Spin and rotation effects in heavy-ion collisions

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Outline

- Spin in non-relativistic quantum theory
- Spin in relativistic quantum theory
- Wigner functions and spin dependent distributions
- Spin kinetic equations for spin-1/2 and spin-one particles

References

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Energy in rotating frame

In the fixed inertial frame S, the frame S' is rotating about a fixed axis (e.g., the z-axis) S with a constant angular velocity $\boldsymbol{\omega}$.
A particle with the mass m, with a position vector $\boldsymbol{r}'$ and velocity $\boldsymbol{v}'$ in frame S', its absolute velocity $\boldsymbol{v}$ in frame S is

$$\boldsymbol{v} = \boldsymbol{v}' + \boldsymbol{\omega} \times \boldsymbol{r}' \tag{1}$$

The kinetic energy T of the particle in S is

$$\begin{aligned} T =& \frac{1}{2} m \boldsymbol{v} \cdot \boldsymbol{v} = \frac{1}{2} m \left(\boldsymbol{v}' + \boldsymbol{\omega} \times \boldsymbol{r}'\right) \cdot \left(\boldsymbol{v}' + \boldsymbol{\omega} \times \boldsymbol{r}'\right) \\ =& \frac{1}{2} m v'^2 + m \boldsymbol{v}' \cdot \left(\boldsymbol{\omega} \times \boldsymbol{r}'\right) + \frac{1}{2} m \left(\boldsymbol{\omega} \times \boldsymbol{r}'\right)^2 \end{aligned} \tag{2}$$

Energy in rotating frame

The Lagrangian is $L = T - V$, and the generalized momentum is

$$\boldsymbol{p}' = \frac{\partial L}{\partial \boldsymbol{v}'} = \frac{\partial T}{\partial \boldsymbol{v}'} = m\left(\boldsymbol{v}' + \boldsymbol{\omega} \times \boldsymbol{r}'\right) = m\boldsymbol{v} \tag{3}$$

The Hamiltonian is

$$\begin{aligned} H_\omega =& \boldsymbol{p}' \cdot \boldsymbol{v}' - L = \frac{1}{2} m v'^2 - \frac{1}{2} m \left(\boldsymbol{\omega} \times \boldsymbol{r}'\right)^2 + V \\ =& \frac{1}{2m}\left(\boldsymbol{p}' - m\boldsymbol{\omega} \times \boldsymbol{r}'\right)^2 - \frac{1}{2} m \left(\boldsymbol{\omega} \times \boldsymbol{r}'\right)^2 + V \\ =& \frac{1}{2m}\boldsymbol{p}'^2 - \boldsymbol{p}' \cdot \left(\boldsymbol{\omega} \times \boldsymbol{r}'\right) + V = \frac{1}{2m}\boldsymbol{p}'^2 + V - \boldsymbol{\omega} \cdot \boldsymbol{L}' \end{aligned} \tag{4}$$

where $\boldsymbol{L}' = \boldsymbol{r}' \times \boldsymbol{p}'$.

Energy in magnetic field

In classical mechanics, the magnetic moment of a particle in an external magnetic field can be derived as follows

$$\begin{aligned} T &= \frac{(\boldsymbol{p}-q\boldsymbol{A})^2}{2m} = \frac{\boldsymbol{p}^2}{2m} - \frac{q}{m}\boldsymbol{p}\cdot\boldsymbol{A} + \frac{q^2}{2m}\boldsymbol{A}^2 \\ -\frac{q}{m}\boldsymbol{p}\cdot\boldsymbol{A} &= \frac{q}{2m}\boldsymbol{p}\cdot(\boldsymbol{r}\times\boldsymbol{B}) = -\frac{q}{2m}\boldsymbol{L}\cdot\boldsymbol{B} = -\boldsymbol{\mu}_L\cdot\boldsymbol{B} \\ H_B &= \frac{\boldsymbol{p}^2}{2m} + \frac{q^2}{2m}\boldsymbol{A}^2 + V - \boldsymbol{\mu}_L\cdot\boldsymbol{B} \end{aligned} \tag{5}$$

Here we choose $\boldsymbol{A} = -\frac{1}{2}\boldsymbol{r}\times\boldsymbol{B}$ (Coulomb gauge condition is satisfied $\nabla\cdot\boldsymbol{A}=0$) and orbital magnetic moment $\boldsymbol{\mu}_L = \frac{q}{2m}\boldsymbol{L}$ with $\mathrm{L} = \boldsymbol{r}\times\boldsymbol{p}$.

Including spin

We can include the spin contribution to the interaction energy

$$
\begin{aligned}
H_\omega =& \frac{1}{2m}\boldsymbol{p}^2 + V - (\boldsymbol{L} + \boldsymbol{S}) \cdot \boldsymbol{\omega} \\
H_B =& \frac{\boldsymbol{p}^2}{2m} + \frac{q^2}{2m}\boldsymbol{A}^2 + V - (\boldsymbol{\mu}_L + \boldsymbol{\mu}_S) \cdot \boldsymbol{B}
\end{aligned} \tag{6}
$$

where we have suppressed the prime symbol in the rotating frame, and $\boldsymbol{\mu}_S = g\frac{q}{2m}\boldsymbol{S}$ is the spin magnetic moment.

Spin in non-relativistic quantum theory

In non-relativistic quantum mechanics, the spin operators of a particle form a 3D vetcor $\hat{\boldsymbol{S}} = \sum_i \hat{S}^i \boldsymbol{e}_i$, where $\boldsymbol{e}_i$ are three basis vectors and $i = 1, 2, 3$ or x, y, z. They satisfy the commutation rule for the angular momentum

$$\left[\hat{S}^i, \hat{S}^j\right] = i\epsilon_{ijk}\hat{S}^k. \tag{7}$$

The spin operators are defined in the frame in which the particle's momentum is much less than its mass.

The mean spin vector is defined as:

$$\boldsymbol{S} = \left\langle \hat{\boldsymbol{S}} \right\rangle = \mathrm{Tr}\left(\hat{\rho}\hat{\boldsymbol{S}}\right) \tag{8}$$

where $\hat{\rho}$ is the density operator of the particle under consideration and $\hat{\boldsymbol{S}}$ is the spin operator. The polarization vector is defined as the mean value of the spin operator normalized to the spin quantum number of the particle:

$$\boldsymbol{P} = \frac{1}{S}\left\langle \hat{\boldsymbol{S}} \right\rangle \tag{9}$$

so that its maximal value is 1, that is $|\boldsymbol{P}| \leq 1$.

Spin in non-relativistic thermal systems

Consider a non-relativistic particle at equilibrium in a thermal bath at temperature T in a rotating vessel at an angular velocity $\boldsymbol{\omega}$, the density operator is

$$\hat{\rho} = \frac{1}{Z} \exp\left[-\beta \hat{H} + \beta \mu \hat{Q} + \beta \boldsymbol{\omega} \cdot \hat{\boldsymbol{J}} + \beta \hat{\boldsymbol{\mu}}_B \cdot \boldsymbol{B} \right] \tag{10}$$

where $\hat{\boldsymbol{J}} = \hat{\boldsymbol{L}} + \hat{\boldsymbol{S}}$, $\hat{Q}$ is a conserved charge with μ being the corresponding chemical potential, and $\boldsymbol{B}$ is a constant and uniform external magnetic field with $\hat{\boldsymbol{\mu}}_B = \mu_B \hat{\boldsymbol{S}}/S$ being the magnetic moment.

If the constant angular velocity $\boldsymbol{\omega}$, as well as the constant magnetic field $\boldsymbol{B}$ are parallel, the above density operator can be diagonalized in the basis of eigenvectors of the spin operator component parallel to $\boldsymbol{\omega}$, $\hat{\boldsymbol{S}} \cdot \boldsymbol{\omega}$, thereby giving rise to a probability distribution for its different eigenvalues m. The different probabilities read

$$w\left[T, B, \omega\right](m) = \frac{\exp\left[\beta\left(\frac{\mu_B B}{S} + \omega\right) m\right]}{\sum_{m=-S}^{S} \exp\left[\beta\left(\frac{\mu_B B}{S} + \omega\right) m\right]} \tag{11}$$

The distribution Eq. (11) may now be used to estimate the spin vector in Eq. (8).

