Thermodynamics of hadrons using the Gaussian functional method in the linear sigma model

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and

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Introduction

Chiral symmetry breaking

- Mass generation for (massless or light) fermion
- Nambu-Goldstone boson
- Restoration at high temperature (phase transition)

Non-perturbative interaction among mesons

The interaction term of linear sigma model ($\lambda \sim \mathcal{O}(10)$)

$$\mathcal{L}_{int} = \frac{\lambda}{4}(\sigma^2 + \boldsymbol{\pi}^2)^2$$

Chiral symmetry with the fluctuations of mesons around their mean field values at finite temperature

- The Cornwall-Jackiw-Tomboulis (CJT) formalism

  J. M. Cornwall, R. Jackiw and E. Tomboulis PRD 10 (1974) ...
- The optimized perturbation theory

  S. Chiku and T. Hatsuda PRD 58 (1998) ...
- The Gaussian Functional Method:
  corresponding to Hartree-Fock approx. + RPA

  T. Barnes and G. Ghandour PRD 22 (1980) ...

Gaussian Functional Method

Barnes and Ghandour PRD 22 (1980), Nakamura and Domitrasinovic PTP 106 (2001)

1. Schrödinger picture in field theory with the Gaussian ground state functional ansatz
2. The minimization condition: determination of the variational parameters
   - The resulting (dressed) mass of Nambu-Goldstone (NG) boson is not zero due to the non-perturbative effect
3. Considering the bound state of mesons (4 quarks state): Bethe-Salpeter equation
   - Emergence of the NG bosons → Physical mass

   ....

   - Fixing the parameters with the sigma meson mass (500 MeV)
   - Dressed mass vs. Physical mass
4. The phase transition at finite temperature

The Gaussian Functional Method I

O(2) Linear sigma model ($\boldsymbol{\phi} = (\phi_0, \phi_1, \phi_2, \phi_3) = (\sigma, \boldsymbol{\pi})$)

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \boldsymbol{\phi})^2 + \frac{1}{2}\mu_0 \boldsymbol{\phi}^2 - \frac{\lambda_0}{4}(\boldsymbol{\phi}^2)^2 + \varepsilon\sigma$$

The Hamiltonian

$$\mathcal{H}[\boldsymbol{\phi}] = \int d\boldsymbol{y}\delta(\boldsymbol{y}-\boldsymbol{x})\left(-\frac{1}{2}\frac{\delta^2}{\delta\phi_i(\boldsymbol{x})\phi_i(\boldsymbol{y})} + \frac{1}{2}\nabla_x\phi_i(\boldsymbol{x})\nabla_y\phi_i(\boldsymbol{y}) - \frac{1}{2}\mu_0\boldsymbol{\phi}^2 + \frac{\lambda_0}{4}(\boldsymbol{\phi}^2)^2 + \varepsilon\sigma\right)$$

The Gaussian ground state functional

$$\Psi[\boldsymbol{\phi}] = \mathcal{N}\exp\left(-\frac{1}{4}\int d\boldsymbol{x}d\boldsymbol{y}\,[\underbrace{\phi_i(\boldsymbol{x})}_{\text{fluctuation}} - \underbrace{\langle\phi_i(\boldsymbol{x})\rangle}_{\text{V.E.V}}]G^{-1}_{ij}(\boldsymbol{x},\boldsymbol{y})[\phi_j(\boldsymbol{y}) - \langle\phi_j(\boldsymbol{y})\rangle]\right)$$

$$G_{ij}(\boldsymbol{x},\boldsymbol{y}) = \frac{1}{2}\delta_{ij}\int\frac{d^3\boldsymbol{k}}{(2\pi)^3}\frac{1}{\sqrt{\boldsymbol{k}^2 + \underbrace{M_i^2}_{\text{dressed mass}}}}e^{i\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{y})}$$

