

From KB equations to spin kinetic equations

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What is the CTP formalism?

- The **Closed-Time Path (CTP)** formalism, also known as the Schwinger-Keldysh formalism, is a powerful tool in quantum field theory for studying systems **out of equilibrium**.
- It generalizes the standard in-out formalism (vacuum) to **non-equilibrium** situations where the initial state is described by a density matrix.
- Key idea: The time contour runs from t_0 to $+\infty$ and then **back** to t_0 , forming a closed loop.

Vacuum vs. non-equilibrium

Aspect	Vacuum (In-Out)	Non-Equilibrium (CTP)
Initial state	$ 0\rangle$ (vacuum)	$\rho(t_0)$ (density matrix)
Time contour	$-\infty \rightarrow +\infty$	$t_0 \rightarrow +\infty \rightarrow t_0$
Final state	$\langle 0 $	$\text{Tr}\rho(t_0)$
Evolution	$U(+\infty, -\infty)$	$U_{\text{CTP}}(t_0)$

Main Difference

In vacuum, we evolve an in-state to an out-state. In non-equilibrium, **only states at t_0** are involved, leading to the CTP.

Two-point Green's function

The two-point Green's function in the full theory is defined as

$$G(x, y) = \langle \Omega | T \phi_{(H)}(x) \phi_{(H)}(y) | \Omega \rangle \quad (1)$$

where:

- $|\Omega\rangle$ is the **ground state** of the full Hamiltonian H
- $\phi_{(H)}(x)$ is the field operator in the **Heisenberg picture**

The Hamiltonian is split into free and interaction parts:

$$H = H_0 + H_{\text{int}} \quad (2)$$

Heisenberg vs Schroedinger picture

At the initial time t_0 , $\phi_{(S)}(\mathbf{x})$ is a field in Schroedinger picture and can be Fourier expanded as

$$\phi_{(S)}(\mathbf{x}) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \left(a_{\mathbf{p}} e^{i\mathbf{p}\cdot\mathbf{x}} + a_{\mathbf{p}}^\dagger e^{-i\mathbf{p}\cdot\mathbf{x}} \right) \quad (3)$$

We can obtain $\phi_{(H)}(t, \mathbf{x})$ from $\phi_{(S)}(\mathbf{x})$ by

$$\phi_{(H)}(x) = e^{iH(t-t_0)} \phi_{(S)}(\mathbf{x}) e^{-iH(t-t_0)} \quad (4)$$

We see $H_{(H)} = H_{(S)} = H$.

Schroedinger vs interaction picture

We can define the field operator in the interaction picture

$$\begin{aligned}\phi_{(I)}(x) &= e^{iH_0(t-t_0)}\phi_{(S)}(\mathbf{x})e^{-iH_0(t-t_0)} \\ &= \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \left(a_{\mathbf{p}} e^{-ip \cdot x} + a_{\mathbf{p}}^\dagger e^{ip \cdot x} \right)\end{aligned}\quad (5)$$

where we have used $t_0 = 0$ and

$$\begin{aligned}a_{\mathbf{p}}^{(I)} &= e^{iH_0 t} a_{\mathbf{p}} e^{-iH_0 t} = a_{\mathbf{p}} e^{-iE_{\mathbf{p}} t} \\ a_{\mathbf{p}}^{\dagger(I)} &= e^{iH_0 t} a_{\mathbf{p}}^\dagger e^{-iH_0 t} = a_{\mathbf{p}}^\dagger e^{iE_{\mathbf{p}} t}\end{aligned}\quad (6)$$

We note that $a_{\mathbf{p}}$ and $a_{\mathbf{p}}^\dagger$ in Eq. (5) are in Schroedinger picture and independent of time. We have used $H_0 \equiv H_0^{(I)} = H_0^{(S)}$.