Spin in non-relativistic thermal systems

For the simpler case with $B = 0$, the spin vector along $\boldsymbol{\omega}$ is

$$
\begin{aligned}
\boldsymbol{S} =& \hat{\boldsymbol{\omega}} \frac{\sum_{m=-S}^{S} m \exp(\beta\omega m)}{\sum_{m=-S}^{S} \exp(\beta\omega m)} = \hat{\boldsymbol{\omega}} \frac{\partial}{\partial(\beta\omega)} \ln\left[\sum_{m=-S}^{S} \exp(\beta\omega m)\right] \\
=& \hat{\boldsymbol{\omega}} \frac{\partial}{\partial(\beta\omega)} \ln\left\{\exp(-\beta\omega S)\left[1+\exp(\beta\omega)+\cdots+\exp(2\beta\omega S)\right]\right\} \\
=& \hat{\boldsymbol{\omega}} \frac{\partial}{\partial(\beta\omega)} \ln\left\{\exp(-\beta\omega S)\frac{1-\exp\left[\beta\omega(2S+1)\right]}{1-\exp(\beta\omega)}\right\} \\
=& \hat{\boldsymbol{\omega}} \frac{\partial}{\partial(\beta\omega)} \ln\frac{\exp\left[-\beta\omega(S+1/2)\right]-\exp\left[\beta\omega(S+1/2)\right]}{\exp(-\beta\omega/2)-\exp(\beta\omega/2)} \\
=& \hat{\boldsymbol{\omega}} \frac{\partial}{\partial(\beta\omega)} \ln\frac{\sinh\left[\beta\omega(S+1/2)\right]}{\sinh[\beta\omega/2]}
\end{aligned}
\tag{12}
$$

where $\hat{\boldsymbol{\omega}}$ is the unit vector along the direction of $\boldsymbol{\omega}$.

Spin in non-relativistic thermal systems

At the limit $\omega \ll T$ or $\beta\omega \ll 1$, the spin vector in Eq. (12) becomes:

$$\begin{aligned} \boldsymbol{S} \approx & \frac{1}{3}S(S+1)\beta\omega \\ \boldsymbol{P} \approx & \frac{1}{3}(S+1)\beta\omega \end{aligned} \tag{13}$$

Then we obtain

$$\begin{aligned} \boldsymbol{S} =& \frac{1}{2}\boldsymbol{P} \approx \frac{1}{4}\beta\omega, \quad (S=1/2) \\ \boldsymbol{S} =& \boldsymbol{P} \approx \frac{2}{3}\beta\omega, \quad (S=1) \\ \boldsymbol{S} =& \frac{3}{2}\boldsymbol{P} \approx \frac{5}{4}\beta\omega, \quad (S=3/2) \end{aligned} \tag{14}$$

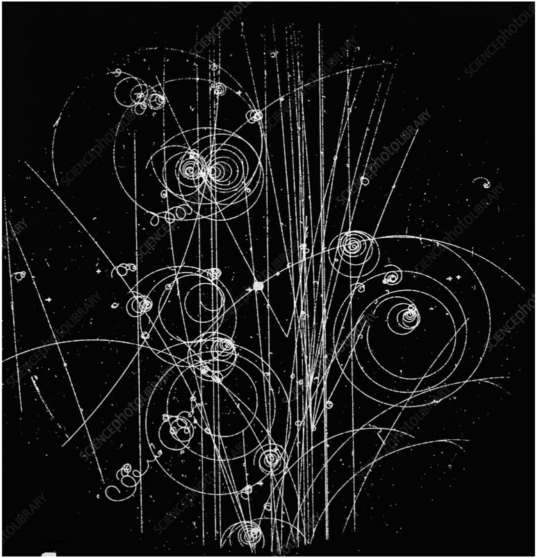
Homework

Prove Eq. (13).

Spin in relativistic quantum theory

- What is a microscopic particle? How to describe it (mass, spin, charge, parity)?
- Quantum + Special Relativity

Particle tracks in bubble chamber



Translation and Lorentz transformation

Let x^μ be the coordinates of a four-vector in frame F and x'^μ is the coordinates of a four-vector in frame F', then the translation and Lorentz transformation is expressed as

$$x'^\mu = \Lambda^\mu_{\ \nu} x^\nu + a^\mu \tag{15}$$

which can be denoted as $\{a, \Lambda\}$ with a^μ being a four vector and Λ a 4×4 matrix satisfying

$$g_{\mu\nu} \Lambda^\mu_{\ \rho} \Lambda^\nu_{\ \sigma} = g_{\rho\sigma} \tag{16}$$

Note that the positions of two indices are sensitive in $\Lambda^\mu_{\ \nu}$, they can be lowered or raised by the metrix tensor $g_{\mu\nu}$ or $g^{\mu\nu}$. The distance is invariant under Lorentz transformation

$$g_{\mu\nu} dx'^\mu dx'^\nu = g_{\mu\nu} \Lambda^\mu_{\ \rho} \Lambda^\nu_{\ \sigma} dx^\rho dx^\sigma = g_{\rho\sigma} dx^\rho dx^\sigma \tag{17}$$

Lorentz transformation

We can also derive from (16)

$$\begin{aligned}
g_{\mu\nu}\Lambda^{\mu}_{\ \rho}\Lambda^{\nu}_{\ \sigma} &= g_{\rho\sigma} \\
\left(g_{\mu\nu}\Lambda^{\mu}_{\ \rho}\Lambda^{\nu}_{\ \sigma}\right) g^{\rho\sigma} &= g_{\rho\sigma}g^{\rho\sigma} \\
g_{\mu\nu}\left(\Lambda^{\mu}_{\ \rho}\Lambda^{\nu}_{\ \sigma}g^{\rho\sigma}\right) &= 4 \\
&\downarrow \\
\Lambda^{\mu}_{\ \rho}\Lambda^{\nu}_{\ \sigma}g^{\rho\sigma} &= g^{\mu\nu}
\end{aligned}$$

We have following property for $\Lambda^{\mu}_{\ \nu}$

$$\begin{aligned}
\det\left(g_{\mu\nu}\Lambda^{\mu}_{\ \rho}\Lambda^{\nu}_{\ \sigma}\right) &= (\det\Lambda)^2 \det g = \det g \\
&\Rightarrow (\det\Lambda)^2 = 1 \Rightarrow \det\Lambda = \pm 1
\end{aligned} \tag{18}$$

Lorentz transformation

The inverse Lorentz transformation is derived as

$$
\begin{aligned}
g_{\mu\nu}\Lambda^{\mu}{}_{\rho}\Lambda^{\nu}{}_{\sigma} =& g_{\rho\sigma} \Rightarrow g_{\mu\nu}\Lambda^{\mu}{}_{\rho}\Lambda^{\nu}{}_{\sigma}g^{\rho\lambda} = g^{\lambda}{}_{\sigma} \\
\Lambda_{\nu}^{\lambda}\Lambda^{\nu}{}_{\sigma} =& g^{\lambda}{}_{\sigma} = \left(\Lambda^{-1}\right)^{\lambda}{}_{\nu}\Lambda^{\nu}{}_{\sigma} \\
& \downarrow \\
\left(\Lambda^{-1}\right)^{\lambda}{}_{\nu} =& \Lambda_{\nu}^{\lambda} = g_{\nu\rho}\Lambda^{\rho}{}_{\alpha}g^{\alpha\lambda} = (g\Lambda g)_{\nu}^{\lambda}
\end{aligned} \tag{19}
$$

or in another way

$$
\begin{aligned}
\Lambda_{\alpha}^{\lambda}\Lambda^{\beta}_{\lambda} =& g_{\alpha}^{\beta} = \Lambda_{\alpha}^{\lambda}\left(\Lambda^{-1}\right)_{\lambda}^{\beta} \\
& \downarrow \\
\left(\Lambda^{-1}\right)_{\lambda}^{\beta} =& \Lambda^{\beta}_{\lambda}
\end{aligned} \tag{20}
$$

The Lorentz transformation for x^μ reads

$$x'^\mu = \Lambda^\mu_{\ \nu} x^\nu \tag{21}$$

The inverse transformation

$$\left(\Lambda^{-1}\right)^\lambda_{\ \mu} x'^\mu = \left(\Lambda^{-1}\right)^\lambda_{\ \mu} \Lambda^\mu_{\ \nu} x^\nu = x^\lambda$$

$$\downarrow$$

$$x^\lambda = \left(\Lambda^{-1}\right)^\lambda_{\ \mu} x'^\mu = x'^\mu \Lambda_\mu^{\ \lambda} \tag{22}$$

Explicit form

The explicit form of the transformation $x'^{\mu} = \Lambda^{\mu}_{\ \nu} x^{\nu}$ is

$$\begin{aligned} t' =& \gamma(1 - \boldsymbol{\beta} \cdot \boldsymbol{x}) \\ \boldsymbol{x}' =& \boldsymbol{x} + (\gamma - 1)(\hat{\boldsymbol{\beta}} \cdot \boldsymbol{x})\hat{\boldsymbol{\beta}} - \gamma \boldsymbol{\beta} t \\ =& \boldsymbol{x}_{\perp} + \gamma \left(\boldsymbol{x}_{\parallel} - \boldsymbol{\beta} \cdot t \right) \end{aligned} \tag{23}$$

Then we have

$$\Lambda^{\mu}_{\ \nu} = \begin{pmatrix} \gamma & -\gamma \boldsymbol{\beta}^T \\ -\gamma \boldsymbol{\beta} & 1_{3\times 3} + (\gamma - 1)\hat{\boldsymbol{\beta}}\hat{\boldsymbol{\beta}}^T \end{pmatrix} \tag{24}$$

where $\boldsymbol{\beta} = (\beta_1, \beta_2, \beta_3)^T$.

