega_k \Delta\rho_s(k). \end{aligned}$$

The individual vacuum energies are formal; use the same reference counting and a common regulator throughout.

Refs: E. Farhi et al., PLB 427, 334 (1998); Graham, Quandt and Weigel, LNP 777 (2009).

From Large-Box Quantization to the Density of States

Consider one scattering channel in a spherical box of radius R . Suppress the *same* boundary/centrifugal phase in both quantization conditions:

Homogeneous background

$$kR = n_0(k)\pi$$

$$\underbrace{n_\Phi(k) - n_0(k)}_{\text{smooth level-count difference}} = \frac{\delta(k)}{\pi} \xrightarrow[\text{continuum } k]{R \rightarrow \infty}$$

Inhomogeneous background

$$kR + \delta(k) = n_\Phi(k)\pi$$

$$\rho_\Phi^{\text{cont}}(k) - \rho_0(k) = \frac{1}{\pi} \frac{d\delta}{dk}.$$

General coupled channels: $\delta_a(k) = (2i)^{-1} \ln \det S_a(k)$, with the phase branch fixed continuously.

In our chiral model: $a = (G, \Pi, s)$, i.e. grand spin, parity, and signed energy branch, with multiplicity $(2G + 1)$. The densities on the preceding pages sum these channels.

Phase shifts count how the *continuum* levels move; discrete bound levels must be added separately.

Refs: S. Kahana and G. Ripka, NPA 429, 462 (1984); Graham, Quandt and Weigel, LNP 777 (2009).

Fourth-Order Born Subtraction of the Phase Shifts

The Born series expands the phase shift in powers of the background interaction: $\delta = \delta^{(1)} + \delta^{(2)} + \dots$.

In 3+1 dimensions, ultraviolet-divergent contributions occur within the first four orders:

$$E_{\text{VPE}}^{\text{formal}} = E_{\text{bound}} - \frac{1}{2\pi} \int_0^\infty dk \, \omega_k \frac{d\delta(k)}{dk} \quad (\text{UV divergent})$$

$$\bar{\delta}(k) = \delta(k) - \sum_{n=1}^4 \delta^{(n)}(k)$$

The four-order-subtracted spectral remainder is

$$E_{\text{spec}}^{\text{sub},(4)} = E_{\text{bound}} - \frac{1}{2\pi} \int_0^\infty dk \, \omega_k \frac{d\bar{\delta}(k)}{dk} \quad (\text{UV convergent}).$$

$\delta(k)$ is the total phase shift, with both signed branches and state multiplicities summed consistently.

Refs: E. Farhi et al., PLB 427, 334 (1998); NPB 630, 241 (2002).

Born–Feynman Matching: Equal Energy Contributions

At each order in the background interaction, the Born term and the corresponding fermion loop give the same vacuum-energy contribution.

$$\underbrace{-\frac{1}{2\pi} \int_0^\infty dk \, \omega_k \frac{d\delta^{(n)}(k)}{dk}}_{E_{\text{Born}}^{(n)}} = E_{F,\text{loop}}^{(n)}, \quad n = 1, 2, 3, 4.$$



$n = 1$: tadpole



$n = 2$: bubble



$n = 3$: triangle



$n = 4$: box

Solid arrows: fermion propagators. Dashed legs: external background insertions.

The energy removed by Born subtraction is restored by the matching loop diagrams.

Same regulator, background and state counting; before adding counterterms.

Refs: E. Farhi et al., NPB 630, 241 (2002); Graham, Quandt and Weigel, LNP 777 (2009).

From Born Subtraction to the Renormalized VPE

Same background, spectral counting and finite prescription in every term.

$$\begin{aligned}
 E_{\text{VPE}}^{\text{ren}} &= \lim_{\text{reg} \rightarrow 0} \left\{ E_{\text{bound}} + E_{\text{cont}}[\delta] + E_{\text{ct}} \right\} \\
 &= \lim_{\text{reg} \rightarrow 0} \left\{ E_{\text{bound}} + E_{\text{cont}}[\bar{\delta}] + \sum_{n=1}^4 E_{F,\text{loop}}^{(n)} + E_{\text{ct}} \right\} \\
 &= E_{\text{bound}} + E_{\text{cont}}[\bar{\delta}] + \boxed{\sum_{n=1}^4 \Gamma_{F,\text{ren}}^{(n)}}.
 \end{aligned}$$

Subtract

Phase shifts: Born 1–4.

Add back

Matching fermion loops 1–4.

Renormalize

Loops + counterterms: finite.

Common regulator in the first two lines; $\Gamma_{F,\text{ren}}^{(n)}$ already includes counterterms. All four orders, including the triangle, are retained. The same background and finite conditions are required throughout.

