From KB equations to spin kinetic equations

Qun Wang

Department of Modern Physics (USTC)
School of Mechanics and Physics (AUST)

2026.09.20-24

Outline

What is the CTP formalism?

- The **Closed-Time Path (CTP)** formalism, also known as the Schwinger-Keldysh formalism, is a powerful tool in quantum field theory for studying systems **out of equilibrium**.
- It generalizes the standard in-out formalism (vacuum) to **non-equilibrium** situations where the initial state is described by a density matrix.
- Key idea: The time contour runs from t_0 to $+\infty$ and then **back** to t_0, forming a closed loop.

Vacuum vs. non-equilibrium

Aspect	**Vacuum (In-Out)**	**Non-Equilibrium (CTP)**
Initial state	$\|0\rangle$ (vacuum)	$\rho(t_0)$ (density matrix)
Time contour	$-\infty \to +\infty$	$t_0 \to +\infty \to t_0$
Final state	$\langle 0\|$	$\mathrm{Tr}\rho(t_0)$
Evolution	$U(+\infty, -\infty)$	$U_{\mathrm{CTP}}(t_0)$

Main Difference

In vacuum, we evolve an in-state to an out-state. In non-equilibrium, **only states at** t_0 are involved, leading to the CTP.

Two-point Green's function

The two-point Green's function in the full theory is defined as

$$G(x,y) = \langle \Omega | \, T\phi_{(H)}(x)\phi_{(H)}(y) \, | \Omega \rangle \tag{1}$$

where:

- $|\Omega\rangle$ is the **ground state** of the full Hamiltonian H
- $\phi_{(H)}(x)$ is the field operator in the **Heisenberg picture**

The Hamiltonian is split into free and interaction parts:

$$H = H_0 + H_{\text{int}} \tag{2}$$

Heisenberg vs Schroedinger picture

At the initial time t_0, $\phi_{(S)}(\mathbf{x})$ is a field in Schroedinger picture and can be Fourier expanded as

$$\phi_{(S)}(\mathbf{x}) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \left(a_{\mathbf{p}} e^{i\mathbf{p}\cdot\mathbf{x}} + a^{\dagger}_{\mathbf{p}} e^{-i\mathbf{p}\cdot\mathbf{x}} \right) \tag{3}$$

We can obtain $\phi_{(H)}(t, \mathbf{x})$ from $\phi_{(S)}(\mathbf{x})$ by

$$\phi_{(H)}(x) = e^{iH(t-t_0)} \phi_{(S)}(\mathbf{x}) e^{-iH(t-t_0)} \tag{4}$$

We see $H_{(H)} = H_{(S)} = H$.

Schroedinger vs interaction picture

We can define the field operator in the interaction picture

$$\begin{aligned}\phi_{(I)}(x) &= e^{iH_0(t-t_0)}\phi_{(S)}(\mathbf{x})e^{-iH_0(t-t_0)} \\ &= \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}\left(a_{\mathbf{p}}e^{-ip\cdot x}+a^{\dagger}_{\mathbf{p}}e^{ip\cdot x}\right)\end{aligned} \tag{5}$$

where we have used $t_0 = 0$ and

$$\begin{aligned}a^{(I)}_{\mathbf{p}} &= e^{iH_0t}a_{\mathbf{p}}e^{-iH_0t} = a_{\mathbf{p}}e^{-iE_pt} \\ a^{\dagger(I)}_{\mathbf{p}} &= e^{iH_0t}a^{\dagger}_{\mathbf{p}}e^{-iH_0t} = a^{\dagger}_{\mathbf{p}}e^{iE_pt}\end{aligned} \tag{6}$$

We note that $a_{\mathbf{p}}$ and $a^{\dagger}_{\mathbf{p}}$ in Eq. (5) are in Schroedinger picture and independent of time. We have used $H_0 \equiv H_0^{(I)} = H_0^{(S)}$.

Heisenberg vs interaction picture

Now we can express the Heisenberg picture field $\phi(x)$ in terms of $\phi_{(I)}(t,\mathbf{x})$

$$\begin{aligned}\phi_{(H)}(x) &= e^{iH(t-t_0)}\phi_{(S)}(\mathbf{x})e^{-iH(t-t_0)} \\ &= e^{iH(t-t_0)}e^{-iH_0(t-t_0)}\phi_{(I)}(x)e^{iH_0(t-t_0)}e^{-iH(t-t_0)} \\ &= U^{\dagger}(t,t_0)\phi_{(I)}(x)U(t,t_0) \end{aligned} \tag{7}$$

where $U(t,t_0)$ is the unitary operator connecting the interaction and Heisenberg picture,

$$\begin{aligned} U(t,t_0) &= e^{iH_0(t-t_0)}e^{-iH(t-t_0)} \\ U(t_0,t) &= U^{\dagger}(t,t_0) = e^{iH(t-t_0)}e^{-iH_0(t-t_0)} \end{aligned} \tag{8}$$

Heisenberg vs interaction picture

So we have

$$\phi_{(H)}(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \left[a_{\mathbf{p}}(t) e^{-ip\cdot x} + a^{\dagger}_{\mathbf{p}}(t) e^{ip\cdot x} \right] \tag{9}$$

where

$$\begin{aligned} a_{\mathbf{p}}(t) &= U^{\dagger}(t, t_0) a_{\mathbf{p}} U(t, t_0) \\ a^{\dagger}_{\mathbf{p}}(t) &= U^{\dagger}(t, t_0) a^{\dagger}_{\mathbf{p}} U(t, t_0) \end{aligned} \tag{10}$$

Evolution operator

The evolution operator $U(t, t_0)$ satisfies the Schroedinger equation

$$i\frac{d}{dt}U(t, t_0) = H_{\rm int}^{(I)}U(t, t_0) \tag{11}$$

where $H_{\rm int}^{(I)}$ is the interaction Hamiltonian in the interaction picture,

$$H_{\rm int}^{(I)} = e^{iH_0(t-t_0)}H_{\rm int}^{(S)}e^{-iH_0(t-t_0)} \tag{12}$$

Evolution operator

The derivation of (11) is

$$\begin{aligned} i\frac{d}{dt}U(t,t_0) &= e^{iH_0(t-t_0)}He^{-iH(t-t_0)} - H_0U(t,t_0) \\ &= e^{iH_0(t-t_0)}H_{\rm int}^{(S)}e^{-iH(t-t_0)} \\ &= e^{iH_0(t-t_0)}H_{\rm int}^{(S)}e^{-iH_0(t-t_0)}e^{iH_0(t-t_0)}e^{-iH(t-t_0)} \\ &= H_{\rm int}^{(I)}U(t,t_0) \end{aligned} \tag{13}$$

The solution to $U(t,t_0)$ is

$$U(t,t_0) = T\left[\exp\left(-i\int_{t_0}^{t}dt' H_{\rm int}^{(I)}(t')\right)\right] \tag{14}$$

Note that $U(t,t_0)$ is a unitary operator which connects the interaction with Heisenberg picture.

Evolution operator

We can define an evolution operator in the interaction picture from t_1 to t_2

$$\begin{aligned} U(t_2,t_1) &= U(t_2,t_0)U^\dagger(t_1,t_0) \\ &= e^{iH_0(t_2-t_0)}e^{-iH(t_2-t_1)}e^{-iH_0(t_1-t_0)} \end{aligned} \tag{15}$$

We can prove that all elements of $U(t_2,t_1)$ consititute a U(1) group:

- The inverse of $U(t_2,t_1)$ is $U(t_1,t_2)$: $U(t_2,t_1)U(t_1,t_2)=1$.
- Unitary condition $U^\dagger(t_2,t_1)=U(t_1,t_2)$.
- The result of two consecutive evolution is still an evolution: $U(t_3,t_2)U(t_2,t_1)=U(t_3,t_1)$.