Explicit form

The explicit form of the inverse transformation

$$
\begin{aligned}
\left(\Lambda^{-1}\right)^{\lambda}_{\ \nu} =& \Lambda_{\nu}^{\ \lambda} = g_{\nu\rho}\Lambda^{\rho}_{\ \alpha}g^{\alpha\lambda} = \left(g\Lambda g\right)_{\nu}^{\ \lambda} \\
=& \begin{pmatrix} 1 & 0^{T} \\ 0 & -1_{3\times 3} \end{pmatrix} \begin{pmatrix} \gamma & -\gamma\boldsymbol{\beta}^{T} \\ -\gamma\boldsymbol{\beta} & 1_{3\times 3} + (\gamma-1)\hat{\boldsymbol{\beta}}\hat{\boldsymbol{\beta}}^{T} \end{pmatrix} \begin{pmatrix} 1 & 0^{T} \\ 0 & -1_{3\times 3} \end{pmatrix} \\
=& \begin{pmatrix} \gamma & \gamma\boldsymbol{\beta}^{T} \\ \gamma\boldsymbol{\beta} & 1_{3\times 3} + (\gamma-1)\hat{\boldsymbol{\beta}}\hat{\boldsymbol{\beta}}^{T} \end{pmatrix} \qquad (25)
\end{aligned}
$$

Poincare group (inhomogeneous Lorentz group)

The translation and Lorentz transformation (15) form Poincare group or inhomogeneous group

$$\begin{aligned} x''^{\mu} =& \Lambda'^{\mu}_{\ \rho} x'^{\rho} + a'^{\mu} = \Lambda'^{\mu}_{\ \rho} \left(\Lambda^{\rho}_{\ \nu} x^{\nu} + a^{\rho} \right) + a'^{\mu} \\ =& \Lambda'^{\mu}_{\ \rho} \Lambda^{\rho}_{\ \nu} x^{\nu} + \Lambda'^{\mu}_{\ \rho} a^{\rho} + a'^{\mu} \end{aligned} \tag{26}$$

One can check that $\Lambda'^{\mu}_{\ \rho}\Lambda^{\rho}_{\ \nu}$ satisfies (16)

$$\begin{aligned} g_{\mu\alpha} \left(\Lambda'^{\mu}_{\ \rho} \Lambda^{\rho}_{\ \nu} \right) \left(\Lambda'^{\alpha}_{\ \sigma} \Lambda^{\sigma}_{\ \beta} \right) =& \left(g_{\mu\alpha} \Lambda'^{\mu}_{\ \rho} \Lambda'^{\alpha}_{\ \sigma} \right) \left(\Lambda^{\rho}_{\ \nu} \Lambda^{\sigma}_{\ \beta} \right) \\ =& g_{\rho\sigma} \Lambda^{\rho}_{\ \nu} \Lambda^{\sigma}_{\ \beta} = g_{\nu\beta} \end{aligned} \tag{27}$$

So $\Lambda'^{\mu}_{\ \rho}\Lambda^{\rho}_{\ \nu}$ is also a Lorentz transformation.

Poincare group (inhomogeneous Lorentz group)

Then two consecutive transformation (26) can be put into a compact form

$$U(\Lambda', a')U(\Lambda, a) = U(\Lambda'\Lambda, \Lambda' a + a') \tag{28}$$

The inverse of the transformation (15) is

$$\begin{aligned} x^\lambda &= \left(\Lambda^{-1}\right)^\lambda_{\ \mu} \Lambda^\mu_{\ \nu} x^\nu = \left(\Lambda^{-1}\right)^\lambda_{\ \mu} \left(x'^\mu - a^\mu\right) \\ &= \left(\Lambda^{-1}\right)^\lambda_{\ \mu} x'^\mu - \left(\Lambda^{-1}\right)^\lambda_{\ \mu} a^\mu \end{aligned} \tag{29}$$

which can be represented by $U\left(\Lambda^{-1}, -\Lambda^{-1}a\right)$. We can check it is really the inverse of (15)

$$\begin{aligned} U(\Lambda, a)U\left(\Lambda^{-1}, -\Lambda^{-1}a\right) =& U\left(\Lambda^{-1}, -\Lambda^{-1}a\right) U(\Lambda, a) \\ =& U(1, 0) \end{aligned} \tag{30}$$

Proper orthochronous Lorentz group

Form (18), we have $\det\Lambda = \pm 1$. The transformation with $\det\Lambda = +1$ form a subgroup of Lorentz group. From (16) we obtain for $\rho = \sigma = 0$

$$\left(\Lambda^0_{\ 0}\right)^2 = 1 + \left(\Lambda^i_{\ 0}\right)^2 > 1 \tag{31}$$

which lead to the inequalities for $\Lambda^0_{\ 0}$:

$$\Lambda^0_{\ 0} > 1, \ \ \Lambda^0_{\ 0} < -1 \tag{32}$$

The conditions

$$\det\Lambda = +1 \quad \text{and} \quad \Lambda^0_{\ 0} > 1 \tag{33}$$

define the proper orthochronous Lorentz group, a subgroup of Poincare group.

It is impossible to jump from $\det\Lambda = +1$ to $\det\Lambda = -1$ or from $\Lambda^0_{\ 0} > 1$ to $\Lambda^0_{\ 0} < -1$ by a continuous change of parameters. Any Lorentz transformation that can be obtained from the identity by a continuous change of parameters must have $\det\Lambda = +1$ and $\Lambda^0_{\ 0} > 1$, the same sign as the identity, and hence belong to the proper orthochronous Lorentz group.

Any Lorentz transformation is either proper and orthochronous, or may be written as the product of an element of the proper orthochronous Lorentz group with one of the discrete transformations $\mathscr{P}$ or $\mathscr{T}$ or $\mathscr{P}\mathscr{T}$, where $\mathscr{P}$ is the space inversion, whose non-zero elements are

$$\mathscr{P}_0^0 = 1, \ \mathscr{P}_1^1 = \mathscr{P}_2^2 = \mathscr{P}_3^3 = -1 \tag{34}$$

and $\mathscr{T}$ is the time-reversal matrix, whose non-zero elements are

$$\mathscr{T}_0^0 = -1, \ \mathscr{T}_1^1 = \mathscr{T}_2^2 = \mathscr{T}_3^3 = 1 \tag{35}$$

Thus the study of the whole Lorentz group reduces to the study of its proper orthochronous subgroup, plus space inversion and time-reversal.

Poincare algebra

Much of the information about any Lie symmetry group is contained in properties of the group elements near the identity, so we want to study those transformations with

$$\Lambda^{\mu}_{\ \nu} = g^{\mu}_{\nu} + \omega^{\mu}_{\ \nu}, \quad a^{\mu} = \epsilon^{\mu} \tag{36}$$

where both $\omega^{\mu}_{\ \nu}$ and ϵ^{μ} are taken infinitesimal. Eq. (16) becomes

$$\begin{aligned} g_{\rho\sigma} =& g_{\mu\nu}\left(g^{\mu}_{\rho} + \omega^{\mu}_{\ \rho}\right)\left(g^{\nu}_{\sigma} + \omega^{\nu}_{\ \sigma}\right) \\ =& g_{\sigma\rho} + \omega_{\sigma\rho} + \omega_{\rho\sigma} + O(\omega^2) \\ \rightarrow& \omega_{\sigma\rho} + \omega_{\rho\sigma} = 0, \ \ \text{at } O(\omega^2) \end{aligned} \tag{37}$$

We have 10 independent components: $\omega_{\sigma\rho}$ (6) and ϵ^{μ} (4).