Refs: E. Farhi et al., NPB 630, 241 (2002); M. Quandt, N. Graham and H. Weigel, NPB 923, 350 (2017).

Results

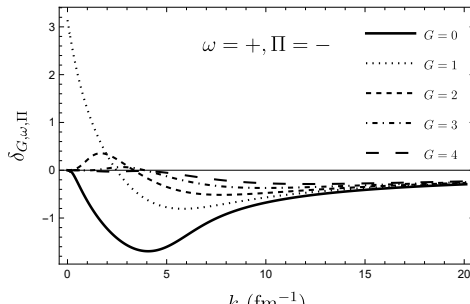
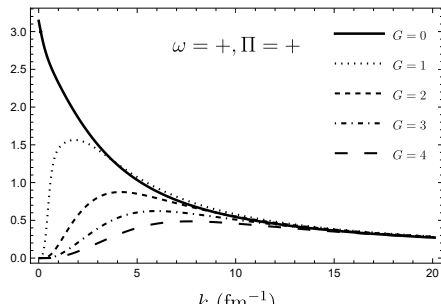
From scattering data to the vacuum-energy budget

- ① Scattering phase shifts in the chiral background
- ② Four identified bound levels and the subtracted continuum
- ③ Fermionic diagram add-back and the preliminary VPE

Direct fourth-order fermionic subtraction on a fixed chiral background.

Scattering Phase Shifts in the Chiral Background

Positive-energy continuum, both parity sectors



Different grand-spin channels show distinct momentum dependence; the VPE also requires $E < 0$ scattering states.

Actual phase-shift curves from the chiral-soliton calculation.

Refs: Xia, Shu & Li, *Phys. Rev. C* **113**, 045202 (2026), Fig. 2.

Direct Calculation: Four Identified Bound Levels

Fixed chiral background; $N_c = 3$. The four tracked levels contribute

$$E_{\text{bound}} = -\frac{N_c}{2} \sum_{G,\Pi} (2G+1) \sum_b |E_b|.$$

Channel (G, Π)	E_b fm^{-1}	$2G+1$	Energy contribution fm^{-1}
(0, +)	+0.215955	1	−0.323933
(0, +)	−2.474932	1	−3.712398
(1, −)	+2.429613	3	−10.933261
(1, +)	−1.862535	3	−8.381409
Total			−23.351001

Each signed level is counted once; no additional factor of two.

Refs: Current direct fermion calculation. Four identified levels; completeness remains open. No extra valence-occupation energy is included.

Continuum Energy and Fermionic Diagram Add-Back

Fixed chiral background; $N_c = 3$; direct Born 1–4 subtraction.

Born-subtracted continuum

$$E_{\text{cont}}^{\text{sub}(1-4)} \simeq +29.2 \text{ fm}^{-1}$$

Computed range: $+(29.19\text{--}29.80) \text{ fm}^{-1}$.

$G = 0, \dots, 4$; both energy signs and parities.

$+29.2$ is an illustrative representative value,
not a converged limit.

Renormalized fermionic diagrams

Contribution	fm^{-1}
$\Gamma_F^{(1),\text{ren}}$	0
$\Gamma_F^{(2),\text{ren}}$	−13.4038
$\Gamma_F^{(3),\text{ren}}$	+33.9475
$\Gamma_F^{(4),\text{ren}}$	−29.8526
Total add-back	−9.3089

The continuum and the diagram add-back are separate contributions to the VPE.

Refs: Current direct fermion calculation. Color counted once; specified finite subtraction (conditions in Backup). Finite-cutoff exploratory results; high-partial-wave convergence and physical finite matching remain open.

Preliminary VPE: A Complete Energy Budget

Direct fermionic Born 1–4 subtraction on the fixed chiral background:

$$E_{\text{VPE}}^{(4)} = E_{\text{bound}} + E_{\text{cont}}^{\text{sub}(1-4)} + \sum_{n=1}^4 \Gamma_{F,\text{ren}}^{(n)}.$$

Contribution	fm^{-1}
Four identified bound levels	−23.351001
Born-subtracted continuum	+29.260522
Renormalized fermionic diagrams, orders 1–4	−9.308918
Exploratory truncated VPE	−3.399396

Current exploratory estimates: $E_{\text{VPE}}^{(4)} = -3.469$ to -2.857 fm^{-1} .

Bound and continuum terms partly cancel; diagram add-back shifts the energy downward.

Refs: Current direct fermion calculation; $N_c = 3$, specified finite subtraction (conditions in Backup). Finite momentum and partial-wave cutoffs; high- G convergence, bound-spectrum completeness and physical finite matching remain open.