Ground state projection

To deal with Eq. (1), we express the ground state of H_0 in terms of eigenstates of H,

$$|0\rangle = \sum_n |n, H\rangle \langle n, H|0\rangle$$
$$\langle 0| = \sum_n \langle 0|n, H\rangle \langle n, H| \tag{16}$$

where $|n, H\rangle$ denote energy eigenstates of H with $|0, H\rangle = |\Omega\rangle$. We let $|0\rangle$ evolve in time with H

$$e^{-iH(t_\infty + t_0)}|0\rangle = \sum_n e^{-iE_n(t_\infty + t_0)}|n, H\rangle \langle n, H|0\rangle$$
$$\langle 0| e^{-iH(t_\infty - t_0)} = \sum_n e^{-iE_n(t_\infty - t_0)} \langle 0|n, H\rangle \langle n, H| \tag{17}$$

where E_n denotes the energy of the state $|n, H\rangle$ with $E_n > E_{n-1}$.

Ground state projection

In order to extract $|\Omega\rangle$ out of other states, we introduce a small imaginary part into t_∞ as $t_\infty \to t_\infty(1-i\epsilon)$, then we have

$$
\begin{aligned}
e^{-iH[t_\infty(1-i\epsilon)+t_0]}\,|0\rangle &= e^{-E_0\epsilon}\left\{\left(e^{-iE_0(t_\infty+t_0)}\,\langle\Omega|0\rangle\right)|\Omega\rangle\right. \\
&\quad \left. + \sum_{n\neq 0} e^{-iE_n(t_\infty+t_0)}e^{-(E_n-E_0)t_\infty\epsilon}\,|n,H\rangle\,\langle n,H|0\rangle\right\} \\
\langle 0|\,e^{-iH[t_\infty(1-i\epsilon)-t_0]} &= e^{-E_0\epsilon}\left\{\left(e^{-iE_0(t_\infty-t_0)}\,\langle 0|\Omega\rangle\right)\langle\Omega|\right. \\
&\quad \left. + \sum_{n\neq 0} e^{-iE_n(t_\infty-t_0)}e^{-(E_n-E_0)t_\infty\epsilon}\,\langle 0|n,H\rangle\,\langle n,H|\right\}
\end{aligned}
\tag{18}
$$

Ground state projection

Since all higher states are suppressed relative to the ground state by a factor $e^{-(E_n-E_0)t_\infty\epsilon}$, so we have

$$
\begin{aligned}
e^{-iH(t_\infty+t_0)}\,|0\rangle &\approx \left(e^{-iE_0(t_\infty+t_0)}\,\langle\Omega|0\rangle\right)|\Omega\rangle \\
\langle 0|\,e^{-iH(t_\infty-t_0)} &\approx \left(e^{-iE_0(t_\infty-t_0)}\,\langle 0|\Omega\rangle\right)\langle\Omega| \qquad (19)
\end{aligned}
$$

We can approximate

$$
\begin{aligned}
|\Omega\rangle &\approx \lim_{t_\infty\to\infty(1-i\epsilon)}\left(e^{iE_0(t_\infty+t_0)}\,\langle\Omega|0\rangle^{-1}\right)e^{-iH(t_\infty+t_0)}\,|0\rangle \\
&= \lim_{t_\infty\to\infty(1-i\epsilon)}\left(e^{iE_0(t_\infty+t_0)}\,\langle\Omega|0\rangle^{-1}\right)U(t_0,-t_\infty)\,|0\rangle \\
\langle\Omega| &\approx \lim_{t_\infty\to\infty(1-i\epsilon)}\left(e^{iE_0(t_\infty-t_0)}\,\langle 0|\Omega\rangle^{-1}\right)\langle 0|\,e^{-iH(t_\infty-t_0)} \\
&\approx \lim_{t_\infty\to\infty(1-i\epsilon)}\left(e^{iE_0(t_\infty-t_0)}\,\langle 0|\Omega\rangle^{-1}\right)\langle 0|\,U(t_\infty,t_0) \qquad (20)
\end{aligned}
$$

where we have used Eq. (8) and $H_0\,|0\rangle = 0$

Gell-Mann-Low Theorem

The normalization condition for the ground state reads

$$\begin{aligned} 1 &= \langle \Omega | \Omega \rangle \\ &= \lim_{t_\infty \to \infty(1-i\epsilon)} \left(e^{2iE_0 t_\infty} |\langle \Omega | 0 \rangle|^{-2} \right) \langle 0 | U(t_\infty, t_0) U(t_0, -t_\infty) | 0 \rangle \end{aligned} \tag{21}$$

which determines the normalization constant

$$e^{2iE_0 t_\infty} |\langle \Omega | 0 \rangle|^{-2} = \lim_{t_\infty \to \infty(1-i\epsilon)} \frac{1}{\langle 0 | U(t_\infty, -t_\infty) | 0 \rangle} \tag{22}$$

Gell-Mann-Low Theorem

Using Eqs. (7,20), Eq. (1) can be written as

$$\begin{aligned} G(x,y) &= \langle\Omega|\, T\phi_{(H)}(x)\phi_{(H)}(y)\,|\Omega\rangle \\ &= \lim_{t_\infty\to\infty(1-i\epsilon)} \frac{1}{\langle 0|\, U(t_\infty,-t_\infty)\,|0\rangle} \\ &\quad \times \langle 0|\, U(t_\infty,t_0)\, T U^\dagger(x_0,t_0)\phi_{(I)}(x) U(x_0,t_0) \\ &\quad \times U^\dagger(y_0,t_0)\phi_{(I)}(y) U(y_0,t_0) U(t_0,-t_\infty)\,|0\rangle \\ &= \lim_{t_\infty\to\infty(1-i\epsilon)} \frac{\langle 0|\, T[\phi_{(I)}(x)\phi_{(I)}(y) U(t_\infty,-t_\infty)]\,|0\rangle}{\langle 0|\, U(t_\infty,-t_\infty)\,|0\rangle} \end{aligned} \tag{23}$$

The numerator contains the time-ordered product of interaction-picture fields **and** the evolution operator over the full time range.

Non-equilibrium Green's function

In non-equilibrium, the two point correlation function is defined through the density operator at the initial time

$$G(x,y) = \left\langle T\phi_{(H)}(x)\phi_{(H)}(y)\right\rangle \equiv \frac{1}{\mathrm{Tr}\rho(t_0)}\mathrm{Tr}[\rho(t_0)T\phi_{(H)}(x)\phi_{(H)}(y)] \tag{24}$$

- The trace is taken over initial states (eigenstates of H_0)
- The interaction H_{Int} is switched on at later times
- No preferred ground state – the system is in a **mixed state**

Non-equilibrium Green's function

Using Eq. (7), we can write Eq. (24) as

$$
\begin{aligned}
G(x,y) &= \frac{1}{\mathrm{Tr}\rho(t_0)}\mathrm{Tr}[\rho(t_0)T\phi_{(H)}(x)\phi_{(H)}(y)] \\
&= \frac{1}{\mathrm{Tr}\rho(t_0)}\mathrm{Tr}[\rho(t_0)TU^{\dagger}(x_0,t_0)\phi_{(I)}(x)U(x_0,t_0) \\
&\quad \times U^{\dagger}(y_0,t_0)\phi_{(I)}(y)U(y_0,t_0)] \\
&= \frac{1}{\mathrm{Tr}\rho(t_0)}\mathrm{Tr}[\rho(t_0)TU(t_0,x_0)\phi_{(I)}(x)U(x_0,y_0)\phi_{(I)}(y)U(y_0,t_0)] \\
&\equiv \frac{1}{\mathrm{Tr}\rho(t_0)}\mathrm{Tr}[\rho(t_0)T_P\phi_{(I)}(x)\phi_{(I)}(y)U_{CTP}(t_0)] \qquad (25)
\end{aligned}
$$