Poincare algebra

For an infinitesimal Lorentz transformation (36), we have

$$U(1+\omega,\epsilon) = 1 - \frac{1}{2} i\omega_{\rho\sigma} J^{\rho\sigma} + i\epsilon_\rho P^\rho + \cdots \tag{38}$$

Here $J^{\rho\sigma}$ and P^ρ are ω- and ϵ-independent operators, and the dots denote terms of higher order in ω and/or ϵ. In order for $U(1+\omega,\epsilon)$ to be unitary, the operators $J^{\rho\sigma}$ and P^ρ must be Hermitian

$$\begin{aligned} U^\dagger(1+\omega,\epsilon) =& U^{-1}(1+\omega,\epsilon) \\ &\downarrow \\ J^{\rho\sigma\dagger} =& J^{\rho\sigma}, \quad P^{\rho\dagger} = P^\rho \end{aligned} \tag{39}$$

Since $\omega_{\rho\sigma}$ is antisymmetric, we can take its coefficient $J^{\rho\sigma}$ to be antisymmetric also

$$J^{\rho\sigma} = -J^{\sigma\rho} \tag{40}$$

Poincare algebra

We now examine the Lorentz transformation properties of $J^{\rho\sigma}$ and P^{ρ}. We consider the product

$$U(\Lambda, a)U(1+\omega, \epsilon)U^{-1}(\Lambda, a) \tag{41}$$

Here the inverse transformation is

$$U^{-1}(\Lambda, a) = U\left(\Lambda^{-1}, -\Lambda^{-1}a\right) \tag{42}$$

Inserting (38) into (41) we obtain

$$\begin{aligned} & U(\Lambda, a)U(1+\omega, \epsilon)U^{-1}(\Lambda, a) \\ \approx & 1 - \frac{1}{2}i\omega_{\rho\sigma}U(\Lambda, a)J^{\rho\sigma}U^{-1}(\Lambda, a) + i\epsilon_{\rho}U(\Lambda, a)P^{\rho}U^{-1}(\Lambda, a) \end{aligned} \tag{43}$$

Poincare algebra

On the other hand, we can work out three transformation by the rule (28)

$$
\begin{aligned}
& U(\Lambda, a)U(1+\omega,\epsilon)U^{-1}(\Lambda, a) \\
=& U\left(\Lambda(1+\omega), \Lambda\epsilon + a\right) U\left(\Lambda^{-1}, -\Lambda^{-1}a\right) \\
=& U\left(\Lambda(1+\omega)\Lambda^{-1}, -\Lambda(1+\omega)\Lambda^{-1}a + \Lambda\epsilon + a\right) \\
=& U\left(1+\Lambda\omega\Lambda^{-1}, \Lambda\epsilon - \Lambda\omega\Lambda^{-1}a\right) \\
\approx& 1 - \frac{1}{2} i(\Lambda\omega\Lambda^{-1})_{\rho\sigma} J^{\rho\sigma} + i\left(\Lambda\epsilon - \Lambda\omega\Lambda^{-1}a\right)_{\rho} P^{\rho}
\end{aligned}
\tag{44}
$$

Poincare algebra

To first order in ω and ϵ, equating coefficients of $\omega_{\rho\sigma}$ and ϵ_ρ on both sides of Eqs. (43) and (44), we obtain

$$\begin{aligned} U(\Lambda, a) J^{\rho\sigma} U^{-1}(\Lambda, a) =& \Lambda_\mu^{\ \rho} \Lambda_\nu^{\ \sigma} \left(J^{\mu\nu} + P^\mu a^\nu - P^\nu a^\mu \right) \\ U(\Lambda, a) P^\rho U^{-1}(\Lambda, a) =& \Lambda_\mu^{\ \rho} P^\mu \end{aligned} \tag{45}$$

In (45), we can also express

$$\begin{aligned} \Lambda_\mu^{\ \rho} P^\mu &= \left(\Lambda^{-1}\right)^\rho_{\ \mu} P^\mu \\ \Lambda_\mu^{\ \rho} \Lambda_\nu^{\ \sigma} J^{\mu\nu} &= \left(\Lambda^{-1}\right)^\rho_{\ \mu} \left(\Lambda^{-1}\right)^\sigma_{\ \nu} J^{\mu\nu} \end{aligned} \tag{46}$$

Now the Lorentz transformation $\Lambda^\mu_{\ \nu}$ in Eq. (45) should be understood as an independent and different from (38).

Homework

From Eq. (43) and (44), prove Eq. (45).

Poincare algebra

In (45) we consider infinitesimal transformation $\Lambda^{\mu}_{\ \nu} = g^{\mu}_{\ \nu} + \omega^{\mu}_{\ \nu}$ and $a^{\mu} = \epsilon^{\mu}$ and keep only terms of first order in $\omega^{\mu}_{\ \nu}$ and ϵ^{μ}, Eq. (45) now becomes

$$
\begin{aligned}
i\left[\frac{1}{2}\omega_{\mu\nu}J^{\mu\nu} - \epsilon_{\mu}P^{\mu},\ \ J^{\rho\sigma}\right] =& \omega_{\mu}^{\ \rho}J^{\mu\sigma} + \omega_{\nu}^{\ \sigma}J^{\rho\nu} - \epsilon^{\rho}P^{\sigma} + \epsilon^{\sigma}P^{\rho} \\
i\left[\frac{1}{2}\omega_{\mu\nu}J^{\mu\nu} - \epsilon_{\mu}P^{\mu},\ \ P^{\rho}\right] =& \omega_{\mu}^{\ \rho}P^{\mu}
\end{aligned}
\tag{47}
$$

Equating coefficients of $\omega_{\mu\nu}$ and ϵ_{μ} on both sides of these equations, we find the commutation rules for the Lie algebra of Poincare group (or Poincare algebra)

$$
\begin{aligned}
[P^{\mu}, P^{\nu}] =& 0 \\
[P^{\mu}, J^{\rho\sigma}] =& -i\left(g^{\mu\sigma}P^{\rho} - g^{\mu\rho}P^{\sigma}\right) \\
[J^{\mu\nu}, J^{\rho\sigma}] =& -i\left(g^{\mu\rho}J^{\nu\sigma} + g^{\nu\sigma}J^{\mu\rho} - g^{\mu\sigma}J^{\nu\rho} - g^{\nu\rho}J^{\mu\sigma}\right)
\end{aligned}
\tag{48}
$$

Homework: Poincare algebra

Apply infinitesimal transformation $\Lambda^{\mu}_{\ \nu} = g^{\mu}_{\ \nu} + \omega^{\mu}_{\ \nu}$ and $a^{\mu} = \epsilon^{\mu}$ in (45), try to prove (47).

Homework: Poincare algebra

The differential operator forms for $J^{\mu\nu}$ and P^μ are

$$\begin{aligned} P^\mu =& i\partial^\mu = (i\partial_t, -i\mathbf{\nabla}) \\ J^{\mu\nu} =& x^\mu P^\nu - x^\nu P^\mu = i\left(x^\mu \partial^\nu - x^\nu \partial^\mu\right) \end{aligned} \tag{49}$$

With the commutation rule $[x^\mu, P^\nu] = -ig^{\mu\nu}$ and $[x_\mu, P_\nu] = -ig_{\mu\nu}$, we can prove the Poincare algebra (48).

We can define two groups of algebra

$$\begin{aligned}\boldsymbol{J} =& (J_{31}, J_{23}, J_{12}) = (J^{31}, J^{23}, J^{12}) \\ \boldsymbol{K} =& (J^{01}, J^{02}, J^{03}) = -(J_{01}, J_{02}, J_{03})\end{aligned} \tag{50}$$

which corresponds to rotation ($\boldsymbol{J}$) and Lorentz boost ($\boldsymbol{K}$). According to (48), they satisfy

$$\begin{aligned}[J_i, J_j] =& i\epsilon_{ijk} J_k \\ [J_i, K_j] =& i\epsilon_{ijk} K_k \\ [K_i, K_j] =& -i\epsilon_{ijk} J_k\end{aligned} \tag{51}$$

Rotation and Lorentz boost

We can define two sets of operators

$$\begin{aligned} A_i =& \frac{1}{2}(J_i + iK_i) \\ B_i =& \frac{1}{2}(J_i - iK_i) \end{aligned} \tag{52}$$

They satisfy

$$\begin{aligned} [A_i, A_j] =& i\epsilon_{ijk} A_k \\ [B_i, B_j] =& i\epsilon_{ijk} B_k \\ [A_i, B_j] =& 0 \end{aligned} \tag{53}$$

The two groups of operators are connected by $A_i^\dagger = B_i$ and by parity operation $\mathscr{P} A_i \mathscr{P}^{-1} = B_i$.