Conclusions and Finite-Temperature Outlook

- ① **Physical setting:** quark vacuum fluctuations on a prescribed, localized mean-field chiral soliton.
- ② **Spectral method:** bound levels and continuum phase shifts express the vacuum-energy difference.
- ③ **Renormalization:** Born subtraction and renormalized diagrammatic add-back complete the calculation.

Outlook

Finite temperature requires the full physical thermal spectrum and minimization over competing backgrounds.

Thank you for your attention.

Refs: Finite-temperature extension: S. Shu, X. Li and J.-R. Li, NPA 1014, 122256 (2021); X. Li, S. Shu and J.-R. Li, PRC 105, 045203 (2022).

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Quantum fluctuation energies over a spatially inhomogeneous field background in a chiral soliton model

arXiv:2503.04015v4 (supplied source manuscript).

B0 | Backup Roadmap

① A1–A7: Born–Feynman correspondence

Determinant expansion, resolvent identity, density of states and energy matching.

② B1–B11: Direct fermion-loop renormalization

Dimensional regularization, explicit local counterterms, finite conditions and loop integrals.

③ C1–C5: Effective action to the spectral VPE

Fermion determinant, static frequency integral, bound states and phase shifts.

All backup traces exclude color; N_c is explicit. Two flavors are included in the isospin trace.

A1 | Operators and Counting Conventions

Put $f = f_\pi$, $s = \sigma - f$, $m = gf$ and $P = \tau \cdot \pi$. Introduce a bookkeeping parameter λ :

$$D_{E,\lambda} = D_{E,0} + \lambda V_E,$$

$$V_E = g(s + i\gamma_5 P),$$

$$H_\lambda = H_0 + \lambda V_H,$$

$$D_{E,0} = \gamma_\mu \partial_\mu + m,$$

$$S_{E,0} = D_{E,0}^{-1},$$

$$V_H = \gamma_4 V_E.$$

For static fields, $D_{E,\lambda} = \gamma_4(\partial_\tau + H_\lambda)$ and $H_\lambda = H_\lambda^\dagger$.

- $S_{E,0}$ is a **four-dimensional Euclidean propagator**.
- $G_\lambda(z) = (z - H_\lambda)^{-1}$ is a **three-dimensional Hamiltonian resolvent**.
- Tr_4 includes Euclidean coordinates, Dirac and isospin indices; Tr_3 excludes time. Neither includes color.

Refs: Farhi et al., PLB 427, 334 (1998); NPB 630, 241 (2002); Graham–Quandt–Weigel, LNP 777 (2009).

A2 | Log Expansion and the Four Fermion Loops

The relative Euclidean fermion action and static energy are

$$\Delta\Gamma_{E,F}(\lambda) = -N_c \text{Tr}_4 \ln(1 + \lambda S_{E,0} V_E), \quad E_F(\lambda) = \lim_{T \rightarrow \infty} \frac{\Delta\Gamma_{E,F}(\lambda)}{\mathcal{T}}.$$

$$E_F(\lambda) = \sum_{n \geq 1} \lambda^n E_{F,\text{loop}}^{(n)},$$

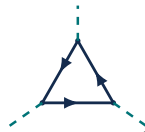
$$E_{F,\text{loop}}^{(n)} = \frac{(-1)^n N_c}{n \mathcal{T}} \text{Tr}_4 (S_{E,0} V_E)^n.$$



$n = 1$: tadpole



$n = 2$: bubble



$n = 3$: triangle



$n = 4$: box

The factor $1/n$ follows from the logarithm; n counts insertions, not loops. A common regulator is understood before expanding.

Refs: Farhi et al., PLB 427, 334 (1998); NPB 630, 241 (2002); Graham–Quandt–Weigel, LNP 777 (2009).

A3 | Resolvent Born Series

For z off the spectrum, define $G_0(z) = (z - H_0)^{-1}$. Then

$$G_\lambda = (G_0^{-1} - \lambda V_H)^{-1} = (1 - \lambda G_0 V_H)^{-1} G_0,$$

$$G_\lambda - G_0 = \sum_{n \geq 1} \lambda^n (G_0 V_H)^n G_0.$$

The corresponding density-of-states difference is

$$\Delta \rho_\lambda(E) = -\frac{1}{\pi} \operatorname{Im} \operatorname{Tr}_3 [G_\lambda(E + i0) - G_0(E + i0)].$$

Therefore the order- n coefficient is

$$\Delta \rho^{(n)}(E) = -\frac{1}{\pi} \operatorname{Im} \operatorname{Tr}_3 [(G_0 V_H)^n G_0]_{z=E+i0}.$$

The Born order is the power of the same background parameter λ .

Refs: Farhi et al., PLB 427, 334 (1998); NPB 630, 241 (2002); Graham–Quandt–Weigel, LNP 777 (2009).