Evolution operator on CTP

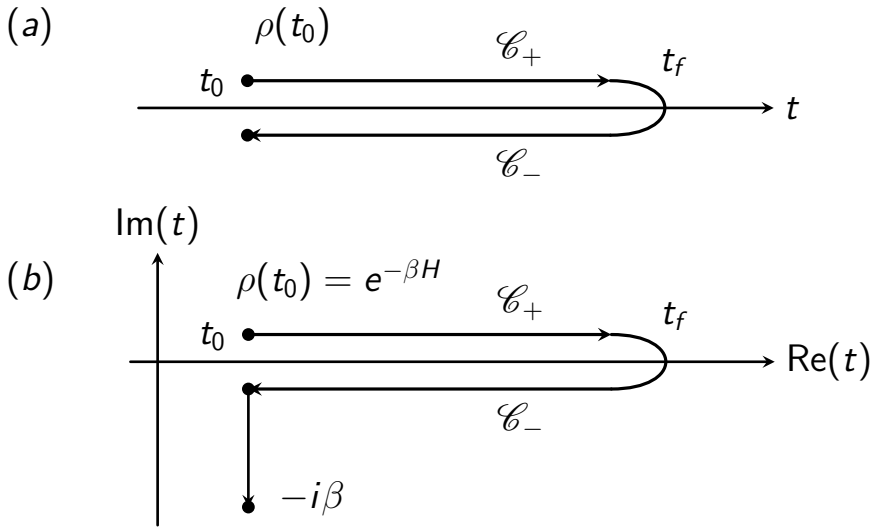
Here $U_{CTP}(t_0)$ is the unitary evolution operator defined on a closed-time path (CTP) starting from t_0 to $+\infty$ and back to t_0

$$\begin{aligned} U_{CTP}(t_0) &\equiv T_P\left[\exp\left(-i\int_{CTP} dt H_{\rm Int}^{(I)}(t)\right)\right] \\ &= T_P\left[\exp\left(-i\int_{t_0}^{\infty} dt_+ H_{\rm Int}^{(I)}(t_+) + i\int_{t_0}^{\infty} dt_- H_{\rm Int}^{(I)}(t_-)\right)\right] \end{aligned} \tag{26}$$

T_P: path ordering on the CTP

- T_P is the ordering operator along the CTP contour.
- Times on the t_+ branch are always **earlier** than times on the t_- branch.
- Within each branch, standard time ordering applies:
  - On t_+: T orders increasing time
  - On t_-: $\bar{T}$ (anti-time ordering) orders decreasing time
- This leads to four types of Green's functions: $G^{++}, G^{+-}, G^{-+}, G^{--}$ (or $G^F, G^<, G^>, G^{\bar{F}}$)

The CTP Contour



Vacuum vs. non-equilibrium: side by side

Vacuum (In-Out)

$$G = \frac{\langle 0|\, T[\phi_{(I)}\phi_{(I)}\, U]\, |0\rangle}{\langle 0|\, U\, |0\rangle} \qquad (27)$$

- Single time contour
- $-\infty \to +\infty$
- Ground state projection
- Denominator cancels disconnected diagrams

Non-Equilibrium (CTP)

$$G = \frac{\mathrm{Tr}[\rho(t_0)\, T_P \phi_{(I)}\phi_{(I)}\, U_{\mathrm{CTP}}]}{\mathrm{Tr}\rho(t_0)} \qquad (28)$$

- Closed time contour
- $t_0 \to +\infty \to t_0$
- Density matrix initial state
- No denominator cancellation (all diagrams contribute)

Two equivalent formulations

In the Schwinger–Keldysh (CTP) formalism, two-point functions can be written in:

1. **Matrix form** – 2×2 matrices in the contour branch space
2. **CTP form** – single time-ordered object on the closed contour

Both are completely equivalent and related by the structure of the time contour.

CTP form

Equivalently, one defines a single Green's function by time-ordering on the closed-time path:

$$G(x_1, x_2) = \langle T_C \psi_\alpha(x_1) \bar{\psi}_\beta(x_2) \rangle \tag{29}$$

where x_{10}, x_{20} lie on the CTP and T_C is the contour-ordering operator.

Connection with Matrix form

The matrix elements G^{ab} are simply the values of G when the time arguments sit on branch a and b of the contour ($a, b \in \{+, -\}$).

Matrix form

The matrix form

$$G=\begin{pmatrix} G^{++} & G^{+-} \\ G^{-+} & G^{--} \end{pmatrix}=\begin{pmatrix} G^{F} & G^{<} \\ G^{>} & G^{\bar{F}} \end{pmatrix}, \qquad \Sigma=\begin{pmatrix} \Sigma^{++} & \Sigma^{+-} \\ \Sigma^{-+} & \Sigma^{--} \end{pmatrix}=\begin{pmatrix} \Sigma^{F} & \Sigma^{<} \\ \Sigma^{>} & \Sigma^{\bar{F}} \end{pmatrix} \tag{30}$$

Definitions (for fermions with Dirac indices α, β):

$$\begin{aligned} G^{F}_{\alpha\beta}(x_1,x_2) &= \langle T\psi_\alpha(x_1)\bar{\psi}_\beta(x_2)\rangle \\ G^{\bar{F}}_{\alpha\beta}(x_1,x_2) &= \langle T_A\psi_\alpha(x_1)\bar{\psi}_\beta(x_2)\rangle \\ G^{<}_{\alpha\beta}(x_1,x_2) &= -\langle\bar{\psi}_\beta(x_2)\psi_\alpha(x_1)\rangle \\ G^{>}_{\alpha\beta}(x_1,x_2) &= \langle\psi_\alpha(x_1)\bar{\psi}_\beta(x_2)\rangle \end{aligned} \tag{31}$$

Unitary transformation to physical basis

Define the unitary matrix

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, \qquad U^{-1} = U^T = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \tag{32}$$

The **physical representation** is obtained by

$$\begin{pmatrix} 0 & G^A \\ G^R & G^C \end{pmatrix} = U \begin{pmatrix} G^F & G^< \\ G^> & G^{\bar{F}} \end{pmatrix} U^{-1} \tag{33}$$

and similarly for Σ (with sign flips on the off-diagonals):

$$\begin{pmatrix} \Sigma^A & 0 \\ \Sigma^C & \Sigma^R \end{pmatrix} = U \begin{pmatrix} \Sigma^F & -\Sigma^< \\ \Sigma^> & -\Sigma^{\bar{F}} \end{pmatrix} U^{-1} \tag{34}$$

Retarded, advanced and correlation functions

The physical components are defined as

$$
\begin{aligned}
O^R &= O^F - O^< = -O^{\bar{F}} + O^> \\
O^A &= O^F - O^> = -O^{\bar{F}} + O^< \\
O^C &= O^F + O^{\bar{F}} = O^> + O^<
\end{aligned}
\tag{35}
$$

for $O = G$ or Σ. Explicitly:

$$
\begin{aligned}
O^R(x_1, x_2) &= \theta(t_1 - t_2)\left[O^>(x_1, x_2) - O^<(x_1, x_2)\right] \\
O^A(x_1, x_2) &= \theta(t_2 - t_1)\left[O^<(x_1, x_2) - O^>(x_1, x_2)\right]
\end{aligned}
\tag{36}
$$

Here O^R and O^A are causal/anti-causal; O^C is the sum of statistical correlators.

Retarded, advanced and correlation Green's functions

In the physical basis, the product $\Sigma \cdot G$ becomes block-structured:

$$U^{-1}\begin{pmatrix}\Sigma^A & 0\\ \Sigma^C & \Sigma^R\end{pmatrix}\begin{pmatrix}0 & G^A\\ G^R & G^C\end{pmatrix}U=\begin{pmatrix}\Sigma^F & -\Sigma^<\\ \Sigma^> & -\Sigma^{\bar{F}}\end{pmatrix}\begin{pmatrix}G^F & G^<\\ G^> & G^{\bar{F}}\end{pmatrix} \tag{37}$$

The physical-basis product is

$$\begin{pmatrix}0 & \Sigma^A G^A\\ \Sigma^R G^R & \Sigma^C G^A+\Sigma^R G^C\end{pmatrix} \tag{38}$$

Advantage

Retarded, advanced and correlation Green's functions **decouple** in the upper-left / lower-right blocks!

Property

The matrix-form integral

$$I = \int_{t_0}^{\infty} dx' \begin{pmatrix} \Sigma^F & -\Sigma^< \\ \Sigma^> & -\Sigma^{\bar{F}} \end{pmatrix} (x_1, x') \begin{pmatrix} G^F & G^< \\ G^> & G^{\bar{F}} \end{pmatrix} (x', x_2) \tag{39}$$

is equivalent to the CTP contour integral

$$I = \int_C \Sigma(x_1, x')\, G(x', x_2) \tag{40}$$

where $C = [t_0^+, +\infty] + [+\infty, t_0^-]$.