Heisenberg vs interaction picture

Now we can express the Heisenberg picture field $\phi(x)$ in terms of $\phi_{(I)}(t, \mathbf{x})$

$$\begin{aligned}\phi_{(H)}(x) &= e^{iH(t-t_0)}\phi_{(S)}(\mathbf{x})e^{-iH(t-t_0)} \\ &= e^{iH(t-t_0)}e^{-iH_0(t-t_0)}\phi_{(I)}(x)e^{iH_0(t-t_0)}e^{-iH(t-t_0)} \\ &= U^\dagger(t, t_0)\phi_{(I)}(x)U(t, t_0)\end{aligned}\tag{7}$$

where $U(t, t_0)$ is the unitary operator connecting the interaction and Heisenberg picture,

$$\begin{aligned}U(t, t_0) &= e^{iH_0(t-t_0)}e^{-iH(t-t_0)} \\ U(t_0, t) &= U^\dagger(t, t_0) = e^{iH(t-t_0)}e^{-iH_0(t-t_0)}\end{aligned}\tag{8}$$

Heisenberg vs interaction picture

So we have

$$\phi_{(H)}(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \left[a_{\mathbf{p}}(t) e^{-ip \cdot x} + a_{\mathbf{p}}^{\dagger}(t) e^{ip \cdot x} \right] \quad (9)$$

where

$$\begin{aligned} a_{\mathbf{p}}(t) &= U^{\dagger}(t, t_0) a_{\mathbf{p}} U(t, t_0) \\ a_{\mathbf{p}}^{\dagger}(t) &= U^{\dagger}(t, t_0) a_{\mathbf{p}}^{\dagger} U(t, t_0) \end{aligned} \quad (10)$$

The evolution operator $U(t, t_0)$ satisfies the Schroedinger equation

$$i \frac{d}{dt} U(t, t_0) = H_{\text{int}}^{(I)} U(t, t_0) \quad (11)$$

where $H_{\text{int}}^{(I)}$ is the interaction Hamiltonian in the interaction picture,

$$H_{\text{int}}^{(I)} = e^{iH_0(t-t_0)} H_{\text{int}}^{(S)} e^{-iH_0(t-t_0)} \quad (12)$$

Evolution operator

The derivation of (11) is

$$\begin{aligned} i \frac{d}{dt} U(t, t_0) &= e^{iH_0(t-t_0)} H e^{-iH(t-t_0)} - H_0 U(t, t_0) \\ &= e^{iH_0(t-t_0)} H_{\text{int}}^{(S)} e^{-iH(t-t_0)} \\ &= e^{iH_0(t-t_0)} H_{\text{int}}^{(S)} e^{-iH_0(t-t_0)} e^{iH_0(t-t_0)} e^{-iH(t-t_0)} \\ &= H_{\text{int}}^{(I)} U(t, t_0) \end{aligned} \tag{13}$$

The solution to $U(t, t_0)$ is

$$U(t, t_0) = T \left[\exp \left(-i \int_{t_0}^t dt' H_{\text{int}}^{(I)}(t') \right) \right] \tag{14}$$

Note that $U(t, t_0)$ is a unitary operator which connects the interaction with Heisenberg picture.

Evolution operator

We can define an evolution operator in the interaction picture from t_1 to t_2

$$\begin{aligned} U(t_2, t_1) &= U(t_2, t_0)U^\dagger(t_1, t_0) \\ &= e^{iH_0(t_2-t_0)}e^{-iH(t_2-t_1)}e^{-iH_0(t_1-t_0)} \end{aligned} \quad (15)$$

We can prove that all elements of $U(t_2, t_1)$ constitute a $U(1)$ group:

- The inverse of $U(t_2, t_1)$ is $U(t_1, t_2)$: $U(t_2, t_1)U(t_1, t_2) = 1$.
- Unitary condition $U^\dagger(t_2, t_1) = U(t_1, t_2)$.
- The result of two consecutive evolution is still an evolution: $U(t_3, t_2)U(t_2, t_1) = U(t_3, t_1)$.