Energy with the variational parameters M_i and $\langle\phi_i\rangle$

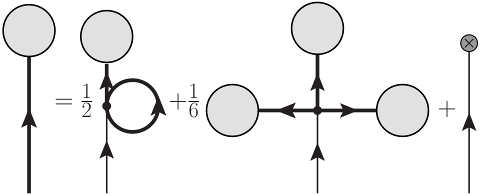
$$\mathcal{E}(M_i, \langle\phi_i\rangle) = \int\mathcal{D}\boldsymbol{\phi}\Psi^*[\boldsymbol{\phi}]\mathcal{H}[\boldsymbol{\phi}]\Psi[\boldsymbol{\phi}]$$

The Gaussian Functional Method II

The minimization condition: Determination of parameters $(\langle\phi_i\rangle, M_i)$

$$\left(\frac{\partial \mathcal{E}(M_i, \langle\phi_i\rangle)}{\partial\langle\phi_i\rangle, M_i}\right)_{\text{min}} = 0\,, \text{ for } i = 0\ldots 3$$

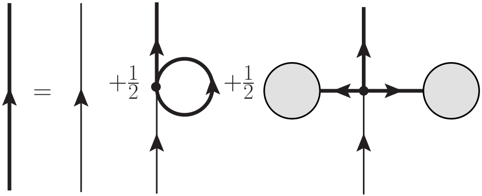
$\Leftrightarrow$ (One- and two-point) Schwinger-Dyson equations



The mean field values: One-point SD equation

$$\begin{aligned}\langle\phi_0\rangle =& v, \quad \langle\phi_i\rangle = 0 \text{ for } i = 1,2,3\\ \mu_0^2 =& -\frac{\varepsilon}{v} + \lambda_0\left[v^2 + 3I_0(M_\sigma) + 3I_0(M_\pi)\right]\\ I_0(M_i) =& i\int\frac{d^4k}{(2\pi)^4}\frac{1}{k^2 - M_i^2 + i\epsilon}\end{aligned}$$

Two-point SD equation: (Determination of masses M_i)



Dressed mass $(M_0 = M_\sigma,\ M_i = M_\pi)$

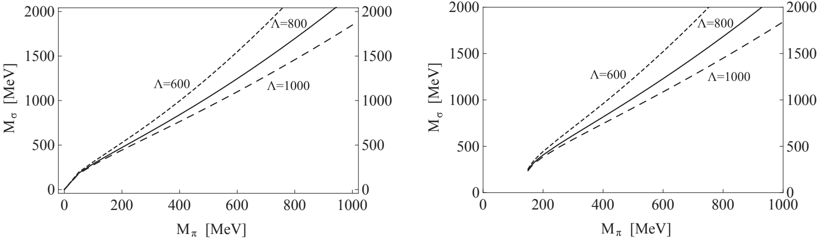
$$
\begin{aligned}
M_\sigma^2 =& -\mu_0^2 + \lambda_0\left[3v^2 + 3I_0(M_\sigma) + 3I_0(M_\pi)\right] \\
=& \frac{\varepsilon}{v} + 2\lambda_0 v^2 \\
M_\pi^2 =& -\mu_0^2 + \lambda_0\left[v^2 + I_0(M_\sigma) + 5I_0(M_\pi)\right] \\
=& \frac{\varepsilon}{v} + \underbrace{2\lambda_0\left[I_0(M_\pi) - I_0(M_\sigma)\right]}_{\text{non-perturbative effect}}
\end{aligned}
$$

The pion mass $M_\pi \neq 0$ even in the chiral limit $\varepsilon \to 0$

$\to$ DO NOT satisfy The Nambu-Goldstone theorem

The dressed mass cannot satisfy the Nambu-Goldstone theorem

Relation between M_σ and M_π with some cutoff Λ



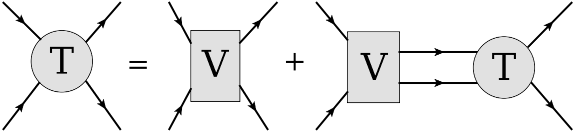
(a) $\varepsilon = 0$ (b) $\varepsilon \neq 0$

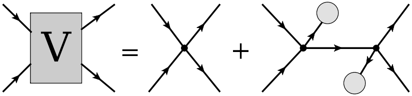
- The dressed masses depend on the cutoff
- They cannot satisfy the NG theorem independently the cutoff
  - There are no finite sigma mass and zero pion mass in the chiral limit $\varepsilon = 0$
  - The physical mass of sigma (600 MeV) and pion (140 MeV) cannot exist in $\varepsilon \neq 0$