Homework: rotation and Lorentz boost

Prove Eqs. (51) and (53) using Eq. (48).

SL(2,C) group

- The SL(2,C) group is the special linear group of 2×2 complex matrices with determinant 1. It is a 6-dimensional Lie group (since 2×2 complex matrices have 4 complex parameters, and det$=1$ imposes one constraint, leaving 6 real parameters). It serves as the universal cover of the proper orthochronous Lorentz group $\mathrm{SO}^+(1,3)$.
- The SL(2,C) group applies to spinor fields. It is crucial for formulating relativistic quantum mechanics and quantum field theory for spin-1/2 particles, as it correctly describes how their spin states transform under Lorentz boosts and rotations.

SL(2,C) group

The group elements are written as

$$
\begin{aligned}
U(\Lambda) =& e^{-i\boldsymbol{J}\cdot\boldsymbol{\theta}-i\boldsymbol{K}\cdot\boldsymbol{\eta}} = e^{-i\boldsymbol{A}\cdot\boldsymbol{\theta}_A-i\boldsymbol{B}\cdot\boldsymbol{\theta}_B} \\
=& \exp\left[-i\frac{1}{2}(\boldsymbol{J}+i\boldsymbol{K})\cdot\boldsymbol{\theta}_A - i\frac{1}{2}(\boldsymbol{J}-i\boldsymbol{K})\cdot\boldsymbol{\theta}_B\right] \\
=& \exp\left[-i\frac{1}{2}\boldsymbol{J}\cdot(\boldsymbol{\theta}_A+\boldsymbol{\theta}_B) + \frac{1}{2}\boldsymbol{K}\cdot(\boldsymbol{\theta}_A-\boldsymbol{\theta}_B)\right] \qquad (54)
\end{aligned}
$$

where

$$
\boldsymbol{\theta}_A = \boldsymbol{\theta} - i\boldsymbol{\eta}, \;\; \boldsymbol{\theta}_B = \boldsymbol{\theta} + i\boldsymbol{\eta} \qquad (55)
$$

One can verify $U^\dagger(\Lambda) = U^{-1}(\Lambda)$, i.e. $U(\Lambda)$ is unitary.

SL(2,C) group

For the irreducible $(S, 0)$ and $(0, S)$ representations (S is the particle's spin quantum number), we have

$$
\begin{aligned}
D^{(S)}(\Lambda) =& e^{-i\boldsymbol{A}\cdot\boldsymbol{\theta}_A} = \exp\left[(-i\theta - \boldsymbol{\eta})\cdot\boldsymbol{A}\right] \\
\overline{D}^{(S)}(\Lambda) =& e^{-i\boldsymbol{B}\cdot\boldsymbol{\theta}_B} = \exp\left[(-i\theta + \boldsymbol{\eta})\cdot\boldsymbol{B}\right]
\end{aligned} \tag{56}
$$

They are related to each other

$$
D^{(S)\dagger}\left(\Lambda^{-1}\right) = \overline{D}^{(S)}(\Lambda) \tag{57}
$$

Note that $D^{(S)}(\Lambda)$ and $D^{(S)\dagger}(\Lambda^{-1})$ are two inequivalent finite-dimensional non-unitary representations of the Lorentz group and correspond to those labeled as $(S, 0)$ and $(0, S)$ respectively.

Pauli-Lubanski pseudo-vector

As usual we shall look for a maximal set of commuting observables. The four energy-momentum operators P^μ commute. Let p^μ be an "eigenvalue" of P^μ. Let us now look for Lorentz transformations which leave p^μ invariant, and write

$$U(\Lambda) = \exp\left(-\frac{i}{2}\omega_{\mu\nu}\mathcal{J}^{\mu\nu}\right) \tag{58}$$

where $\mathcal{J}^{\mu\nu}$ are 4×4 matrices as a particular representation for the generators $J^{\mu\nu}$

$$\begin{aligned}
(\mathcal{J}^{\mu\nu})_{\alpha\beta} &= i\left(g^\mu_\alpha g^\nu_\beta - g^\mu_\beta g^\nu_\alpha\right) \\
(\mathcal{J}^{\mu\nu})_\alpha^{\ \ \beta} &= i\left(g^\mu_{\ \alpha} g^{\nu\beta} - g^{\mu\beta} g^\nu_{\ \alpha}\right) \\
(\mathcal{J}^{\mu\nu})^\alpha_{\ \ \beta} &= i\left(g^{\mu\alpha} g^\nu_{\ \beta} - g^\mu_{\ \beta} g^{\nu\alpha}\right)
\end{aligned} \tag{59}$$

One can check that the above satisfies the commutation rules in (48).

Pauli-Lubanski pseudo-vector

The invariance of p implies $\omega_{\mu\nu}p^{\nu}$ which is solved according to $\omega_{\mu\nu} = -\epsilon_{\mu\nu\sigma\rho}p^{\sigma}s^{\rho} = \epsilon_{\rho\mu\nu\sigma}p^{\sigma}s^{\rho}$ where $\epsilon_{\rho\mu\nu\sigma}$ is the totally antisymmetric tensor $\epsilon_{0123} = -\epsilon^{0123} = -1$, and s^{μ} a four-vector whose component along p^{μ} is irrelevant. One can check

$$\omega_{\mu\nu}\left(\mathcal{J}^{\mu\nu}\right)_{\alpha\beta}p^{\beta} = i\omega_{\mu\nu}\left(g_{\alpha}^{\mu}p^{\nu} - g_{\alpha}^{\nu}p^{\mu}\right) = 0 \tag{60}$$

Thus the Lorentz transformations which leave p^{μ} invariant are represented as

$$U(\Lambda(p)) = \exp\left(-\frac{i}{2}\epsilon_{\rho\mu\nu\sigma}\mathcal{J}^{\mu\nu}p^{\sigma}s^{\rho}\right) \tag{61}$$

Pauli-Lubanski pseudo-vector

Here we defined Pauli-Lubanski pseudo-vector

$$W_\rho = -\frac{1}{2}\epsilon_{\rho\mu\nu\sigma} J^{\mu\nu} P^\sigma = -\frac{1}{2}\epsilon_{\rho\mu\nu\sigma} P^\sigma J^{\mu\nu} \tag{62}$$

Due to $\epsilon_{\rho\mu\nu\sigma}[P^\sigma, J^{\mu\nu}] = 0$ from (48), the position of two operators $J^{\mu\nu}$ and P^σ does not matter. We have the identity

$$W(P) \cdot P = 0 \tag{63}$$

One easily finds from (48)

$$\begin{aligned} [W_\mu, P_\nu] =& 0, \quad [W_\mu, W_\nu] = -i\epsilon_{\mu\nu\rho\sigma} W^\rho P^\sigma \\ [W_\rho, J_{\mu\nu}] =& -i\left(g_{\rho\nu} W_\mu - g_{\rho\mu} W_\nu\right) \end{aligned} \tag{64}$$

The latter indicating that W_μ behaves as a four-vector under Lorentz transformations.

Homework: Pauli-Lubanski operator

Prove Eq. (64) using Eq. (48).

Pauli-Lubanski pseudo-vector

Let then $|p,\sigma\rangle$ be a vector of representation space such that $P^{\mu}|p,\sigma\rangle = p^{\mu}$, so we have

$$\begin{aligned} U(\Lambda(p))\,|p,\sigma\rangle &= \exp\left(iW_{\rho}(P)s^{\rho}\right)|p,\sigma\rangle \\ &= \exp\left(iW_{\rho}(p)s^{\rho}\right)|p,\sigma\rangle \end{aligned} \tag{65}$$

It can be shown that

$$\begin{aligned} P^{\mu}\exp\left(iW_{\rho}(P)s^{\rho}\right)|p,\sigma\rangle =& \exp\left(iW_{\rho}(P)s^{\rho}\right)P^{\mu}\,|p,\sigma\rangle \\ =& p^{\mu}\exp\left(iW_{\rho}(p)s^{\rho}\right)|p,\sigma\rangle \end{aligned} \tag{66}$$

because $[W_{\mu},P_{\nu}] = 0$.