Ground state projection

To deal with Eq. (1), we express the ground state of H_0 in terms of eigenstates of H ,

$$\begin{aligned} |0\rangle &= \sum_n |n, H\rangle \langle n, H|0\rangle \\ \langle 0| &= \sum_n \langle 0|n, H\rangle \langle n, H| \end{aligned} \quad (16)$$

where $|n, H\rangle$ denote energy eigenstates of H with $|0, H\rangle = |\Omega\rangle$. We let $|0\rangle$ evolve in time with H

$$\begin{aligned} e^{-iH(t_\infty+t_0)} |0\rangle &= \sum_n e^{-iE_n(t_\infty+t_0)} |n, H\rangle \langle n, H|0\rangle \\ \langle 0| e^{-iH(t_\infty-t_0)} &= \sum_n e^{-iE_n(t_\infty-t_0)} \langle 0|n, H\rangle \langle n, H| \end{aligned} \quad (17)$$

where E_n denotes the energy of the state $|n, H\rangle$ with $E_n > E_{n-1}$.

Ground state projection

In order to extract $|\Omega\rangle$ out of other states, we introduce a small imaginary part into t_∞ as $t_\infty \rightarrow t_\infty(1 - i\epsilon)$, then we have

$$\begin{aligned} e^{-iH[t_\infty(1-i\epsilon)+t_0]} |0\rangle &= e^{-E_0\epsilon} \left\{ \left(e^{-iE_0(t_\infty+t_0)} \langle\Omega|0\rangle \right) |\Omega\rangle \right. \\ &\quad \left. + \sum_{n \neq 0} e^{-iE_n(t_\infty+t_0)} e^{-(E_n-E_0)t_\infty\epsilon} |n, H\rangle \langle n, H|0\rangle \right\} \\ \langle 0| e^{-iH[t_\infty(1-i\epsilon)-t_0]} &= e^{-E_0\epsilon} \left\{ \left(e^{-iE_0(t_\infty-t_0)} \langle 0|\Omega\rangle \right) \langle\Omega| \right. \\ &\quad \left. + \sum_{n \neq 0} e^{-iE_n(t_\infty-t_0)} e^{-(E_n-E_0)t_\infty\epsilon} \langle 0|n, H\rangle \langle n, H| \right\} \end{aligned} \quad (18)$$

Ground state projection

Since all higher states are suppressed relative to the ground state by a factor $e^{-(E_n-E_0)t_\infty\epsilon}$, so we have

$$\begin{aligned} e^{-iH(t_\infty+t_0)} |0\rangle &\approx \left(e^{-iE_0(t_\infty+t_0)} \langle \Omega | 0 \rangle \right) |\Omega\rangle \\ \langle 0 | e^{-iH(t_\infty-t_0)} &\approx \left(e^{-iE_0(t_\infty-t_0)} \langle 0 | \Omega \rangle \right) \langle \Omega | \end{aligned} \quad (19)$$

We can approximate using $H_0 |0\rangle = 0$ as

$$\begin{aligned} |\Omega\rangle &\approx \lim_{t_\infty \rightarrow \infty(1-i\epsilon)} \left(e^{iE_0(t_\infty+t_0)} \langle \Omega | 0 \rangle^{-1} \right) e^{-iH(t_\infty+t_0)} |0\rangle \\ &= \lim_{t_\infty \rightarrow \infty(1-i\epsilon)} \left(e^{iE_0(t_\infty+t_0)} \langle \Omega | 0 \rangle^{-1} \right) U(t_0, -t_\infty) |0\rangle \\ \langle \Omega | &\approx \lim_{t_\infty \rightarrow \infty(1-i\epsilon)} \left(e^{iE_0(t_\infty-t_0)} \langle 0 | \Omega \rangle^{-1} \right) \langle 0 | e^{-iH(t_\infty-t_0)} \\ &\approx \lim_{t_\infty \rightarrow \infty(1-i\epsilon)} \left(e^{iE_0(t_\infty-t_0)} \langle 0 | \Omega \rangle^{-1} \right) \langle 0 | U(t_\infty, t_0) \end{aligned} \quad (20)$$

Gell-Mann-Low Theorem

The normalization condition for the ground state reads

$$\begin{aligned} 1 &= \langle \Omega | \Omega \rangle \\ &= \lim_{t_\infty \rightarrow \infty (1-i\epsilon)} \left(e^{2iE_0 t_\infty} | \langle \Omega | 0 \rangle |^{-2} \right) \langle 0 | U(t_\infty, t_0) U(t_0, -t_\infty) | 0 \rangle \end{aligned} \quad (21)$$

which determines the normalization constant

$$e^{2iE_0 t_\infty} | \langle \Omega | 0 \rangle |^{-2} = \lim_{t_\infty \rightarrow \infty (1-i\epsilon)} \frac{1}{\langle 0 | U(t_\infty, -t_\infty) | 0 \rangle} \quad (22)$$