A4 | Deriving the Resolvent–Determinant Identity

Define the regulated relative logarithm

$$F_\lambda(z) = \text{Tr}_3 \ln(1 - \lambda G_0(z) V_H) = \text{Tr}_3 \ln(z - H_\lambda) - \text{Tr}_3 \ln(z - H_0).$$

Since $\partial_z G_0 = -G_0^2$, cyclicity of the regulated trace gives

$$F_\lambda(z) = - \sum_{n \geq 1} \frac{\lambda^n}{n} \text{Tr}_3 (G_0 V_H)^n,$$

$$\partial_z \text{Tr}_3 (G_0 V_H)^n = -n \text{Tr}_3 [(G_0 V_H)^n G_0],$$

$$\boxed{\partial_z F_\lambda(z)} = \boxed{\text{Tr}_3 (G_\lambda - G_0)}.$$

Work first off the spectrum with a common regulator; only then take boundary values. The logarithm branch is continued from $\lambda = 0$.

Refs: Farhi et al., PLB 427, 334 (1998); NPB 630, 241 (2002); Graham–Quandt–Weigel, LNP 777 (2009).

A5 | From the Determinant Phase to Born Energy

Use an **energy-oriented** total spectral phase on the full real-energy axis:

$$\delta_{E,\lambda}(E) = -\operatorname{Im} F_\lambda(E + i0), \quad \Delta\rho_\lambda(E) = \frac{1}{\pi} \frac{d\delta_{E,\lambda}}{dE}.$$

Coefficient by coefficient,

$$\delta_E^{(n)}(E) = \frac{1}{n} \operatorname{Im} \operatorname{Tr}_3(G_0 V_H)^n, \quad \Delta\rho^{(n)} = \frac{1}{\pi} \frac{d\delta_E^{(n)}}{dE}.$$

The regulated charge-even energy coefficient is

$$E_{\text{spec}}^{(n)} = -\frac{N_c}{2} \int_{-\infty}^{\infty} dE |E| \Delta\rho^{(n)}(E).$$

Here the full spectral measure includes bound-state residues where present. A continuum-only version requires consistent threshold and endpoint bookkeeping.

Refs: Farhi et al., PLB 427, 334 (1998); NPB 630, 241 (2002); Graham–Quandt–Weigel, LNP 777 (2009).

A6 | Energy Matching: A Static Frequency Proof

Differentiate the determinant energy before removing the regulator:

$$\begin{aligned}\partial_\lambda E_F &= -N_c \int \frac{d\omega}{2\pi} \text{Tr}_3[(i\omega + H_\lambda)^{-1} V_H] \\ &= -\frac{N_c}{2} \text{Tr}_3[\text{sgn}(H_\lambda) V_H],\end{aligned}$$

using $\int d\omega (i\omega + e)^{-1}/(2\pi) = \frac{1}{2} \text{sgn}(e)$ in the symmetric frequency prescription.

$$\partial_\lambda \left[-\frac{N_c}{2} \text{Tr}_3(|H_\lambda| - |H_0|) \right] = -\frac{N_c}{2} \text{Tr}_3[\text{sgn}(H_\lambda) V_H].$$

Both relative energies vanish at $\lambda = 0$. Their regulated Taylor coefficients therefore agree:

$$E_{F,\text{loop}}^{(n)} = E_{\text{spec}}^{(n)}.$$

Refs: Static Hermitian background; differentiate near $\lambda = 0$ away from zero crossings. Threshold rearrangements are included in the full spectral measure.

A7 | Four-Order Subtraction and Add-Back

Let $a = (G, \Pi, \eta)$, $\eta = \pm$, $d_a = 2G + 1$, and $\omega_k = \sqrt{k^2 + m^2}$.

$$\bar{\delta}_a = \delta_a - \sum_{n=1}^4 \delta_a^{(n)}, \quad E_{\text{cont}}[\delta] = -\frac{N_c}{2\pi} \sum_a d_a \int_0^\infty dk \omega_k \delta'_a(k).$$

$$\begin{aligned} E_{\text{VPE}}^{\text{ren}} &= \lim_{\text{reg} \rightarrow 0} \{E_{\text{bound}} + E_{\text{cont}}[\delta] + E_{\text{ct}}\} \\ &= \lim_{\text{reg} \rightarrow 0} \{E_{\text{bound}} + E_{\text{cont}}[\bar{\delta}] + \sum_{n=1}^4 E_{F,\text{loop}}^{(n)} + E_{\text{ct}}\} \\ &= E_{\text{bound}} + E_{\text{cont}}[\bar{\delta}] + \sum_{n=1}^4 \Gamma_F^{(n),\text{ren}}. \end{aligned}$$

The final $\Gamma_F^{(n),\text{ren}}$ already includes counterterms. Four-order subtraction improves UV behavior; it does not certify finite- G , k numerical convergence.