Pauli-Lubanski pseudo-vector

From these commutation rules one observes that $P_\mu P^\mu = P^2$ and $W_\mu W^\mu = W^2$ commnute with all P^μ and $J^{\mu\nu}$, so $P^2 = m^2$ and W^2 are two commutable invariant operators. The proof is simple

$$\begin{aligned}
[J_{\mu\nu}, g^{\rho\sigma} W_\rho W_\sigma] =& i g^{\rho\sigma} \left(g_{\nu\rho} W_\mu - g_{\mu\rho} W_\nu \right) W_\sigma + i g^{\rho\sigma} W_\rho \left(g_{\nu\sigma} W_\mu - g_{\mu\sigma} W_\nu \right) \\
=& i \left(W_\mu W_\nu - W_\nu W_\mu \right) + i \left(W_\nu W_\mu - W_\mu W_\nu \right) = 0 \\
[J_{\mu\nu}, g^{\rho\sigma} P_\rho P_\sigma] =& i g^{\rho\sigma} \left(g_{\nu\rho} P_\mu - g_{\mu\rho} P_\nu \right) P_\sigma + i g^{\rho\sigma} W_\rho \left(g_{\nu\sigma} P_\mu - g_{\mu\sigma} P_\nu \right) \\
=& i \left(P_\mu P_\nu - P_\nu P_\mu \right) + i \left(P_\nu P_\mu - P_\mu P_\nu \right) = 0
\end{aligned} \tag{67}$$

So $P^2 = m^2$ and $W^2/m^2 = S(S+1)$ (spin quantum number) can be regarded as two intrinsic constants to characterize a particle.

Homework: Pauli-Lubanski operator

Prove Eq. (67) using Eqs. (48) and (64).

One can attach to each p three space-like vectors $n^{(i)\mu}(p)$ $(i=1,2,3)$

$$
\begin{aligned}
n^{(i)}(p)\cdot n^{(j)}(p) =& -\delta_{ij}\\
p\cdot n^{(i)}(p) =& 0\\
g_{ij}n^{(i)}_{\mu}(p)n^{(j)}_{\nu}(p) =& g_{\mu\nu}\\
\epsilon^{\mu\nu\rho\sigma}n^{(\alpha)}_{\mu}n^{(\beta)}_{\nu}n^{(\gamma)}_{\rho}n^{(\delta)}_{\sigma} =& -\epsilon^{\alpha\beta\gamma\delta}
\end{aligned}
\tag{68}
$$

where we assume $n^{(0)\mu}(p)=p^{\mu}/m$. We can expand $W^{\mu}(p)$ as

$$
W^{\mu}(p)=\sum_{i=1}^{3}W_{i}(p)n^{(i)\mu}(p) \tag{69}
$$

Spin vector

Here we can extract $W_i(p)$ as

$$\begin{aligned} W_i(p) &= - W(p) \cdot n^{(i)}(p) \\ &= - \frac{1}{2} \epsilon_{\mu\nu\sigma\rho} J^{\mu\nu} p^\sigma n^{(i)\rho}(p) \end{aligned} \tag{70}$$

So one can define three spin operators

$$S_i(p) = \frac{1}{m} W_i(p) = -\frac{1}{2m} \epsilon_{\mu\nu\sigma\rho} J^{\mu\nu} p^\sigma n^{(i)\rho}(p) \tag{71}$$

One can verify the commutators

$$[S_i(p), S_j(p)] = i\epsilon_{ijk} S_k(p) \tag{72}$$

Spin vector

Here $S_i(p)$ are generators of SU(2) subgroup of proper orthochronous Lorentz group which leaves p invariant. Thus we have

$$\sum_{i=1}^{3} S_i^2(p) = \frac{1}{m^2} \sum_{i=1}^{3} W_i^2(p) = -\frac{1}{m^2} W^2(p) \tag{73}$$

which has possible eigenvalues $S(S+1)$ with S being integer or half-integer. So $P^2 = m^2$ and $\hat{S}^2 = S(S+1)$ can characterize a particle.

Homework: spin operators

Prove Eqs. (72,73) using Eqs. (48) and (73). Note that p in $S_i(p)$ and $W^\mu(p)$ is not operator.

One particle spin-momentum state

A particle is defined as a spin-momentum quantum state in its rest frame

$$|m, \sigma\rangle \tag{74}$$

where σ is the spin quantum number along the spin quantization direction (magnetic quantum number)

$$\sigma = -S, -S+1, -S+2, \cdots, S-1, S \tag{75}$$

Here S is the spin quantum number which is defined through

$$\begin{aligned} \hat{S}^2 |m, \sigma\rangle =& S(S+1) |m, \sigma\rangle \\ \hat{S}_z |m, \sigma\rangle =& \sigma |m, \sigma\rangle \end{aligned} \tag{76}$$

where we choose the z-axis as the spin quantization direction.

One particle spin-momentum state

Here m is the particle's mass defined through

$$\begin{aligned} P^{\mu}\left|m,\sigma\right\rangle =& p^{\mu}\left|m,\sigma\right\rangle = (m,0)\left|m,\sigma\right\rangle \\ P^{2}\left|m,\sigma\right\rangle =& p^{2}\left|m,\sigma\right\rangle = m^{2}\left|m,\sigma\right\rangle \end{aligned} \tag{77}$$

where $p^{\mu} = (m,0)$ is the four-momentum of the particle at rest.

One particle spin-momentum state

When the particle is not at rest but has an on-shell four-momentum $p^\mu = (E_p, \boldsymbol{p})$, its spin-momemtum state can be represented by $U\left(L(p)\right)$ as

$$|p,\sigma\rangle \equiv |\boldsymbol{p},\sigma\rangle = U\left(L(p)\right)|m,\sigma\rangle \tag{78}$$

where $L(p) \equiv L^\mu_{\ \nu}(p)$ denotes the Lorentz transformation that transforms a particle at rest to $p^\mu = (E_p, \mathrm{p})$,

$$\begin{aligned} L^0_{\ 0} &= \frac{E_p}{m} = \gamma,\ L^0_{\ i} = L^i_{\ 0} = \frac{1}{m}\boldsymbol{p}_i = \gamma\boldsymbol{\beta}_i \\ L^i_{\ j} &= \delta_{ij} - \hat{\boldsymbol{p}}_i\hat{\boldsymbol{p}}_j + \frac{E_p}{m}\hat{\boldsymbol{p}}_i\hat{\boldsymbol{p}}_j \end{aligned} \tag{79}$$

The state is normalized as

$$\left\langle p',\sigma'|p,\sigma\right\rangle = 2E_p(2\pi)^3\delta^{(3)}\left(\boldsymbol{p}'-\boldsymbol{p}\right)\delta_{\sigma'\sigma} \tag{80}$$

Lorentz transformation of one particle state

The Lorentz transformation of a spin-momentum state is

$$
\begin{aligned}
U(\Lambda)\left|p,\sigma\right\rangle &= U(\Lambda)\,U(L(p))\left|m,\sigma\right\rangle \\
&= U(L(\Lambda p))\,U\left(L^{-1}(\Lambda p)\,\Lambda L(p)\right)\left|m,\sigma\right\rangle \\
&= \Sigma_{\sigma'} U(L(\Lambda p))\left|m,\sigma'\right\rangle\left\langle m,\sigma'\right| U\left(L^{-1}(\Lambda p)\,\Lambda L(p)\right)\left|m,\sigma\right\rangle \\
&= \Sigma_{\sigma'}\left|\Lambda p,\sigma'\right\rangle\left\langle m,\sigma'\right| U\left[L^{-1}(\Lambda p)\,\Lambda L(p)\right]\left|m,\sigma\right\rangle \\
&= \Sigma_{\sigma'}\left|\Lambda p,\sigma'\right\rangle D^{(S)}_{\sigma'\sigma}\left[L^{-1}(\Lambda p)\,\Lambda L(p)\right] \qquad (81)
\end{aligned}
$$

where we defined D-matrix element for the Wigner rotation R

$$
D^{(S)}_{\sigma'\sigma}(R) = \left\langle m,\sigma'\right| U(R)\left|m,\sigma\right\rangle, \quad R = L^{-1}(\Lambda p)\,\Lambda L(p) \qquad (82)
$$

The representation of the little group (Wigner rotation) must be unitary, so we have

$$
D^{(S)\dagger}(R) = D^{(S)-1}(R) = D^{(S)}(R^{-1}) \qquad (83)
$$

Angular momentum operators

The adjoint representation for angular momentum operators in terms of SO(3) structure constants is defined as

$$(S_i)_{jk} = -i\epsilon_{ijk}, \tag{84}$$

which can be explicitly expressed as

$$S_1 = -i\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \; S_2 = -i\begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

$$S_3 = -i\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{85}$$

One can check that S_i $(i = 1, 2, 3)$ satisfy $[S_i, S_j] = i\epsilon_{ijk}S_k$ and that the Casimir operator $\sum_{i=1}^{3} S_i^2$ is 2 which corresponds to spin-1 operators.