Gell-Mann-Low Theorem

Using Eqs. (7,20), Eq. (1) can be written as

$$\begin{aligned} G(x, y) &= \langle \Omega | T \phi_{(H)}(x) \phi_{(H)}(y) | \Omega \rangle \\ &= \lim_{t_\infty \rightarrow \infty (1-i\epsilon)} \frac{1}{\langle 0 | U(t_\infty, -t_\infty) | 0 \rangle} \\ &\quad \times \langle 0 | U(t_\infty, t_0) T U^\dagger(x_0, t_0) \phi_{(I)}(x) U(x_0, t_0) \\ &\quad \times U^\dagger(y_0, t_0) \phi_{(I)}(y) U(y_0, t_0) U(t_0, -t_\infty) | 0 \rangle \\ &= \lim_{t_\infty \rightarrow \infty (1-i\epsilon)} \frac{\langle 0 | T [\phi_{(I)}(x) \phi_{(I)}(y) U(t_\infty, -t_\infty)] | 0 \rangle}{\langle 0 | U(t_\infty, -t_\infty) | 0 \rangle} \end{aligned} \quad (23)$$

The numerator contains the time-ordered product of interaction-picture fields **and** the evolution operator over the full time range.

Non-equilibrium Green's function

In non-equilibrium, the two point correlation function is defined through the density operator at the initial time

$$G(x, y) = \langle T \phi_{(H)}(x) \phi_{(H)}(y) \rangle \equiv \frac{1}{\text{Tr} \rho(t_0)} \text{Tr} [\rho(t_0) T \phi_{(H)}(x) \phi_{(H)}(y)] \quad (24)$$

- The trace is taken over initial states (eigenstates of H_0)
- The interaction H_{Int} is switched on at later times
- No preferred ground state – the system is in a **mixed state**

Non-equilibrium Green's function

Using Eq. (7), we can write Eq. (24) as

$$\begin{aligned} G(x, y) &= \frac{1}{\text{Tr}\rho(t_0)} \text{Tr}[\rho(t_0) T \phi_{(H)}(x) \phi_{(H)}(y)] \\ &= \frac{1}{\text{Tr}\rho(t_0)} \text{Tr}[\rho(t_0) T U^\dagger(x_0, t_0) \phi_{(I)}(x) U(x_0, t_0) \\ &\quad \times U^\dagger(y_0, t_0) \phi_{(I)}(y) U(y_0, t_0)] \\ &= \frac{1}{\text{Tr}\rho(t_0)} \text{Tr}[\rho(t_0) T U(t_0, x_0) \phi_{(I)}(x) U(x_0, y_0) \phi_{(I)}(y) U(y_0, t_0)] \\ &\equiv \frac{1}{\text{Tr}\rho(t_0)} \text{Tr}[\rho(t_0) T_P \phi_{(I)}(x) \phi_{(I)}(y) U_{CTP}(t_0)] \end{aligned} \quad (25)$$

Evolution operator on CTP

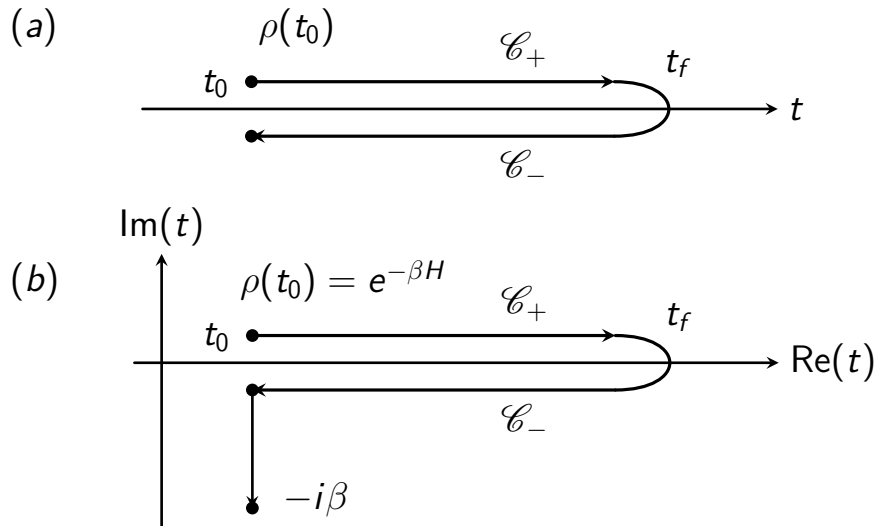
Here $U_{CTP}(t_0)$ is the unitary evolution operator defined on a closed-time path (CTP) starting from t_0 to $+\infty$ and back to t_0