Refs: Farhi et al., PLB 427, 334 (1998); NPB 630, 241 (2002); Graham–Quandt–Weigel, LNP 777 (2009).

B1 | Local Counterterms and Normalization

For two flavors define $X = \sigma^2 + \pi^2 - f^2 = 2fs + s^2 + \pi^2$.

$$E_{\text{ct}} = \int d^3x \{ 2a[(\nabla s)^2 + (\nabla \pi)^2] + 2bX + 2cX^2 \}.$$

- a : wave-function counterterm; b : quadratic invariant; c : quartic invariant.
- The field-independent vacuum constant is removed by the relative reference normalization.
- Write $a = a_{\text{pole}} + a_T$, and likewise for b, c . Factors of two follow the displayed definition.

Pole cancellation is fixed by UV structure; the finite coefficients are fixed by stated renormalization conditions.

Refs: Two-flavor direct-fermion convention used here. Do not transplant coefficients between differently normalized counterterm actions.

B2 | Dimensional Regularization: Basic Integrals

Use $d = 4 - 2\epsilon$, $\Delta_\epsilon = 1/\epsilon - \gamma_E + \ln 4\pi$ and $L = \ln(\mu^2/m^2)$.

$$\begin{aligned}
 I_1 &= \mu^{2\epsilon} \int \frac{d^d p}{(2\pi)^d} \frac{1}{p^2 + m^2} = -\frac{m^2}{16\pi^2} (\Delta_\epsilon + L + 1) + O(\epsilon), \\
 I_2(z) &= \mu^{2\epsilon} \int \frac{d^d p}{(2\pi)^d} \frac{1}{(p^2 + m^2)((p+q)^2 + m^2)} \\
 &= \frac{1}{16\pi^2} \left[\Delta_\epsilon + L - \int_0^1 dx \ln \left(1 + \frac{z}{m^2} x(1-x) \right) \right] + O(\epsilon),
 \end{aligned}$$

where $z = q_E^2$. Set $B(z) = L - \int_0^1 dx \ln[1 + zx(1-x)/m^2]$. The negative pole of I_1 is a dimensional-regularization result, not a positive-cutoff integral. $\overline{\text{MS}}$ removes Δ_ϵ .

Refs: Euclidean integrals; the timelike continuation used below stays below $z = -4m^2$.

B3 | Tadpole and Two-Point Dirac Traces

Let $S(p) = (-i\gamma \cdot p + m)/(p^2 + m^2)$ and $\text{tr}_I 1 = 2$.

$$t_{\text{loop}} = -8N_c g m l_1,$$

$$\Pi_\pi(z) = -8N_c g^2 l_1 + 4N_c g^2 z l_2(z),$$

$$\Pi_\sigma(z) = -8N_c g^2 l_1 + 4N_c g^2 (z + 4m^2) l_2(z).$$

These kernels are defined by

$$E_{1,\text{loop}} = t_{\text{loop}} \int d^3x s, \quad E_{2,\text{loop}} = \frac{1}{2} \int_{\mathbf{q}} [\tilde{s}(-q) \Pi_\sigma(z) \tilde{s}(q) + \tilde{\pi}_a(-q) \Pi_\pi(z) \tilde{\pi}_a(q)].$$

For example, $\text{tr}_{D,I}[S(p)ig\gamma_5\tau_a S(p+q)ig\gamma_5\tau_b] = -8g^2\delta_{ab}[p \cdot (p+q) + m^2]/(AB)$, with $A = p^2 + m^2$, $B = (p+q)^2 + m^2$.

Refs: Repeated pion index summed; $\int_{\mathbf{q}} = \int d^3q/(2\pi)^3$, static $q_4 = 0$.

B4 | UV Poles and Explicit Pole Counterterms

The two-flavor local pole of the relative static fermion energy is

$$E_{F,\text{pole}} = \frac{N_c \Delta_\epsilon}{8\pi^2} \int d^3x \{ g^2 [(\nabla s)^2 + (\nabla \pi)^2] + g^4 [2f^2 X + X^2] \}.$$

Thus $E_{\text{ct,pole}} = -E_{F,\text{pole}}$ is obtained with

$$a_{\text{pole}} = -\frac{N_c g^2}{16\pi^2} \Delta_\epsilon, \quad b_{\text{pole}} = -\frac{N_c g^2 m^2}{8\pi^2} \Delta_\epsilon, \quad c_{\text{pole}} = -\frac{N_c g^4}{16\pi^2} \Delta_\epsilon.$$

Checks: $\Pi_{\pi,\text{pole}} = N_c g^2 \Delta_\epsilon (z + 2m^2)/(4\pi^2)$ and $\Pi_{\sigma,\text{pole}} = N_c g^2 \Delta_\epsilon (z + 6m^2)/(4\pi^2)$ are cancelled by $4az + 4b$ and $4az + 4b + 16cf^2$, respectively.