Proof sketch: I_{+-}

For $t_1 \in C_+$, $t_2 \in C_-$:

$$
\begin{aligned}
I_{+-} &= \int_{t_0}^{\infty} dx' \left[\Sigma^F G^< - \Sigma^< G^{\bar{F}}\right] \\
&= \int_{t_0}^{t_1} dx' \, \Sigma^> G^< + \int_{t_1}^{\infty} dx' \, \Sigma^< G^< \\
&\quad - \int_{t_2}^{\infty} dx' \, \Sigma^< G^< - \int_{t_0}^{t_2} dx' \, \Sigma^< G^> \\
&= \int_{t_0}^{\infty} dx' \left[\Sigma^R G^< + \Sigma^< G^A\right]
\end{aligned}
\tag{41}
$$

The same result is obtained by evaluating the CTP contour integral using the branch-dependent θ_C functions.

Result for all components

Matrix element	Equivalent CTP expression
I_{++}	$\int_{t_0}^{\infty} dx' \left[\Sigma^F G^F - \Sigma^< G^>\right]$
I_{+-}	$\int_{t_0}^{\infty} dx' \left[\Sigma^R G^< + \Sigma^< G^A\right]$
I_{-+}	$\int_{t_0}^{\infty} dx' \left[\Sigma^R G^> + \Sigma^> G^A\right]$
I_{--}	$\int_{t_0}^{\infty} dx' \left[\Sigma^> G^< - \Sigma^{\bar{F}} G^{\bar{F}}\right]$

All four components match the CTP contour integral, proving equivalence.

Wigner coordinates

Introduce relative and center-of-mass coordinates:

$$y = x_1 - x_2, \qquad X = \tfrac{1}{2}(x_1 + x_2) \tag{42}$$

with

$$y' = x' - x_2, \quad X_1 = X + \tfrac{1}{2}y', \quad X_2 = X - \tfrac{1}{2}(y - y') \tag{43}$$

At leading order in the gradient expansion (zeroth order in ∂_X):

$$I^{(0)}_{+-}(y, X) = \int_{-\infty}^{\infty} dy' \left[\Sigma^R(y - y', X) G^<(y', X) + \Sigma^<(y - y', X) G^A(y', X)\right] \tag{44}$$

Fourier transform to momentum space

Convention:

$$\tilde{A}(p) = \int d^4x\, e^{ip\cdot x/\hbar} A(x), \qquad A(x) = \int [dp]\, e^{-ip\cdot x/\hbar} \tilde{A}(p) \tag{45}$$

where $[dp] \equiv d^4p/(2\pi\hbar)^4$. **Zeroth-order transform:**

$$F\left[I^{(0)}_{+-}\right] = \Sigma^R(p,X)G^<(p,X) + \Sigma^<(p,X)G^A(p,X) \tag{46}$$

First-order (gradient) correction

The next-to-leading order gives Poisson-bracket structures:

$$F\left[I_{+-}^{(1)}\right] = -\frac{i\hbar}{2}\left\{\Sigma^{R}, G^{<}\right\}_{P.B.} - \frac{i\hbar}{2}\left\{\Sigma^{<}, G^{A}\right\}_{P.B.}$$
$$\left\{A, B\right\}_{P.B.} \equiv \partial_{X}^{\mu} A\, \partial_{\mu}^{p} B - \partial_{\mu}^{p} A\, \partial_{X}^{\mu} B \tag{47}$$

Physical Meaning

These terms encode **spatial inhomogeneity** and are suppressed by $\hbar$ relative to the local collision term.

Free Dirac structure

For free fermions, $G^<$ and $G^>$ satisfy the Dirac equation:

$$\begin{aligned}(i\gamma_\mu \partial^\mu_{x_1} - m)\, G^<(x_1, x_2) &= 0 \\ G^<(x_1, x_2)\, (i\gamma_\mu \overleftarrow{\partial}^\mu_{x_2} + m) &= 0\end{aligned} \tag{48}$$

In Wigner variables:

$$\partial^\mu_{x_1} = \tfrac{1}{2}\partial^\mu_X + \partial^\mu_y, \qquad \partial^\mu_{x_2} = \tfrac{1}{2}\partial^\mu_X - \partial^\mu_y \tag{49}$$

Fourier-transformed kinetic terms

After Fourier transformation $\int d^4y\, e^{ip\cdot y}$:

$$
\begin{aligned}
I_1 &= \left(i\gamma_\mu \tfrac{1}{2}\partial_X^\mu + \gamma_\mu p^\mu - m\right) G^<(p,X) \\
I_2 &= G^<(p,X)\left(i\gamma_\mu \tfrac{1}{2}\overleftarrow{\partial}_X^\mu - \gamma_\mu p^\mu + m\right)
\end{aligned}
\tag{50}
$$

These are the **kinetic (streaming) terms** that will appear on the left-hand side of the Kadanoff–Baym equation.

From Dyson–Schwinger to KB equations

Schematically, on the CTP contour:

$$G^{-1} = G_0^{-1} - \Sigma \quad \Longrightarrow \quad G_0^{-1} G = 1 + \Sigma G, \quad G G_0^{-1} = 1 + G\Sigma \tag{51}$$

The explicit Kadanoff–Baym equation on the CTP is

$$-i(i\gamma_\mu \partial^\mu_{x_1} - m)\, G(x_1, x_2) = \delta_C^{(4)}(x_1 - x_2) + \int_C dx'\, \Sigma(x_1, x')\, G(x', x_2) \tag{52}$$

where δ_C is the contour delta function (sign depends on the branch).

KB equation for $G^<$ and $G^>$

For $(x_1, x_2) \in (t_+, t_-)$:

$$
\begin{aligned}
&- i(i\hbar\gamma_\mu\partial^\mu_{x_1} - m)\, G^<(x_1, x_2) \\
&= \hbar\int_{-\infty}^{\infty} dx' \left[\Sigma^R G^< + \Sigma^< G^A\right](x_1, x'; x', x_2)
\end{aligned}
\tag{53}
$$

For $(x_1, x_2) \in (t_-, t_+)$:

$$
\begin{aligned}
&- i(i\gamma_\mu\partial^\mu_{x_1} - m)\, G^>(x_1, x_2) \\
&= \int_{-\infty}^{\infty} dx' \left[\Sigma^R G^> + \Sigma^> G^A\right](x_1, x'; x', x_2)
\end{aligned}
\tag{54}
$$

KB equation in momentum space

Fourier transforming and keeping terms up to $O(\hbar^2)$:

$$\Big(i\gamma_\mu \tfrac{1}{2}\hbar\partial_X^\mu + \gamma_\mu p^\mu - m\Big) G^<(p,X)$$
$$= i\hbar\big[\Sigma^R G^< + \Sigma^< G^A\big] + \frac{\hbar^2}{2}\big[\{\Sigma^R, G^<\}_{PB} + \{\Sigma^<, G^A\}_{PB}\big] \tag{55}$$

where $\{A,B\}_{PB} = \partial_x A\cdot\partial_p B - \partial_p A\cdot\partial_x B$ is Poisson bracket.

Counting powers of $\hbar$

Each propagator $\sim \hbar$, each vertex $\sim \hbar^{-1}$. The collision term is $O(\hbar)$; the Poisson-bracket term is $O(\hbar^2)$.

On-shell approximation

Taking only the imaginary (on-shell) part of the spectral functions:

$$\frac{1}{k_0 - p_0 \pm i\epsilon} \approx \mp i\pi\delta(k_0 - p_0) \tag{56}$$

Retarded and advanced functions simplify to

$$O^R \approx \tfrac{1}{2}(O^> - O^<), \qquad O^A \approx -\tfrac{1}{2}(O^> - O^<) \tag{57}$$

The KB equation for $G^<$ becomes

$$\begin{aligned}&\left(i\gamma_\mu \tfrac{1}{2}\hbar\partial_X^\mu + \gamma_\mu p^\mu - m\right) G^<(p, X) \\ &= -\frac{i\hbar}{2}\left[\Sigma^< G^> - \Sigma^> G^<\right] - \frac{\hbar^2}{4}\left[\{\Sigma^<, G^>\}_{PB} - \{\Sigma^>, G^<\}_{PB}\right]\end{aligned} \tag{58}$$

The same collision term for $G^>$

The equation for $G^>$ has the **identical** right-hand side:

$$\left(i\gamma_\mu \tfrac{1}{2}\hbar\partial_X^\mu + \gamma_\mu p^\mu - m\right) G^>(p,X)$$
$$= -\frac{i\hbar}{2}\left[\Sigma^< G^> - \Sigma^> G^<\right] - \frac{\hbar^2}{4}\left[\{\Sigma^<, G^>\}_{PB} - \{\Sigma^>, G^<\}_{PB}\right] \tag{59}$$

Subtracting the two equations yields the quantum Boltzmann equation for the distribution function encoded in $G^< - G^>$.