$$\begin{aligned} U_{CTP}(t_0) &\equiv T_P \left[\exp \left(-i \int_{CTP} dt H_{\text{Int}}^{(I)}(t) \right) \right] \\ &= T_P \left[\exp \left(-i \int_{t_0}^{\infty} dt_+ H_{\text{Int}}^{(I)}(t_+) + i \int_{t_0}^{\infty} dt_- H_{\text{Int}}^{(I)}(t_-) \right) \right] \end{aligned} \quad (26)$$

T_P : path ordering on the CTP

- T_P is the ordering operator along the CTP contour.
- Times on the t_+ branch are always **earlier** than times on the t_- branch.
- Within each branch, standard time ordering applies:
 - On t_+ : T orders increasing time
 - On t_- : $\bar{T}$ (anti-time ordering) orders decreasing time
- This leads to four types of Green's functions: $G^{++}, G^{+-}, G^{-+}, G^{--}$ (or $G^F, G^<, G^>, G^{\bar{F}}$)

The CTP Contour



Vacuum vs. non-equilibrium: side by side

Vacuum (In-Out)

$$G = \frac{\langle 0 | T[\phi(I)\phi(I) U] | 0 \rangle}{\langle 0 | U | 0 \rangle} \quad (27)$$

- Single time contour
- $-\infty \rightarrow +\infty$
- Ground state projection
- Denominator cancels disconnected diagrams

Non-Equilibrium (CTP)

$$G = \frac{\text{Tr}[\rho(t_0) T_P \phi(I)\phi(I) U_{\text{CTP}}]}{\text{Tr}\rho(t_0)} \quad (28)$$

- Closed time contour
- $t_0 \rightarrow +\infty \rightarrow t_0$
- Density matrix initial state
- No denominator cancellation (all diagrams contribute)

Two equivalent formulations

In the Schwinger–Keldysh (CTP) formalism, two-point functions can be written in:

- ① **Matrix form** – 2×2 matrices in the contour branch space
- ② **CTP form** – single time-ordered object on the closed contour

Both are completely equivalent and related by the structure of the time contour.

Equivalently, one defines a single Green's function by time-ordering on the closed-time path:

$$G(x_1, x_2) = \langle T_C \psi_\alpha(x_1) \bar{\psi}_\beta(x_2) \rangle \quad (29)$$

where x_{10}, x_{20} lie on the CTP and T_C is the contour-ordering operator.

Connection with Matrix form

The matrix elements G^{ab} are simply the values of G when the time arguments sit on branch a and b of the contour ($a, b \in \{+, -\}$).

Matrix form

The matrix form

$$G = \begin{pmatrix} G^{++} & G^{+-} \\ G^{-+} & G^{--} \end{pmatrix} = \begin{pmatrix} G^F & G^< \\ G^> & G^{\bar{F}} \end{pmatrix}, \quad \Sigma = \begin{pmatrix} \Sigma^{++} & \Sigma^{+-} \\ \Sigma^{-+} & \Sigma^{--} \end{pmatrix} = \begin{pmatrix} \Sigma^F & \Sigma^< \\ \Sigma^> & \Sigma^{\bar{F}} \end{pmatrix} \quad (30)$$