We cannot identify these masses as physcal mass (NG boson)

The Bethe-Salpeter Equation I

Physical masses appear as pole of the Bethe-Salpeter (four-point SD) euqarion (bound state of mesons)





$\sigma - \pi$ channel $\rightarrow$ Physical pion mass m_π $(s = p^2)$

$$G_{\sigma\pi\to\sigma\pi}(p^2) = i\int \frac{d^4k}{(2\pi)^4} \frac{1}{[k^2 - M_\sigma^2 + i\epsilon]\,[(k-p)^2 - M_\pi^2 + i\epsilon]}$$

$$V_{\sigma\pi\to\sigma\pi}(s) = 2\lambda_0 \left[1 + \left(2\lambda_0 \frac{v^2}{s - M_\pi^2}\right)\right]$$

$$T_{\sigma\pi\to\sigma\pi}(s) = V_{\sigma\pi\to\sigma\pi}(s) + V_{\sigma\pi\to\sigma\pi}(s) G_{\sigma\pi\to\sigma\pi}(s) T_{\sigma\pi\to\sigma\pi}(s)$$

$$= \frac{V_{\sigma\pi\to\sigma\pi}(s)}{1 - V_{\sigma\pi\to\sigma\pi}(s) G_{\sigma\pi\to\sigma\pi}(s)}$$

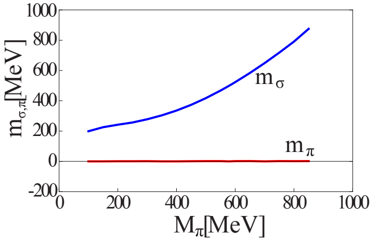
The coupled channel $\sigma - \sigma$ and $\pi - \pi$ $\rightarrow$ Physical sigma mass m_σ

$$\boldsymbol{V} = \begin{pmatrix} V_{\sigma\sigma\to\sigma\sigma} & V_{\sigma\sigma\to\pi\pi} \\ V_{\pi\pi\to\sigma\sigma} & \frac{1}{3}V_{\pi\pi\to\pi\pi} \end{pmatrix} = 2\lambda_0 \begin{pmatrix} 3\left[1+3\frac{2\lambda_0 v^2}{s-M_\sigma^2}\right] & \left[1+3\frac{2\lambda_0 v^2}{s-M_\sigma^2}\right] \\ \left[1+3\frac{2\lambda_0 v^2}{s-M_\sigma^2}\right] & \frac{1}{3}\left[5+3\frac{2\lambda_0 v^2}{s-M_\sigma^2}\right] \end{pmatrix}$$

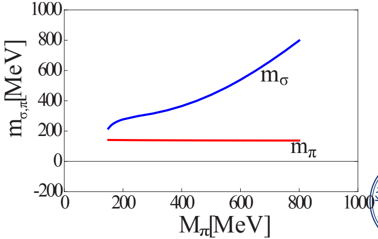
$$\boldsymbol{T} = \begin{pmatrix} T_{\sigma\sigma\to\sigma\sigma} & T_{\sigma\sigma\to\pi\pi} \\ T_{\pi\pi\to\sigma\sigma} & \frac{1}{3}T_{\pi\pi\to\pi\pi} \end{pmatrix}, \quad \boldsymbol{G} = \begin{pmatrix} G_{\sigma\sigma\to\sigma\sigma} & 0 \\ 0 & 3G_{\pi\pi\to\pi\pi} \end{pmatrix}$$

$$\boldsymbol{T} = \boldsymbol{V} + \frac{1}{2}\boldsymbol{V}\boldsymbol{G}\boldsymbol{T} = (1 - \frac{1}{2}\boldsymbol{V}\boldsymbol{G})^{-1}\boldsymbol{V}$$

Physical mass m_σ, m_π vs. Dressed mass M_π (NG boson)