Refs: Obtained from B2–B3 and the invariant constant-background potential; total derivatives vanish for localized smooth fields.

B5 | Counterterms at Each Background Order

Expand the same E_{ct} in powers of (s, π) :

$$E_{\text{ct}}^{(1)} = \int d^3x \, 4bf \, s,$$

$$E_{\text{ct}}^{(2)} = \int d^3x \, [2a((\nabla s)^2 + (\nabla \pi)^2) + 2b(s^2 + \pi^2) + 8cf^2 s^2],$$

$$E_{\text{ct}}^{(3)} = \int d^3x \, 8cf \, s(s^2 + \pi^2),$$

$$E_{\text{ct}}^{(4)} = \int d^3x \, 2c(s^2 + \pi^2)^2.$$

The same three coefficients fix all four orders. In particular, the triangle has a nonzero local counterterm and cannot be omitted by importing a vector-current cancellation argument.

Refs: Both pole and finite parts follow this expansion. The vacuum constant has already been subtracted.

B6 | Finite Conditions: Solve for the Three Coefficients

After $\overline{\text{MS}}$ pole subtraction, impose loop-correction conditions at $z_* = -m_s^2$:

$$0 = t_{\overline{\text{MS}}} + 4b_T f,$$

$$0 = \Pi'_{\sigma, \overline{\text{MS}}}(z_*) + 4a_T,$$

$$0 = \Pi_{\sigma, \overline{\text{MS}}}(z_*) + 4a_T z_* + 4b_T + 16c_T f^2.$$

Consequently,

$$b_T = -\frac{t_{\overline{\text{MS}}}}{4f}, \quad a_T = -\frac{1}{4}\Pi'_{\sigma, \overline{\text{MS}}}(z_*),$$

$$c_T = -\frac{\Pi_{\sigma, \overline{\text{MS}}}(z_*) + 4a_T z_* + 4b_T}{16f^2}.$$

The pion Ward check $\Pi_{\pi, \text{ren}}(0) = 0$ follows from the same tadpole condition in this invariant loop sector; it is not a fourth independent condition.

Refs: Conditions act on loop corrections. They do not set the full physical pion mass to zero or identify m_s with a physical sigma pole.

B7 | Explicit T-like Finite Coefficients

Put $r = m_s^2/m^2$, $u = x(1-x)$ and $L = \ln(\mu^2/m^2)$, with $0 < r < 4$.

$$a_T = -\frac{N_c g^2}{16\pi^2} \left[L - \frac{2}{3} - 6 \int_0^1 dx u \ln(1 - ru) \right],$$

$$b_T = -\frac{N_c g^2 m^2}{8\pi^2} (L + 1),$$

$$c_T = -\frac{N_c g^4}{16\pi^2} \left[L - \frac{r}{4} - \frac{3}{2} \int_0^1 dx \ln(1 - ru) \right].$$

These follow from B6 with $B'(z) = -\int_0^1 dx u/(m^2 + zu)$. Useful reduction identities are

$$J_0 + (4 - r)J_1 = \frac{2}{3} + 6J_u, \quad r(4 - r)J_1 = r + 2J_0,$$

where $J_0 = \int \ln(1 - ru)$, $J_u = \int u \ln(1 - ru)$ and $J_1 = \int u/(1 - ru)$ on $[0, 1]$.

Refs: Current subtraction point $m_s \simeq 842.7$ MeV; $m = 491$ MeV. This is a finite scheme, not completed physical on-shell matching.

B8 | Renormalized Tadpole and Bubble

With $\hat{\Pi}_\alpha = \Pi_{\alpha, \overline{\text{MS}}}$ from B3,

$$\Gamma_F^{(1), \text{ren}} = (t_{\overline{\text{MS}}} + 4b_T f) \int d^3x s = 0,$$

$$\Pi_{\pi, \text{ren}}(z) = \hat{\Pi}_\pi(z) + 4a_T z + 4b_T,$$

$$\begin{aligned} \Pi_{\sigma, \text{ren}}(z) &= \hat{\Pi}_\sigma(z) + 4a_T z + 4b_T + 16c_T f^2 \\ &= \hat{\Pi}_\sigma(z) - \hat{\Pi}_\sigma(z_*) - (z - z_*) \hat{\Pi}'_\sigma(z_*). \end{aligned}$$

The finite second-order energy is

$$\Gamma_F^{(2), \text{ren}} = \frac{1}{2} \int_{\mathbf{q}} \left[|\tilde{s}(q)|^2 \Pi_{\sigma, \text{ren}}(\mathbf{q}^2) + \sum_a |\tilde{\pi}_a(q)|^2 \Pi_{\pi, \text{ren}}(\mathbf{q}^2) \right].$$

The double subtraction is explicit in the scalar kernel; the pion kernel uses the same invariant counterterms, not an independently adjusted finite term.