Basis vectors in particle's rest frame

Consider the eigenvalue problem of $\hat{\boldsymbol{n}} \cdot \hat{\boldsymbol{S}}$ where $\hat{\boldsymbol{n}}$ is a unit vector. Its eigenvectors corresponding to eigenvalues $\lambda = \pm 1, 0$ are

$$\begin{aligned} \boldsymbol{e}_{\pm} &= \mp \frac{1}{\sqrt{2}}(\hat{\boldsymbol{\theta}} \pm i\hat{\boldsymbol{\phi}}), \\ \boldsymbol{e}_0 &= \hat{\boldsymbol{n}}, \end{aligned} \tag{86}$$

where $\left(\hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\phi}}, \hat{\boldsymbol{n}}\right)$ form a Cardesian coordiante

$$\begin{aligned} \hat{\boldsymbol{\theta}} &= (\cos\theta\cos\phi, \cos\theta\sin\phi - \sin\theta), \\ \hat{\boldsymbol{\phi}} &= (-\sin\phi, \cos\phi, 0), \\ \hat{\boldsymbol{n}} &= (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta). \end{aligned} \tag{87}$$

Basis vectors in particle's moving frame

We define $\left(n^{(1)\mu}, n^{(2)\mu}, n^{(3)\mu}\right)$ as the Lorentz boost of three orthorgonal basis directions $\left(\hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\phi}}, \hat{\boldsymbol{n}}\right)$. With $n^{(0)\mu} = p^\mu/m = (E_p, \boldsymbol{p})/m$, $\left(n^{(0)\mu}, n^{(1)\mu}, n^{(2)\mu}, n^{(3)\mu}\right)$ forms a tetrad and satisfy the relations [Florescu:1968zz]

$$\begin{aligned} n_\mu^{(\alpha)} n_\nu^{(\beta)} g^{\mu\nu} =& g^{\alpha\beta}, \\ n_\mu^{(\alpha)} n_\nu^{(\beta)} g_{\alpha\beta} =& g_{\mu\nu}, \\ \epsilon^{\mu\nu\rho\sigma} n_\mu^{(\alpha)} n_\nu^{(\beta)} n_\rho^{(\gamma)} n_\sigma^{(\delta)} =& -\epsilon^{\alpha\beta\gamma\delta}, \end{aligned} \tag{88}$$

where the indices in parenthesis should always be superscripts. For example, the explicit form of $n^{(3)\mu}$ is

$$n^{(3)\mu} = \left(\frac{\hat{\boldsymbol{n}} \cdot \boldsymbol{p}}{m}, \hat{\boldsymbol{n}} + \frac{(\hat{\boldsymbol{n}} \cdot \boldsymbol{p})\boldsymbol{p}}{m(m+E_p)}\right). \tag{89}$$

Homework

Prove Eq. (88) using

$$n^{(i)\mu} = \left(\frac{\boldsymbol{n}_i \cdot \boldsymbol{p}}{m}, \boldsymbol{n}_i + \frac{(\boldsymbol{n}_i \cdot \boldsymbol{p})\boldsymbol{p}}{m(m+E_p)} \right)$$

for $i = 1, 2, 3$ and $(\boldsymbol{n}_1, \boldsymbol{n}_2, \boldsymbol{n}_3) = \left(\hat{\theta}, \hat{\phi}, \hat{\boldsymbol{n}} \right)$.

Projection operator for spin-1/2 particles

The projection operator of a massive spin-1/2 particle with positive energy,

$$u(s,p)\bar{u}(s,p)=\frac{1}{2}(p\cdot\gamma+m)\left[1+s\gamma_5\gamma\cdot n^{(3)}\right], \tag{90}$$

was generalized by Bouchiat and Michel [Bouchiat:1958yui] to

$$\begin{aligned} u(s,p)\bar{u}(s',p) =&\frac{1}{2}(p\cdot\gamma+m)\left[\delta_{s's}+\gamma_5\gamma\cdot n^{(i)}(\tau_i)_{s's}\right] \\ =&\frac{1}{2}\left[\delta_{s's}+\gamma_5\gamma\cdot n^{(i)}(\tau_i)_{s's}\right](p\cdot\gamma+m), \end{aligned} \tag{91}$$

where p is the on-shell four-momentum and orthogonal to $n^{(i)}$ $(i=1,2,3)$, $s,s'=\pm 1$ denote the spin states of a spin-1/2 particle, and τ_i are the Pauli matrices in spin index space.

Spin density matrices for spin-1/2 particles

To prove Eqs. (90) and (91) one check both sides of equations in the particle's rest frame with $p^\mu = (m, 0)$ and then make Lorentz boost for the spinors and Dirac matrices. For the antiparticle, we have similar formula to Eq. (91)

$$\begin{aligned} v(s,p)\bar{v}(s',p) =& \frac{1}{2}(p\cdot\gamma - m)\left[\delta_{ss'} + \gamma_5\gamma\cdot n^{(i)}(\tau_i)_{ss'}\right] \\ =& \frac{1}{2}\left[\delta_{ss'} + \gamma_5\gamma\cdot n^{(i)}(\tau_i)_{ss'}\right](p\cdot\gamma - m). \end{aligned} \tag{92}$$

the Pauli matrices in spin index space. Equations (91) and (92) are useful in the derivation of spin density matrix elements of vector mesons

Projection operator for spin-1 particles

The spin-1 particle wave vectors $\epsilon^{\mu}(\lambda)$ can be written as

$$\begin{aligned}\epsilon^{\mu}(\pm 1, p) =& \mp \frac{1}{\sqrt{2}} \left(n^{(1)\mu} \pm i n^{(2)\mu} \right), \\ \epsilon^{\mu}(0, p) =& n^{(3)\mu},\end{aligned} \tag{93}$$

which satisfy

$$\begin{aligned}p \cdot \epsilon(\lambda, p) =& 0, \\ \epsilon(\lambda, p) \cdot \epsilon^{*}(\lambda', p) =& -\delta_{\lambda\lambda'}, \\ \epsilon(\lambda, p) \cdot \epsilon^{*}(-\lambda, p) =& (-1)^{\lambda} \epsilon(\lambda, p) \cdot \epsilon(\lambda, p) = -\delta_{\lambda 0},\end{aligned} \tag{94}$$

Projection operator for spin-1 particles

and also

$$\begin{aligned}\sum_{\lambda}\epsilon^{\mu}(\lambda,p)\epsilon^{\nu*}(\lambda,p) &= \sum_{\lambda}(-1)^{\lambda}\epsilon^{\mu}(\lambda)\epsilon^{\nu}(-\lambda) = \sum_{i=1}^{3}n^{(i)\mu}n^{(i)\nu} \\ &= -g^{\mu\nu}+\frac{p^{\mu}p^{\nu}}{m^2} \equiv -\Delta_p^{\mu\nu},\end{aligned} \tag{95}$$

where we used $n^{(i)}\cdot n^{(j)} = -\delta_{ij}$ for $i,j=1,2,3$.

Projection operator for spin-1 particles

Starting from an alternative form of Eq. (93)

$$\epsilon^{\mu}(\lambda,p)=(1-|\lambda|)\,n_3^{\mu}-\frac{1}{\sqrt{2}}\lambda\left[n^{(1)\mu}+i\lambda n^{(2)\mu}\right]. \tag{96}$$

Then the projection operator can be obtained as

$$\epsilon^{\mu}(\lambda,p)\epsilon^{\nu*}(\lambda',p)=\frac{1}{3}\left[-\Delta_p^{\mu\nu}I_3+i\frac{3}{2m}\epsilon^{\mu\nu\sigma\tau}p_{\sigma}n_{\tau}^{(i)}\Sigma^i-\frac{3}{2}n^{(i)\mu}n^{(j)\nu}\Sigma^{ij}\right]_{\lambda'\lambda}, \tag{97}$$

where $i,j=1,2,3$, I_3 is the 3×3 unit matrix, Σ^i denote the standard matrix representation of the spin-1 angular momentum operator.