Clifford decomposition

The Wigner function $G^<(x,p)$ is a 4×4 matrix in Dirac space. Expand in the 16 independent generators of the Clifford algebra:

$$\Gamma_a \in \left\{1,\ \gamma^\mu,\ i\gamma^5,\ \gamma^5\gamma^\mu,\ \sigma^{\mu\nu}=\frac{i}{2}[\gamma^\mu,\gamma^\nu]\right\} \tag{60}$$

$$G^<(x,p)=\frac{1}{4}\left(\mathcal{F}+i\gamma^5\mathcal{P}+\gamma^\mu\mathcal{V}_\mu+\gamma^5\gamma^\mu\mathcal{A}_\mu+\frac{1}{2}\sigma^{\mu\nu}\mathcal{S}_{\mu\nu}\right) \tag{61}$$

- $\mathcal{F}$: scalar, $\mathcal{P}$: pseudoscalar, $\mathcal{V}_\mu$: vector, $\mathcal{A}_\mu$: axial-vector, $\mathcal{S}_{\mu\nu}$: tensor components
- Real-valued coefficient functions $\Rightarrow$ obtained by tracing $G^<$ with the corresponding generator
- $\mathcal{F}\leftrightarrow$ phase-space distribution; $\mathcal{A}_\mu\leftrightarrow$ spin-polarization distribution

Conventions: $g_{\mu\nu}=\mathrm{diag}(+,-,-,-)$, $\epsilon^{0123}=-\epsilon_{0123}=1$, $\mathcal{A}^\mu=\mathrm{Tr}(\gamma^\mu\gamma^5 G^<)$.

Equations of motion: real parts

Insert the decomposition into the KB equation

$$(\gamma\cdot K - m)G^{<} = I_{\rm coll},\quad K^{\mu}\equiv p^{\mu}+\frac{i\hbar}{2}\partial_x^{\mu} \tag{62}$$

and take traces. Real parts are

$$\begin{aligned}
p^{\mu}\mathcal{V}_{\mu} - m\mathcal{F} &= {\rm ReTr}\left(I_{\rm coll}\right),\\
m\mathcal{P} + \frac{\hbar}{2}\partial_x^{\mu}\mathcal{A}_{\mu} &= {\rm ReTr}\left(i\gamma^5 I_{\rm coll}\right),\\
p_{\mu}\mathcal{F} - m\mathcal{V}_{\mu} + \frac{\hbar}{2}\partial_x^{\nu}\mathcal{S}_{\mu\nu} &= {\rm ReTr}\left(\gamma_{\mu} I_{\rm coll}\right),\\
\frac{1}{2}\epsilon_{\mu\nu\alpha\beta}p^{\nu}\mathcal{S}^{\alpha\beta} + m\mathcal{A}_{\mu} - \frac{\hbar}{2}\partial_{x,\mu}\mathcal{P} &= {\rm ReTr}\left(\gamma^5\gamma_{\mu} I_{\rm coll}\right),\\
\epsilon_{\mu\nu\alpha\beta}p^{\alpha}\mathcal{A}^{\beta} + m\mathcal{S}_{\mu\nu} - \frac{\hbar}{2}\partial_{x,[\mu}\mathcal{V}_{\nu]} &= -{\rm ReTr}\left(\sigma_{\mu\nu} I_{\rm coll}\right),
\end{aligned} \tag{63}$$

Collision term

Collision term (up to $\mathcal{O}(\hbar^2)$):

$$I_{\rm coll} = -\frac{i\hbar}{2}\left(\Sigma^< G^> - \Sigma^> G^<\right) - \frac{\hbar^2}{4}\left(\left\{\Sigma^<, G^>\right\}_{PB} - \left\{\Sigma^>, G^<\right\}_{PB}\right). \tag{64}$$

Equations of motion: imaginary parts

$$
\begin{aligned}
\frac{\hbar}{2}\partial_x^\mu \mathcal{V}_\mu &= \mathrm{Im}\mathrm{Tr}\left(I_{\mathrm{coll}}\right), \\
p^\mu \mathcal{A}_\mu &= \mathrm{Im}\mathrm{Tr}\left(-i\gamma^5 I_{\mathrm{coll}}\right), \\
p^\nu \mathcal{S}_{\nu\mu} + \frac{\hbar}{2}\partial_{x,\mu}\mathcal{F} &= \mathrm{Im}\mathrm{Tr}\left(\gamma_\mu I_{\mathrm{coll}}\right), \\
p_\mu \mathcal{P} + \frac{\hbar}{4}\epsilon_{\mu\nu\alpha\beta}\partial_x^\nu \mathcal{S}^{\alpha\beta} &= \mathrm{Im}\mathrm{Tr}\left(\gamma^5\gamma_\mu I_{\mathrm{coll}}\right), \\
p_{[\mu}\mathcal{V}_{\nu]} + \frac{\hbar}{2}\epsilon_{\mu\nu\alpha\beta}\partial_x^\alpha \mathcal{A}^\beta &= -\mathrm{Im}\mathrm{Tr}\left(\sigma_{\mu\nu} I_{\mathrm{coll}}\right).
\end{aligned}
\tag{65}
$$

- Without collisions: equivalent to kinetic theory for massive fermions
- Collision terms from a *different derivation* in earlier work reproduce these equations
- Written perturbatively, order by order in $\hbar$

Mass-shell and Boltzmann-type equations

Apply $\gamma \cdot K + m$ to the KB equation and combine with its Hermitian conjugate (using $\gamma^0 (G^<)^\dagger \gamma^0 = G^<$):

On-shell (Klein–Gordon type)

$$\left(p^2 - \frac{\hbar^2}{4}\partial_x^2 - m^2\right) \mathrm{Tr}\left(\Gamma_a G^<\right) = \mathrm{ReTr}\left[\Gamma_a(\gamma \cdot K + m) I_{\mathrm{coll}}\right] \tag{66}$$

$$\hbar p \cdot \partial_x \mathrm{Tr}\left(\Gamma_a G^<\right) = \mathrm{ImTr}\left[\Gamma_a(\gamma \cdot K + m) I_{\mathrm{coll}}\right] \tag{67}$$

Equivalent to the ten component equations of the previous slides.

Off-shell terms

Decompose into on-shell and off-shell part,

$$G^< = G^<_{\rm on} + G^<_{\rm off}$$

with $(p^2 - m^2)G^<_{\rm on} \equiv 0$. The on-shell condition (66) gives

$$\mathrm{Tr}\left(\Gamma_a G^<_{\rm off}\right) = \frac{1}{p^2 - m^2}\mathrm{ReTr}\left[\Gamma_a(\gamma\cdot K + m)I_{\rm coll}\right] + \mathcal{O}(\hbar^2).$$

The adjoint KB equation implies (∂_x acts only on $G^\lessgtr$ not on $\Sigma^\lessgtr$, $I_{\rm coll} \sim g^2$)

$$I_{\rm coll}\left(\gamma\cdot p - m - \frac{i\hbar}{2}\gamma\cdot\overleftarrow{\partial}_x\right) = \mathcal{O}(\hbar^2 g^4)$$

$$\begin{aligned} I_{\rm coll}\left(\gamma\cdot p - m - \frac{i\hbar}{2}\gamma\cdot\overleftarrow{\partial}_x\right)\left(\gamma\cdot p + m - \frac{i\hbar}{2}\gamma\cdot\overleftarrow{\partial}_x\right) &= I_{\rm coll}\left(p^2 - m^2 - \frac{\hbar^2}{4}\overleftarrow{\partial}_x^2 - \frac{i\hbar}{2}\gamma\cdot\overleftarrow{\partial}_x\right) \\ &= \mathcal{O}(\hbar^2 g^4) \end{aligned}$$