Definitions (for fermions with Dirac indices α, β):

$$\begin{aligned} G_{\alpha\beta}^F(x_1, x_2) &= \langle T \psi_\alpha(x_1) \bar{\psi}_\beta(x_2) \rangle \\ G_{\alpha\beta}^{\bar{F}}(x_1, x_2) &= \langle T_A \psi_\alpha(x_1) \bar{\psi}_\beta(x_2) \rangle \\ G_{\alpha\beta}^<(x_1, x_2) &= -\langle \bar{\psi}_\beta(x_2) \psi_\alpha(x_1) \rangle \\ G_{\alpha\beta}^>(x_1, x_2) &= \langle \psi_\alpha(x_1) \bar{\psi}_\beta(x_2) \rangle \end{aligned} \quad (31)$$

Unitary transformation to physical basis

Define the unitary matrix

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, \quad U^{-1} = U^T = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \quad (32)$$

The **physical representation** is obtained by

$$\begin{pmatrix} 0 & G^A \\ G^R & G^C \end{pmatrix} = U \begin{pmatrix} G^F & G^{<} \\ G^{>} & G^{\bar{F}} \end{pmatrix} U^{-1} \quad (33)$$

and similarly for Σ (with sign flips on the off-diagonals):

$$\begin{pmatrix} \Sigma^A & 0 \\ \Sigma^C & \Sigma^R \end{pmatrix} = U \begin{pmatrix} \Sigma^F & -\Sigma^{<} \\ \Sigma^{>} & -\Sigma^{\bar{F}} \end{pmatrix} U^{-1} \quad (34)$$

Retarded, advanced and correlation functions

The physical components are defined as

$$\begin{aligned}O^R &= O^F - O^< = -O^{\bar{F}} + O^> \\O^A &= O^F - O^> = -O^{\bar{F}} + O^< \\O^C &= O^F + O^{\bar{F}} = O^> + O^<\end{aligned}\tag{35}$$

for $O = G$ or Σ . Explicitly:

$$\begin{aligned}O^R(x_1, x_2) &= \theta(t_1 - t_2) [O^>(x_1, x_2) - O^<(x_1, x_2)] \\O^A(x_1, x_2) &= \theta(t_2 - t_1) [O^<(x_1, x_2) - O^>(x_1, x_2)]\end{aligned}\tag{36}$$

Here O^R and O^A are causal/anti-causal; O^C is the sum of statistical correlators.

Retarded, advanced and correlation Green's functions

In the physical basis, the product $\Sigma \cdot G$ becomes block-structured:

$$U^{-1} \begin{pmatrix} \Sigma^A & 0 \\ \Sigma^C & \Sigma^R \end{pmatrix} \begin{pmatrix} 0 & G^A \\ G^R & G^C \end{pmatrix} U = \begin{pmatrix} \Sigma^F & -\Sigma^< \\ \Sigma^> & -\Sigma^{\bar{F}} \end{pmatrix} \begin{pmatrix} G^F & G^< \\ G^> & G^{\bar{F}} \end{pmatrix} \quad (37)$$

The physical-basis product is

$$\begin{pmatrix} 0 & \Sigma^A G^A \\ \Sigma^R G^R & \Sigma^C G^A + \Sigma^R G^C \end{pmatrix} \quad (38)$$

Advantage

Retarded, advanced and correlation Green's functions **decouple** in the upper-left / lower-right blocks!

Two-point Green's functions in free theory

From Eq. (31) we obtain two-point Green's functions in free theory

$$\begin{aligned} G_{(0)}^{<}(x, p) &\approx - (2\pi\hbar)\theta(p_0)\delta(p^2 - m^2) u(r, \mathbf{p}) \bar{u}(s, \mathbf{p}) f_{rs}^{(0)}(x, \mathbf{p}) \\ &\quad - (2\pi\hbar)\theta(-p_0)\delta(p^2 - m^2) v(r, -\mathbf{p}) \bar{v}(s, -\mathbf{p}) \left[\delta_{sr} - \bar{f}_{sr}^{(0)}(x, -\mathbf{p}) \right] \\ G_{(0)}^{>}(x, p) &\approx (2\pi\hbar)\theta(p_0)\delta(p^2 - m^2) u(r, \mathbf{p}) \bar{u}(s, \mathbf{p}) \left[\delta_{rs} - f_{rs}^{(0)}(x, \mathbf{p}) \right] \\ &\quad + (2\pi\hbar)\theta(-p_0)\delta(p^2 - m^2) v(r, -\mathbf{p}) \bar{v}(s, -\mathbf{p}) \bar{f}_{sr}^{(0)}(x, -\mathbf{p}) \end{aligned}$$