Refs: Fourier transform $\tilde{s}(q) = \int d^3x e^{-iq \cdot x} s(x)$, likewise for π .

B9 | Triangle and Box: Momentum-Space Integrals

Write $V_i = \tilde{V}_E(\mathbf{q}_i)$, $q_i = (0, \mathbf{q}_i)$, and $S_j = S(p + Q_j)$.

$$E_{3,\text{loop}} = -\frac{N_c}{3} \int_{\mathbf{q}_1, \mathbf{q}_2} \int_p \text{tr}_{D,I} [S(p) V_1 S(p + q_1) V_2 S(p + q_1 + q_2) V_{-12}],$$

$$E_{4,\text{loop}} = \frac{N_c}{4} \int_{\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3} \int_p \text{tr}_{D,I} [S(p) V_1 S(p + q_1) V_2 \\ \times S(p + q_1 + q_2) V_3 S(p + q_1 + q_2 + q_3) V_{-123}].$$

Here $\int_p = \mu^{2\epsilon} \int d^{4-2\epsilon} p / (2\pi)^{4-2\epsilon}$.

$$\Gamma_F^{(3),\text{ren}} = [E_{3,\text{loop}}]_{\overline{\text{MS}}} + 8c_T f \int d^3x s(s^2 + \pi^2),$$

$$\Gamma_F^{(4),\text{ren}} = [E_{4,\text{loop}}]_{\overline{\text{MS}}} + 2c_T \int d^3x (s^2 + \pi^2)^2.$$

Refs: The negative triangle and positive box prefactors are from the logarithm. No momentum-independent replacement of the finite nonlocal loops is made.

B10 | Evaluating the Loop Kernels by Parameters

For denominators $A_j = (p + Q_j)^2 + m^2$, introduce $x_j \geq 0$, $\sum_j x_j = 1$:

$$\frac{1}{\prod_{j=1}^n A_j} = \Gamma(n) \int [dx]_n \frac{1}{(\ell^2 + \mathcal{M}^2)^n}, \quad \ell = p + \bar{Q}, \quad \bar{Q} = \sum_j x_j Q_j,$$

$$\mathcal{M}^2 = m^2 + \sum_j x_j Q_j^2 - \bar{Q}^2.$$

Take the Dirac/isospin trace, shift the numerator, and reduce even tensors with $\ell_\mu \ell_\nu \mapsto \delta_{\mu\nu} \ell^2/d$.

$$\mu^{2\epsilon} \int \frac{d^d \ell}{(2\pi)^d} \frac{(\ell^2)^j}{(\ell^2 + \mathcal{M}^2)^n} = \frac{\mu^{2\epsilon}}{(4\pi)^{d/2}} \frac{\Gamma(j + d/2) \Gamma(n - j - d/2)}{\Gamma(d/2) \Gamma(n)} (\mathcal{M}^2)^{d/2+j-n}.$$

Expand in ϵ , remove the local poles of B4–B5, add the finite terms of B7, then integrate the remaining external momenta against the fixed profiles.

Refs: Use d -dimensional tensor reduction before expanding poles; otherwise $O(\epsilon) \times 1/\epsilon$ finite terms can be lost. Non-anomalous two-point traces use anticommuting γ_5 .

B11 | Numerical Add-Back and Its Interpretation

Renormalized contribution	fm^{-1}
$\Gamma_F^{(1),\text{ren}}$	0
$\Gamma_F^{(2),\text{ren}}$	-13.4038
$\Gamma_F^{(3),\text{ren}}$	+33.9475
$\Gamma_F^{(4),\text{ren}}$	-29.8526
Total add-back	-9.3089

- Fixed legacy9.5 profile; $N_c = 3$ once; stated T-like finite conditions.
- These are supplied numerical values, not a newly repeated loop integration.
- Diagram sum $\neq$ total VPE; full physical finite matching remains open.

Refs: V2.1 numerical handoff. The derivation above specifies the analytic organization; it does not independently certify the supplied solver or profile matching.