On-shell equations

We obtain

$$I_{\rm coll}^{\rm (off)} = i\frac{\hbar}{p^2-m^2}p\cdot\partial_x I_{\rm coll} + \mathcal{O}(\hbar^2 g^4) + \mathcal{O}(\hbar^4 g^2) \tag{68}$$

where ∂_x acts only on $G^{\lessgtr}$ not on $\Sigma^{\lessgtr}$. From Eq. (67) we obtain

$$\begin{aligned}
\hbar p\cdot\partial_x {\rm Tr}\left(\Gamma_a G_{\rm on}^<\right) &= {\rm Im Tr}\left[\Gamma_a(\gamma\cdot K+m)I_{\rm coll}\right] - \hbar p\cdot\partial_x {\rm Tr}\left(\Gamma_a G_{\rm off}^<\right)\\
&= {\rm Im Tr}\left[\Gamma_a(\gamma\cdot K+m)I_{\rm coll}^{\rm (on)}\right]\\
&+ \frac{\hbar}{p^2-m^2}p\cdot\partial_x {\rm Im Tr}\left[i\Gamma_a(\gamma\cdot K+m)I_{\rm coll}\right]\\
&- \frac{\hbar}{p^2-m^2}p\cdot\partial_x {\rm Re Tr}\left[\Gamma_a(\gamma\cdot K+m)I_{\rm coll}\right]\\
&+ \mathcal{O}(\hbar^3) + \mathcal{O}(\hbar^2 g^4) + \mathcal{O}(\hbar^4 g^2)
\end{aligned} \tag{69}$$

On-shell equations

Off-shell pieces cancel in the Boltzmann equation to lowest order in g:

$$\hbar p\cdot\partial_x \mathrm{Tr}\left(\Gamma_a G_{\mathrm{on}}^{<}\right) = \mathrm{Im}\mathrm{Tr}\left[\Gamma_a(\gamma\cdot K+m)I_{\mathrm{coll}}^{(\mathrm{on})}\right] + \mathcal{O}(\hbar^3)+\mathcal{O}(\hbar^2 g^4)+\mathcal{O}(\hbar^4 g^2) \tag{70}$$

Expansion scheme

Expand every function and operator:

$$F = \sum_{n=0}^{\infty} \hbar^n F^{(n)} \tag{71}$$

- $I_{\rm coll}$ starts at $\mathcal{O}(\hbar)$: $I_{\rm coll}^{(0)} = 0$
- Zeroth order: free, on-shell kinetic theory
- First order: local collision term $\rightarrow$ leading-order Boltzmann equations
- Second order: nonlocal (Poisson-bracket) collision terms

Independent components: $\mathcal{F}$ (scalar) and $\mathcal{A}_\mu$ (axial-vector); all others are expressed through them.

Zeroth order $\mathcal{O}(\hbar^0)$

With $I^{(0)}_{\rm coll} = 0$, all component at $O(\hbar^0)$ are on-shell,

$$
\begin{aligned}
&p^\mu \mathcal{V}^{(0)}_\mu - m\mathcal{F}^{(0)} = 0, \qquad && \mathcal{P}^{(0)} = 0, \qquad && p^\mu \mathcal{A}^{(0)}_\mu = 0, \\
&p_\mu \mathcal{F}^{(0)} - m\mathcal{V}^{(0)}_\mu = 0, \qquad && p^\nu \mathcal{S}^{(0)}_{\mu\nu} = 0, \qquad && p_{[\mu}\mathcal{V}^{(0)}_{\nu]} = 0,
\end{aligned} \tag{72}
$$

$$
\tfrac{1}{2}\epsilon_{\mu\nu\alpha\beta} p^\nu \mathcal{S}^{(0)\alpha\beta} + m\mathcal{A}^{(0)}_\mu = 0, \qquad \epsilon_{\mu\nu\alpha\beta} p^\alpha \mathcal{A}^{(0)\beta} + m\mathcal{S}^{(0)}_{\mu\nu} = 0. \tag{73}
$$

Solution in terms of the independent components:

$$
\mathcal{P}^{(0)} = 0, \qquad \mathcal{V}^{(0)}_\mu = \frac{p_\mu}{m}\mathcal{F}^{(0)}, \qquad \mathcal{S}^{(0)}_{\mu\nu} = -\frac{1}{m}\epsilon_{\mu\nu\alpha\beta} p^\alpha \mathcal{A}^{(0)\beta}. \tag{74}
$$

All components satisfy on-shell conditions and kinetic equations.

First order $\mathcal{O}(\hbar)$: collision term

$$I_{\rm coll}^{(1)} = -\frac{i}{2}\left[\Sigma^{<(0)} G^{>(0)} - \Sigma^{>(0)} G^{<(0)}\right] \tag{75}$$

- Only *zeroth-order* Wigner functions appear $\Rightarrow$ on-shell
- Local in space-time: collisions at a single space-time point
- Conserves energy-momentum at the four-fermion vertex

Equations of motion at this order couple gradients of $\mathcal{O}(\hbar^0)$ components to $\mathcal{O}(\hbar)$ components.

First order: Boltzmann equations for $\mathcal{F}^{(0)}$ and $\mathcal{A}^{(0)\mu}$

From $\partial_x\cdot$ Eq. (vector, real) + Eq. (scalar, imaginary):

$$p\cdot\partial_x\mathcal{F}^{(0)} = 2m\,\mathrm{ImTr}\left(I_{\mathrm{coll}}^{(1)}\right) \tag{76}$$

From $\epsilon^{\rho\sigma\mu\nu}p_\sigma\times$ Eq. (tensor, imaginary):

$$\begin{aligned} p\cdot\partial_x\mathcal{A}^{(0)\mu} &= -\epsilon^{\mu\nu\alpha\beta}p_\nu\mathrm{ImTr}\left(\sigma_{\alpha\beta}I_{\mathrm{coll}}^{(1)}\right) \\ &= -2p^\mu\mathrm{ImTr}\left(\gamma_5 I_{\mathrm{coll}}^{(1)}\right) - 2m\mathrm{ImTr}\left(\gamma_5\gamma^\mu I_{\mathrm{coll}}^{(1)}\right) \end{aligned} \tag{77}$$

- On-shell property of $I_{\mathrm{coll}}^{(1)}$ allows $p\cdot\gamma\to m$ under traces
- These drive the *leading-order* Boltzmann equations for the MVSDs

First order: mass-shell constraints

$$
\begin{aligned}
\left(p^2-m^2\right)\mathcal{F}^{(1)} &= 2m\mathrm{ReTr}\left(I_{\mathrm{coll}}^{(1)}\right), \\
\left(p^2-m^2\right)\mathcal{P}^{(1)} &= 0, \\
\left(p^2-m^2\right)\mathcal{V}_{\mu}^{(1)} &= 2p_{\mu}\mathrm{ReTr}\left(I_{\mathrm{coll}}^{(1)}\right), \\
\left(p^2-m^2\right)\mathcal{A}_{\mu}^{(1)} &= -\epsilon_{\mu\nu\alpha\beta}p^{\nu}\mathrm{ReTr}\left(\sigma^{\alpha\beta}I_{\mathrm{coll}}^{(1)}\right), \\
\left(p^2-m^2\right)\mathcal{S}_{\mu\nu}^{(1)} &= 2p_{[\mu}\mathrm{ImTr}\left(\gamma_{\nu]}I_{\mathrm{coll}}^{(1)}\right)+2m\mathrm{ReTr}\left(\sigma_{\mu\nu}I_{\mathrm{coll}}^{(1)}\right),
\end{aligned}
$$

- $\mathcal{F}^{(1)}$, $\mathcal{V}^{(1)}$, $\mathcal{A}^{(1)}$, $\mathcal{S}^{(1)}$ can have off-shell parts at $\mathcal{O}(\hbar)$
- $\mathcal{P}^{(1)}$ stays on-shell $\Rightarrow$ off-shell effects enter at most at $\mathcal{O}(\hbar^2)$