Two-point Green's functions in free theory

Here matrix valued spin distributions (MVSD) are defined as

$$\begin{aligned} f_{rs}^{(0)}(x, \mathbf{p}) &= \int \frac{d^4 u}{(2\pi\hbar)^3 2} \delta(p \cdot u) e^{-iu \cdot x / \hbar} \left\langle a^\dagger \left(s, \mathbf{p} - \frac{1}{2} \mathbf{u} \right) a \left(r, \mathbf{p} + \frac{1}{2} \mathbf{u} \right) \right\rangle \\ f_{sr}^{(-)}(x, -\mathbf{p}) &= \int \frac{d^4 u}{(2\pi\hbar)^3 2} \delta(\bar{p} \cdot u) e^{-iu \cdot x / \hbar} \left\langle b^\dagger \left(r, -\mathbf{p} - \frac{1}{2} \mathbf{u} \right) b \left(s, -\mathbf{p} + \frac{1}{2} \mathbf{u} \right) \right\rangle \end{aligned} \quad (39)$$

where $p^\mu = (E_p, \mathbf{p})$ and $\bar{p}^\mu = (E_p, -\mathbf{p})$

Two-point Green's functions in free theory

The retarded and advanced two-point Green's functions

$$\begin{aligned} G_{(0)}^R(p) &= \frac{i\hbar(p_\mu\gamma^\mu - m)}{p^2 - m^2 + i\text{sign}(p_0)\epsilon}, \quad (\text{poles : } p_0 = \pm E_p - i\epsilon) \\ G_{(0)}^A(p) &= \frac{i\hbar(p_\mu\gamma^\mu - m)}{p^2 - m^2 + i\text{sign}(p_0)\epsilon}, \quad (\text{poles : } p_0 = \pm E_p + i\epsilon) \end{aligned} \quad (40)$$

Two-point Green's functions in free theory

The Feynman two-point Green's function is

$$\begin{aligned} G_{(0)}^F(x, p) &= G_{(0)}^R(p) + G_{(0)}^<(x, p) = G_{\text{vac}}^{(0)F}(p) + G_{\text{medium}}^{(0)}(x, p) \\ G_{\text{vac}}^{(0)F}(p) &= \frac{i\hbar(p_\mu \gamma^\mu - m)}{p^2 - m^2 + i\epsilon} \quad (\text{poles : } p_0 = \pm E_p \mp i\epsilon) \\ G_{\text{medium}}^{(0)}(x, p) &= - (2\pi\hbar)\theta(p_0)\delta(p^2 - m^2) u(r, \mathbf{p}) \bar{u}(s, \mathbf{p}) f_{rs}^{(0)}(x, \mathbf{p}) \\ &\quad + (2\pi\hbar)\theta(-p_0)\delta(p^2 - m^2) v(r, -\mathbf{p}) \bar{v}(s, -\mathbf{p}) \bar{f}_{sr}^{(0)}(x, -\mathbf{p}) \end{aligned} \quad (41)$$

Two-point Green's functions in free theory

Anti-Feynman two-point Green's function is

$$\begin{aligned} G_{(0)}^{\bar{F}}(x, p) &= -G_{(0)}^R(p) + G_{(0)}^>(x, p) = G_{\text{vac}}^{(0)\bar{F}}(p) + G_{\text{medium}}^{(0)}(x, p) \\ G_{\text{vac}}^{(0)\bar{F}}(p) &= -\frac{i\hbar(p_\mu\gamma^\mu - m)}{p^2 - m^2 - i\epsilon} \quad (\text{poles : } p_0 = \pm E_p \pm i\epsilon) \end{aligned} \quad (42)$$

Note that the medium part for Feynman and anti-Feynman Green's functions are the same.