(a) $\varepsilon = 0$



(b) $\varepsilon \neq 0$

Parameters

We accept the parameters below ($\varepsilon = f_\pi m_{\pi 0}^2 = 93 \times 142^2$ MeV^3)

chiral limit	breaking case
$\lambda_0 = 83.6$	$\lambda_0 = 75.5$
$\mu_0 = 1680$ MeV	$\mu_0 = 1610$ MeV
$\Lambda = 800$ MeV	$\Lambda = 800$ MeV
$\varepsilon = 0$ MeV^3	$\varepsilon = 1.86 \times 10^6$ MeV^3
$\Downarrow$	$\Downarrow$
$M_\sigma = 1200$ MeV	$M_\sigma = 1150$ MeV
$M_\pi = 580$ MeV	$M_\pi = 564$ MeV
$m_\sigma = 500$ MeV	$m_\sigma = 500$ MeV
$v = f_\pi = 93$ MeV	$v = f_\pi = 93$ MeV
$m_\pi = 0$ MeV	$m_\pi = 138$ MeV

We fit the parameters to reproduce the pion decay constant f_π and the pion mass m_π

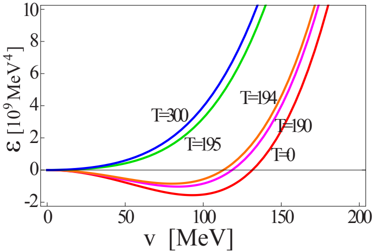
Finite Temperature I

Finite temperature with the Matsubara formalism

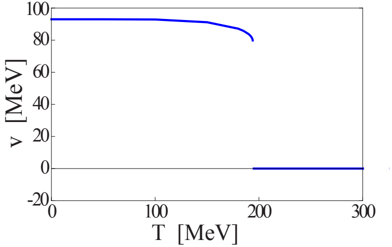
$\mathcal{E}(v, M_\sigma, M_\pi) \to \mathcal{E}(v(T), M_\sigma(T), M_\pi(T); T)$

The behavior of the free energy as a function of the mean field value v (fixing M_σ, M_π)

In the case of the chiral limit



(a) Free energy

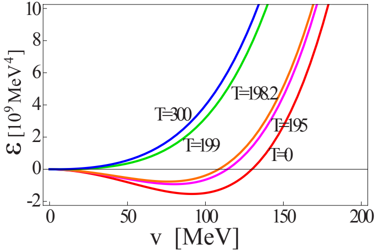


(b) Mean field values v

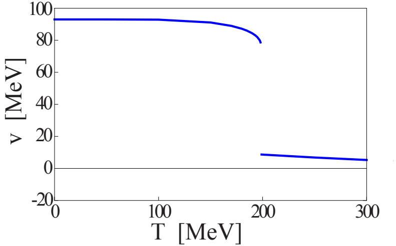
The free energy is suddenly change at 195 MeV and the first order phase transition

Finite Temperature II

In the case of the explicit chiral symmetry breaking $\mathcal{E}_{\chi SB} = \varepsilon v$



(a) Free energy



(b) Mean field values v

Similar behavior even in the case of explicit symmetry breaking:
Suddenly change at 195 MeV and the first order phase transition

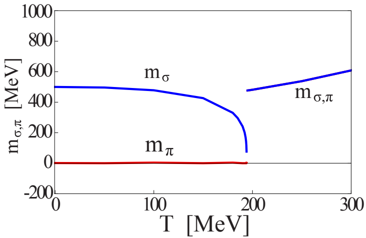
- The whole energy at transition temperature $\mathcal{E} \sim -10^8$MeV^4
- The chiral symmetry breaking term $\mathcal{E}_{\chi SB} \sim -10^7$MeV^4

Since the contribution of $\mathcal{E}_{\chi SB}$ is 10 times smaller, the free energy suddenly change

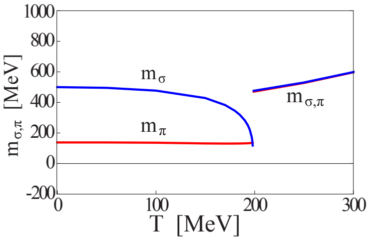
$\rightarrow$ In the MFA, they are comparable $\sim -10^8$ MeV^4