Other components at $\mathcal{O}(\hbar)$

Once $\mathcal{F}^{(1)}$ and $\mathcal{A}^{(1)}_{\mu}$ are known, We obtain all other components at order $\mathcal{O}(\hbar)$,

$$
\begin{aligned}
\mathcal{P}^{(1)} =& -\frac{1}{2m}\partial_x^{\mu}\mathcal{A}^{(0)}_{\mu} + \frac{1}{m}\mathrm{Re}\,\mathrm{Tr}\left(i\gamma^5 I^{(1)}_{\mathrm{coll}}\right) \\
\mathcal{V}^{(1)}_{\mu} =& \frac{1}{m}p_{\mu}\mathcal{F}^{(1)} - \frac{1}{2m}\partial_x^{\nu}\mathcal{S}^{(0)}_{\nu\mu} - \frac{1}{m}\mathrm{Re}\,\mathrm{Tr}\left(\gamma_{\mu} I^{(1)}_{\mathrm{coll}}\right) \\
\mathcal{S}^{(1)}_{\mu\nu} =& -\frac{1}{m}\epsilon_{\mu\nu\alpha\beta}p^{\alpha}\mathcal{A}^{(1)\beta} + \frac{1}{2m}\,\partial_{x[\mu}\mathcal{V}^{(0)}_{\nu]} - \frac{1}{m}\mathrm{Re}\,\mathrm{Tr}\left(\sigma_{\mu\nu} I^{(1)}_{\mathrm{coll}}\right)
\end{aligned}
\tag{78}
$$

Second order: collision terms

The second order collision terms

$$I_{\rm coll}^{(2)} = \Delta I_{\rm coll}^{(1)} + I_{\rm coll,PB}^{(0)} \tag{79}$$

Local part (first-order expansion of $\Sigma^{\lessgtr} G^{\gtrless}$):

$$\Delta I_{\rm coll}^{(1)} = -\frac{i}{2}\left[\Sigma^{<(1)} G^{>(0)} - \Sigma^{>(1)} G^{<(0)} + \Sigma^{<(0)} G^{>(1)} - \Sigma^{>(0)} G^{<(1)}\right] \tag{80}$$

Nonlocal part (Poisson brackets $\rightarrow$ collisions at different space-time points):

$$I_{\rm coll,PB}^{(0)} = -\frac{1}{4}\left[\left\{\Sigma^{<(0)}, G^{>(0)}\right\}_{PB} - \left\{\Sigma^{>(0)}, G^{<(0)}\right\}_{PB}\right],$$
$$\{A, B\}_{PB} \equiv (\partial_x A)\cdot(\partial_p B) - (\partial_p A)\cdot(\partial_x B). \tag{81}$$

The nonlocal term is the seed of spin–vorticity coupling: it converts orbital into spin angular momentum.

Second order: Boltzmann equations

Evolution equations for scalar and axial vector components

$$p\cdot\partial_x\mathcal{F}^{(1)}=2m\mathrm{Im}\mathrm{Tr}\left(I^{(2)}_{\mathrm{coll}}\right)+\mathrm{Re}\mathrm{Tr}\left(\gamma\cdot\partial_x I^{(1)}_{\mathrm{coll}}\right) \tag{82}$$

$$\begin{aligned}p\cdot\partial_x\mathcal{A}^{(1)\mu}=&-\epsilon^{\mu\nu\alpha\beta}p_\nu\,\mathrm{Im}\,\mathrm{Tr}\left(\sigma_{\alpha\beta}I^{(2)}_{\mathrm{coll}}\right)-\mathrm{Re}\,\mathrm{Tr}\left(\gamma^5\partial^\mu_x I^{(1)}_{\mathrm{coll}}\right)\\=&-2p^\mu\,\mathrm{Im}\,\mathrm{Tr}\left(\gamma^5 I^{(2)}_{\mathrm{coll}}\right)-2\,\mathrm{Im}\,\mathrm{Tr}\left(\gamma\cdot p\,\gamma^5\gamma^\mu I^{(2)}_{\mathrm{coll}}\right)\\&-\mathrm{Re}\,\mathrm{Tr}\left(\gamma^5\partial^\mu_x I^{(1)}_{\mathrm{coll}}\right)\end{aligned} \tag{83}$$

Consistency between the two derivations (divergence vs. momentum contraction) yields constraint equations,

$$\mathrm{Re}\mathrm{Tr}\left(\gamma\cdot\partial_x I^{(1)}_{\mathrm{coll}}\right)=2\mathrm{Im}\mathrm{Tr}\left[(\gamma\cdot p-m)I^{(2)}_{\mathrm{coll}}\right], \tag{84}$$

satisfied up to $\mathcal{O}(\hbar g^4)$ using the adjoint KB equation.

Decomposition: on-shell and off-shell part

In general, the Wigner function at $\mathcal{O}(\hbar)$ can be decomposed into

$$\begin{aligned} G^{\lessgtr(1)} &= G^{\lessgtr(1)}_{(\mathrm{on})} + G^{\lessgtr(1)}_{(\mathrm{off})} \\ I^{(2)}_{\mathrm{coll}} &= I^{(2,\mathrm{on})}_{\mathrm{coll}} + I^{(2,\mathrm{off})}_{\mathrm{coll}} \\ &= \Delta I^{(1,\mathrm{on})}_{\mathrm{coll}} + I^{(0,\mathrm{on})}_{\mathrm{coll,PB}} + \Delta I^{(1,\mathrm{off})}_{\mathrm{coll}} + I^{(0,\mathrm{off})}_{\mathrm{coll,PB}} \end{aligned} \tag{85}$$

The evolution equations (82) and (83) can be decomposed into on-shell and off-shell parts. Equation (78) can also be decomposed into on-shell and off-shell parts as

Decomposition: on-shell and off-shell part

On-shell part:

$$
\begin{aligned}
\mathcal{P}^{(1,\mathrm{on})} =& -\frac{1}{2m}\partial_x^\mu \mathcal{A}_\mu^{(0)} + \frac{1}{m}\mathrm{Re}\,\mathrm{Tr}\left(i\gamma^5 I_{\mathrm{coll}}^{(1)}\right) \\
\mathcal{V}_\mu^{(1,\mathrm{on})} =& \frac{1}{m}p_\mu \mathcal{F}_{(\mathrm{on})}^{(1)} - \frac{1}{2m}\partial_x^\nu \mathcal{S}_{\nu\mu}^{(0)} - \frac{1}{m}\mathrm{Re}\,\mathrm{Tr}\left(\gamma_\mu I_{\mathrm{coll}}^{(1)}\right) \\
\mathcal{S}_{\mu\nu}^{(1,\mathrm{on})} =& -\frac{1}{m}\epsilon_{\mu\nu\alpha\beta}p^\alpha \mathcal{A}_{(\mathrm{on})}^{(1)\beta} + \frac{1}{2m}\partial_{x[\mu}\mathcal{V}_{\nu]}^{(0)} - \frac{1}{m}\mathrm{Re}\,\mathrm{Tr}\left(\sigma_{\mu\nu} I_{\mathrm{coll}}^{(1)}\right)
\end{aligned}
\tag{86}
$$

Off-shell part:

$$
\begin{aligned}
\mathcal{P}^{(1,\mathrm{off})} =& 0, \quad \mathcal{V}_\mu^{(1,\mathrm{off})} = \frac{1}{m}p_\mu \mathcal{F}_{(\mathrm{off})}^{(1)} \\
\mathcal{S}_{\mu\nu}^{(1,\mathrm{off})} =& -\frac{1}{m}\epsilon_{\mu\nu\alpha\beta}p^\alpha \mathcal{A}_{(\mathrm{off})}^{(1)\beta}
\end{aligned}
\tag{87}
$$

On-shell evolution equations

We only need to solve on-shell evolution equations. The zeroth order evolution equations

$$\begin{aligned} p\cdot\partial_x\mathcal{F}^{(0)} &= 2m\,\mathrm{Im}\mathrm{Tr}\left(I_{\mathrm{coll}}^{(1)}\right) \\ p\cdot\partial_x\mathcal{A}^{(0)\mu} &= -\epsilon^{\mu\nu\alpha\beta}p_\nu\mathrm{Im}\mathrm{Tr}\left(\sigma_{\alpha\beta}I_{\mathrm{coll}}^{(1)}\right) \end{aligned} \tag{88}$$