C1 | Integrating Out Quarks: Relative Effective Action

For a prescribed Euclidean background,

$$Z_F[\Phi] = \int \mathcal{D}\bar{\psi} \mathcal{D}\psi e^{-\int \bar{\psi} D_E[\Phi] \psi} = (\det D_E[\Phi])^{N_c}.$$

Define the relative fermionic effective action:

$$\Delta\Gamma_{E,F} = -\ln \frac{Z_F[\Phi]}{Z_F[\Phi_0]} = -N_c \{ \text{Tr}_4 \ln D_E[\Phi] - \text{Tr}_4 \ln D_E[\Phi_0] \}.$$

$$E_F = \lim_{T \rightarrow \infty} \frac{\Delta\Gamma_{E,F}}{\mathcal{T}}. \quad \text{Minkowski: } E_F = - \lim_{T \rightarrow \infty} \frac{\Delta S_{M,F}}{T}.$$

This is the fermion sector on a fixed background. Classical meson energy, valence occupation and meson loops are separate ingredients.

Refs: Xia–Shu–Li, arXiv:2503.04015v4, Sec. III.A; Graham–Quandt–Weigel (2009).

C2 | Static Hamiltonian and the Charge-Even Determinant

For a static Hermitian Hamiltonian,

$$D_E = \gamma_4(\partial_\tau + H), \quad D_E^\dagger D_E = (-\partial_\tau + H)(\partial_\tau + H) = -\partial_\tau^2 + H^2.$$

The real, charge-even part of the relative action is

$$\text{Re } \Delta\Gamma_{E,F} = -\frac{N_c}{2} \text{Tr}_4 [\ln(-\partial_\tau^2 + H^2) - \ln(-\partial_\tau^2 + H_0^2)].$$

- Staticity removes $[\partial_\tau, H]$; the γ_4 factors cancel in $D_E^\dagger D_E$.
- Squaring retains $|E|$ information but does not determine a possible determinant phase or induced charge.
- Use the same boundary conditions and UV regulator in both operators.

Refs: This is a charge-even vacuum-energy derivation, not a claim that every fermion determinant has a trivial phase.

C3 | The Frequency Integral Produces Absolute Energies

At zero temperature, separate the Euclidean frequency:

$$E_F = -\frac{N_c}{2} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \text{Tr}_3[\ln(\omega^2 + H^2) - \ln(\omega^2 + H_0^2)].$$

For $A, B > 0$, define $J(A, B) = \int d\omega / (2\pi) \ln[(\omega^2 + A^2)/(\omega^2 + B^2)]$.

$$\partial_A J = \int \frac{d\omega}{2\pi} \frac{2A}{\omega^2 + A^2} = 1, \quad J(B, B) = 0 \quad \Rightarrow \quad J(A, B) = A - B.$$

Diagonalizing H and H_0 therefore gives the regulated formal sum

$$E_F^{\text{formal}} = -\frac{N_c}{2} \left[\sum_{\alpha} |E_{\alpha}| - \sum_{\alpha} |E_{\alpha}^{(0)}| \right].$$

Refs: Frequency integration does not remove the remaining spatial UV divergence. Color is explicit; both signed branches are included once.

C4 | Separate Bound and Continuum Spectral Measures

Let $\omega_k = \sqrt{k^2 + m^2}$, $a = (G, \Pi, \eta)$, $\eta = \pm$ and $d_a = 2G + 1$.

$$\Delta\rho_a(k) = \rho_a(k) - \rho_{0,a}(k) = \frac{1}{\pi} \frac{d\delta_a(k)}{dk}.$$

The formal relative vacuum energy becomes

$$E_F^{\text{formal}} = -\frac{N_c}{2} \sum_b d_b |E_b| - \frac{N_c}{2\pi} \sum_a d_a \int_0^\infty dk \omega_k \delta'_a(k).$$

- The bound index lists each signed physical level once, with its multiplicity.
- For coupled channels $\delta_a = (2i)^{-1} \ln \det S_a$ with continuous branches.
- This k -oriented scattering phase is distinguished from the full-axis energy-oriented phase in A5.

Refs: Threshold and endpoint terms must be retained if rewriting this formula by integration by parts.

C5 | Final Structure and the Limits of the Present Numbers

The spectral formula and the renormalized diagram calculation combine as

$$E_{\text{VPE}}^{(4)} = E_{\text{bound}} + E_{\text{cont}} \left[\delta - \sum_{n=1}^4 \delta^{(n)} \right] + \sum_{n=1}^4 \Gamma_F^{(n), \text{ren}}.$$

- ① A1–A7 show why subtracting Born energy requires matching fermion loops.
- ② B1–B10 specify pole removal and the adopted finite counterterms.
- ③ C1–C4 explain the action-to-spectrum relation and the signed-state counting.

A formally finite organization is distinct from a numerically converged, physically matched prediction.

Refs: Current finite- G , K values remain exploratory. The 19-slide main presentation and its numerical inputs are unchanged.