Finite Temperature III

Solutions of the BS equation at finite temperature $G(s) \to G(s, T)$

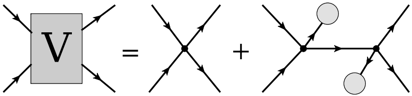


(a) $\varepsilon = 0$



(b) $\varepsilon \neq 0$

- Mesons bound state picture holds only in the symmetry broken phase
- In the symmetric phase, they are unbound and their masses coincide the dressed mass $m_\pi^2 = M_\pi^2 + \frac{4\lambda_0 v^2 G_{\sigma\pi\to\sigma\pi}(m_\pi^2)}{1-2\lambda_0 G_{\sigma\pi\to\sigma\pi}(m_\pi^2)}$



The second term vanishes due to the symmetry restoration $(v \to 0)$

Summary

1. We treat the non-perturbative effect using Gaussian ground state functional ansatz
2. Determination of the variational parameters (One- and Two-point Schwinger-Dyson Equation)
   - There are NO NG bosons: unphysical particle
3. Mesons bound state in the Bethe-Salpeter Equation
   - Emergence of the NG bosons
   - 4 quarks picture of mesons
4. The behavior of the chiral symmetry at finite temperature
   - The phase transition
   - The meson-meson bound state

Future work

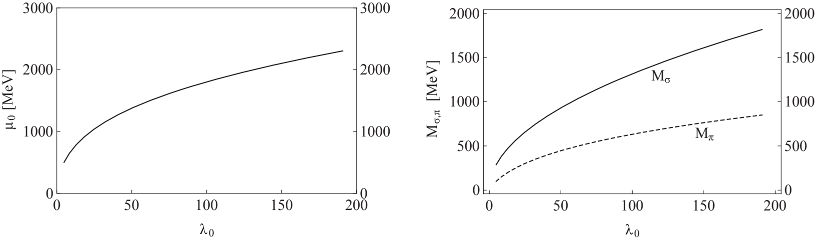
- Investigation of 3 flavor case
- Application to 2 color system at finite density

The end

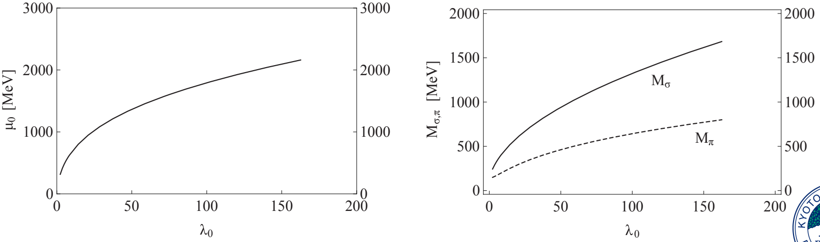
Thank you for your kind attention

Back up slides

Chiral limit($\varepsilon = 0$)



Explicit breaking($\varepsilon \neq 0$)

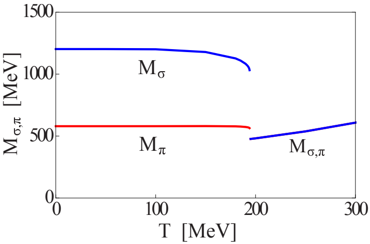


(a) μ_0 vs. λ_0 (b) M_σ, M_π vs. λ_0

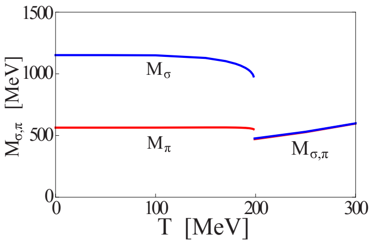
Dressed mass at finite temperature

$$M_\sigma^2(T) = \frac{\varepsilon}{v(T)} + 2\lambda_0 v^2(T)$$

$$M_\pi^2(T) = \frac{\varepsilon}{v(T)} + 2\lambda_0 \left[I_0(M_\pi(T)) - I_0(M_\sigma(T)) \right]$$



(a) $\varepsilon = 0$



(b) $\varepsilon \neq 0$