The first order evolution equations

$$\begin{aligned} p\cdot\partial_x\mathcal{F}_{(\mathrm{on})}^{(1)} &= 2m\mathrm{Im}\mathrm{Tr}\left(I_{\mathrm{coll}}^{(2,\mathrm{on})}\right)+\mathrm{Re}\mathrm{Tr}\left(\gamma\cdot\partial_x I_{\mathrm{coll}}^{(1)}\right) \\ p\cdot\partial_x\mathcal{A}_{(\mathrm{on})}^{(1)\mu} &= -\epsilon^{\mu\nu\alpha\beta}p_\nu\mathrm{Im}\mathrm{Tr}\left(\sigma_{\alpha\beta}I_{\mathrm{coll}}^{(2,\mathrm{on})}\right)-\mathrm{Re}\mathrm{Tr}\left(\gamma^5\partial_x^\mu I_{\mathrm{coll}}^{(1)}\right) \end{aligned} \tag{89}$$

Matrix valued spin distributions

The MVSD as 2×2 matrices can be parametrized as

$$
\begin{aligned}
f_{rs}(x, \boldsymbol{p}) &= \frac{1}{2}\left[\delta_{rs} f_{(\mathrm{unpol})}(x, \boldsymbol{p}) + \boldsymbol{\tau}_{rs} \cdot \boldsymbol{f}_{(\mathrm{pol})}(x, \boldsymbol{p})\right], \\
\bar{f}_{rs}(x, \boldsymbol{p}) &= \frac{1}{2}\left[\delta_{rs} \bar{f}_{(\mathrm{unpol})}(x, \boldsymbol{p}) + \boldsymbol{\tau}_{rs} \cdot \bar{\boldsymbol{f}}_{(\mathrm{pol})}(x, \boldsymbol{p})\right],
\end{aligned}
\tag{90}
$$

where $\boldsymbol{\tau} = (\tau_1, \tau_2, \tau_3)$ are Pauli matrices in spin space, $f_{(\mathrm{unpol})}$ and $\boldsymbol{f}_{(\mathrm{pol})}$ are unpolarized and polarized distributions for particles respectively, and $\bar{f}_{(\mathrm{unpol})}$ and $\bar{\boldsymbol{f}}_{(\mathrm{pol})}$ are those for antiparticles respectively.

Matrix valued spin distributions

They are given by

$$
\begin{aligned}
f_{(\mathrm{unpol})} =& \mathrm{Tr} f, \quad \bar{f}_{(\mathrm{unpol})} = \mathrm{Tr}\bar{f}, \\
\boldsymbol{f}_{(\mathrm{pol})} =& \left(f_{(1,\mathrm{pol})}, f_{(2,\mathrm{pol})}, f_{(3,\mathrm{pol})}\right) = \left(\mathrm{Tr}(\tau_1 f), \mathrm{Tr}(\tau_2 f), \mathrm{Tr}(\tau_3 f)\right), \\
\bar{\boldsymbol{f}}_{(\mathrm{pol})} =& \left(\bar{f}_{(1,\mathrm{pol})}, \bar{f}_{(2,\mathrm{pol})}, \bar{f}_{(3,\mathrm{pol})}\right) = \left(\mathrm{Tr}(\tau_1 \bar{f}), \mathrm{Tr}(\tau_2 \bar{f}), \mathrm{Tr}(\tau_3 \bar{f})\right).
\end{aligned} \tag{91}
$$

We see that polarized distributions are three-vector that gives the polarization vector.

Two-point Green's function at $\mathcal{O}(\hbar^0)$

The zeroth order (on-shell) function $G^<_{(0)}(x,p)$ can be expressed in terms of the zeroth order MVSDs as

$$\begin{aligned} G^<_{(0)}(x,p) \approx & -(2\pi\hbar)\theta(p_0)\delta\left(p^2-m^2\right) u\left(r,\boldsymbol{p}\right)\bar{u}\left(s,\boldsymbol{p}\right) f^{(0)}_{rs}\left(x,\boldsymbol{p}\right) \\ & -(2\pi\hbar)\theta(-p_0)\delta\left(p^2-m^2\right) v\left(r,-\boldsymbol{p}\right)\bar{v}\left(s,-\boldsymbol{p}\right)\left[\delta_{sr}-\bar{f}^{(0)}_{sr}\left(x,-\boldsymbol{p}\right)\right]. \end{aligned} \tag{92}$$

Note that $G^>_{(0)}(x,p)$ can be obtained from (92) by replacing $f^{(0)} \to -(1-f^{(0)})$ and $1-\bar{f}^{(0)} \to -\bar{f}^{(0)}$.

Two-point Green's function at $\mathcal{O}(\hbar)$

In terms of the first order MVSDs $f_{rs}^{(1)}$ and $\bar{f}_{sr}^{(1)}$, The first order on-shell function $G_{(0)}^{<}(x,p)$ can be expressed as

$$\begin{aligned} G_{(1,\mathrm{on})}^{<}(x,p) \approx & -(2\pi\hbar)\theta(p_0)\delta\left(p^2-m^2\right) u\left(r,\boldsymbol{p}\right)\bar{u}\left(s,\boldsymbol{p}\right) f_{rs}^{(1)}\left(x,\boldsymbol{p}\right) \\ & +(2\pi\hbar)\theta(-p_0)\delta\left(p^2-m^2\right) v\left(r,-\boldsymbol{p}\right)\bar{v}\left(s,-\boldsymbol{p}\right) \bar{f}_{sr}^{(1)}\left(x,-\boldsymbol{p}\right) \end{aligned} \tag{93}$$

Extracting scalar and axial vector components

We can extract the scalar and axial vector components of $G^{<}_{(0)}(x,p)$ in Eq. (92) at $\mathcal{O}(\hbar^0)$ according to the decomposition in Eq. (61)

$$\begin{aligned}
\mathcal{F}_{(0)} =& -(2\pi\hbar)(2m)\delta\left(p^2-m^2\right) \\
&\times\left\{\theta(p_0)f^{(0)}_{(\mathrm{unpol})}(x,\boldsymbol{p})-\theta(-p_0)\left[2-\bar{f}^{(0)}_{(\mathrm{unpol})}(x,-\boldsymbol{p})\right]\right\}, \\
\mathcal{A}^{\mu}_{(0)} =& -(2\pi\hbar)(2m)\delta\left(p^2-m^2\right) \\
&\times\left\{\theta(p_0)n^{(j)\mu}(\boldsymbol{p})f^{(0)}_{(j,\mathrm{pol})}(x,\boldsymbol{p})+\theta(-p_0)n^{(j)\mu}(-\boldsymbol{p})\bar{f}^{(0)}_{(j,\mathrm{pol})}(x,-\boldsymbol{p})\right\},
\end{aligned} \tag{94}$$

where $n^{(j)\mu}(\boldsymbol{p})$ with $j=1,2,3$ are Lorentz boost of three orthorgonal vectors $\left(\hat{\boldsymbol{\theta}},\hat{\boldsymbol{\phi}},\hat{\boldsymbol{n}}\right)$ in the particle's rest frame.

Evolution equations for MVSDs

From evolution equations (88) and (89) we can derive evolution equations for MVSDs at $\mathcal{O}(\hbar^0)$

$$\begin{aligned} p\cdot\partial_x f^{(0)}_{(\mathrm{unpol})} &= \mathscr{C}_{\mathrm{scalar}}(f^{(0)}) \\ p\cdot\partial_x\left[n^{(j)\mu} f^{(0)}_{(j,\mathrm{pol})}\right] &= \mathscr{C}^{\mu}_{\mathrm{av}}(f^{(0)}) \end{aligned} \tag{95}$$

and at $\mathcal{O}(\hbar^1)$

$$\begin{aligned} p\cdot\partial_x f^{(1)}_{(\mathrm{unpol})} &= \mathscr{C}_{\mathrm{scalar}}(f^{(0)}, \partial_x f^{(0)}, f^{(1)}) \\ p\cdot\partial_x\left[n^{(j)\mu} f^{(0)}_{(j,\mathrm{pol})}\right] &= \mathscr{C}^{\mu}_{\mathrm{av}}(f^{(0)}, \partial_x f^{(0)}, f^{(1)}) \end{aligned} \tag{96}$$