Explicit form for Σ^i and Σ^{ij}

The explicit form for Σ^i

$$\Sigma^1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \Sigma^2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \Sigma^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \tag{98}$$

In Eq. (97) $\Delta_p^{\mu\nu}$ is defined in the last equality of Eq. (94), and Σ^{ij} are traceless and symmetric matrices defined as

$$(\Sigma^{ij})_{\lambda\lambda'} = \left(\Sigma^i \Sigma^j + \Sigma^j \Sigma^i - \frac{4}{3} I_3 \delta_{ij} \right)_{\lambda\lambda'} . \tag{99}$$

Explicit form for Σ^i and Σ^{ij}

The explicit forms of Σ^{ij} are

$$
\begin{aligned}
\Sigma^{12} &= \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \qquad & \Sigma^{23} &= \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & i \\ 0 & -i & 0 \end{pmatrix} \\
\Sigma^{31} &= \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}, \qquad & \Sigma^{11} &= \begin{pmatrix} -\frac{1}{3} & 0 & 1 \\ 0 & \frac{2}{3} & 0 \\ 1 & 0 & -\frac{1}{3} \end{pmatrix} \\
\Sigma^{22} &= \begin{pmatrix} -\frac{1}{3} & 0 & -1 \\ 0 & \frac{2}{3} & 0 \\ -1 & 0 & -\frac{1}{3} \end{pmatrix}, \qquad & \Sigma^{33} &= \begin{pmatrix} \frac{2}{3} & 0 & 0 \\ 0 & -\frac{4}{3} & 0 \\ 0 & 0 & \frac{2}{3} \end{pmatrix}
\end{aligned} \tag{100}
$$

Explicit form for Σ^i and Σ^{ij}

These matrices are traceless and satisfy

$$\begin{aligned}
\mathrm{Tr}\left(\Sigma^i\right) =& \mathrm{Tr}\left(\Sigma^{ij}\right) = 0, \\
\mathrm{Tr}\left(\Sigma^i\Sigma^j\Sigma^k\right) =& i\epsilon_{ijk}, \\
\mathrm{Tr}\left(\Sigma^i\Sigma^j\right) =& 2\delta_{ij},\ \mathrm{Tr}\left(\Sigma^i\Sigma^{jk}\right) = 0, \\
\mathrm{Tr}\left(\Sigma^{ij}\Sigma^{kl}\right) =& 2\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} - \frac{2}{3}\delta_{ij}\delta_{kl}\right).
\end{aligned} \tag{101}$$

In the sense of trace, Σ^i are orthogonal to Σ^{jk}, Σ^{ij} $(i \neq j)$ are orthogonal to Σ^{kk} (no summation over k), $\left\{\Sigma^1, \Sigma^2, \Sigma^3\right\}$ and $\left\{\Sigma^{12}, \Sigma^{23}, \Sigma^{31}\right\}$ are two orthogonal sets (any two elements in each set are orthogonal), but $\left\{\Sigma^{11}, \Sigma^{22}, \Sigma^{33}\right\}$ is not an orthogonal set.

Projection operator for spin-1 particles

In the diagonal case with $\lambda = \lambda'$, we obtain from Eq. (97)

$$\epsilon^{\mu}(\lambda, p)\epsilon^{\nu *}(\lambda, p) = -\frac{1}{3}\Delta_p^{\mu\nu} + i\frac{\lambda}{2m}\epsilon^{\mu\nu\sigma\tau}p_\sigma n_\tau^{(3)} + \left(\frac{1}{3} - \frac{1}{2}|\lambda|\right)\left(\Delta_p^{\mu\nu} + 3n^{(3)\mu}n^{(3)\nu}\right). \tag{102}$$

Spin density matrices for spin-1 particles

We consider an example for the spin density matrix in a production process of the spin-1 particle with the amplitude

$$\mathcal{M}(\lambda) = \mathcal{T}_\mu \epsilon^{\mu *}(\lambda, p), \tag{103}$$

where the repetition of two indices implies a summartion. The spin density matrix is defined as

$$\begin{aligned}\rho_{\lambda\lambda'} =& \frac{\mathcal{M}(\lambda)\mathcal{M}^*(\lambda')}{\sum_\lambda |\mathcal{M}(\lambda)|^2} = \frac{\epsilon_\mu^*(p,\lambda)\mathcal{T}^\mu \mathcal{T}^{\nu *}\epsilon_\nu(p,\lambda')}{\sum_\lambda \epsilon_\mu^*(p,\lambda)\mathcal{T}^\mu \mathcal{T}^{\nu *}\epsilon_\nu(p,\lambda)} \\ \equiv& \epsilon_\mu^*(\lambda, p)\rho^{\mu\nu}\epsilon_\nu(\lambda', p),\end{aligned} \tag{104}$$

Spin density matrices for spin-1 particles

where $\rho^{\mu\nu}$ in turn is given by

$$\begin{aligned}\rho^{\mu\nu} =& \frac{\mathcal{T}^{\mu}\mathcal{T}^{\nu *}}{\sum_{\lambda}|\mathcal{M}(\lambda)|^2} = \sum_{\lambda,\lambda'} \epsilon^{\mu}(p,\lambda)\rho_{\lambda\lambda'}\epsilon^{\nu *}(p,\lambda') \\ =& -\frac{1}{3}\Delta_p^{\mu\nu} + \frac{1}{2m} i\epsilon^{\mu\nu\rho\sigma}p_{\rho}P_{\sigma} - \frac{1}{2}Q^{\mu\nu}.\end{aligned} \tag{105}$$

Here the vector polarization P_σ and tensor polarization $Q_{\mu\nu}$ (symmetric in $\mu\nu$) are related to the spin density matrix as

$$\begin{aligned}P_{\sigma} =& \mathrm{Tr}\left(\rho\Sigma^{i}\right) n_{\sigma}^{(i)}, \\ Q_{\mu\nu} =& \mathrm{Tr}\left(\rho\Sigma^{ij}\right) n_{\mu}^{(i)} n_{\nu}^{(j)},\end{aligned} \tag{106}$$

which can be obtained explicitly for a specific process.

Spin density matrices for spin-1 particles

Then inserting Eq. (97) into Eq. (104), one obtains

$$\begin{aligned}\rho_{\lambda\lambda'} &= \sum_{\mu,\nu} \epsilon_\nu(p,\lambda')\epsilon^*_\mu(p,\lambda)\rho^{\mu\nu} \\ &= \left[\frac{1}{3}I_3 - \frac{1}{2}P^\tau n^{(i)}_\tau \Sigma^i + \frac{1}{4}n^{(i)\nu}n^{(j)\mu}Q_{\mu\nu}\Sigma^{ij}\right]_{\lambda\lambda'},\end{aligned} \tag{107}$$

where we used $\Delta_p^{\nu\mu}Q_{\mu\nu} = 0$, $\delta_{ij}\Sigma^{ij} = 0$ and $\epsilon^{\mu\nu\rho\tau}\epsilon_{\mu\nu\eta\sigma} = -2\left(\delta^\rho_\eta\delta^\tau_\sigma - \delta^\rho_\sigma\delta^\tau_\eta\right)$. We can rewrite Eq. (107) as

$$\rho = \frac{1}{3}I_3 + \frac{1}{2}\boldsymbol{P}_i\Sigma^i + \frac{1}{4}\boldsymbol{Q}_{ij}\Sigma^{ij}, \tag{108}$$

where the 3D polarization vector and tensor are defined as

$$\begin{aligned} \boldsymbol{P}_i \equiv & -\left(P \cdot n^{(i)}\right) = \mathrm{Tr}\left(\rho \Sigma^i\right), \\ \boldsymbol{Q}_{ij} \equiv & n^{(i)\mu} n^{(j)\nu} Q_{\mu\nu} = \mathrm{Tr}\left(\rho \Sigma^{ij}\right). \end{aligned} \tag{109}$$

These 3D vector and tensors are Lorentz invariant: they can actually be evaluated in the particle's rest frame in which $n^{(i)\mu}$ $(i = 1, 2, 3)$ become $\left(\hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\phi}}, \hat{\boldsymbol{n}}\right)$ in (87). Equation (107) with (109) is a general expression for the spin density matrix of spin-1 particles and widely used in literature.