Projectors for spinors

We can the formula for projectors (particles)

$$\begin{aligned} u(s, p) \bar{u}(s', p) &= \frac{1}{2} (p \cdot \gamma + m) \left[\delta_{s's} + \gamma_5 \gamma \cdot n^{(i)} (\tau_i)_{s's} \right] \\ &= \frac{1}{2} \left[\delta_{s's} + \gamma_5 \gamma \cdot n^{(i)} (\tau_i)_{s's} \right] (p \cdot \gamma + m) \\ &= \frac{1}{2} \delta_{s's} (p \cdot \gamma + m) - i \frac{1}{2} p_\nu n_\mu^{(i)} (\tau_i)_{s's} \gamma_5 \sigma^{\mu\nu} \\ &\quad + \frac{1}{2} m n_\mu^{(i)} (\tau_i)_{s's} (\gamma_5 \gamma^\mu) \\ &= \frac{1}{2} \delta_{s's} (p \cdot \gamma + m) + \frac{1}{2} (\tau_i)_{s's} \epsilon^{\mu\nu\rho\sigma} p_\nu n_\mu^{(i)} \sigma_{\rho\sigma} \\ &\quad + \frac{1}{2} m n_\mu^{(i)} (\tau_i)_{s's} (\gamma_5 \gamma^\mu) \end{aligned}$$

which contains scalar, vector, axial vector and tensor component.

Projectors for spinors (anti-particles)

$$\begin{aligned} v(s, p) \bar{v}(s', p) &= \frac{1}{2} (p \cdot \gamma - m) \left[\delta_{ss'} + \gamma_5 \gamma \cdot n^{(i)} (\tau_i)_{ss'} \right] \\ &= \frac{1}{2} \left[\delta_{ss'} + \gamma_5 \gamma \cdot n^{(i)} (\tau_i)_{ss'} \right] (p \cdot \gamma - m). \end{aligned} \quad (43)$$

Schematically, on the CTP contour:

$$G^{-1} = G_0^{-1} - \Sigma \quad \Rightarrow \quad G_0^{-1} G = 1 + \Sigma G, \quad G G_0^{-1} = 1 + G \Sigma \quad (44)$$

The explicit Kadanoff–Baym equation on the CTP is

$$-i(i\gamma_\mu \partial_{x_1}^\mu - m) G(x_1, x_2) = \delta_C^{(4)}(x_1 - x_2) + \int_C dx' \Sigma(x_1, x') G(x', x_2) \quad (45)$$

where δ_C is the contour delta function (sign depends on the branch).

The CTP contour integral

$$I = \int_C \Sigma(x_1, x') G(x', x_2) \quad (46)$$

where $C = [t_0^+, +\infty] + [+ \infty, t_0^-]$ is equivalent to the matrix-form integral

$$I = \int_{t_0}^{\infty} dx' \begin{pmatrix} \Sigma^F & -\Sigma^< \\ \Sigma^> & -\Sigma^{\bar{F}} \end{pmatrix} (x_1, x') \begin{pmatrix} G^F & G^< \\ G^> & G^{\bar{F}} \end{pmatrix} (x', x_2) \quad (47)$$

Another form of I_{+-}

For $t_1 \in C_+$, $t_2 \in C_-$, we have

$$\begin{aligned} I_{+-} &= \int_{t_0}^{\infty} dx' \left[\Sigma^F(x_1, x') G^<(x', x_2) - \Sigma^<(x_1, x') G^{\bar{F}}(x', x_2) \right] \\ &= \int_{t_0}^{\infty} dx' \left\{ \left[\theta(t_1 - t') \Sigma^>(x_1, x') + \theta(t' - t_1) \Sigma^<(x_1, x') \right] G^<(x', x_2) \right. \\ &\quad \left. - \Sigma^<(x_1, x') \left[\theta(t' - t_2) G^<(x', x_2) + \theta(t_2 - t') G^>(x', x_2) \right] \right\} \\ &= \int_{t_0}^{t_1} dx' \Sigma^>(x_1, x') G^<(x', x_2) + \int_{t_1}^{\infty} dx' \Sigma^<(x_1, x') G^<(x', x_2) \\ &\quad - \int_{t_2}^{\infty} dx' \Sigma^<(x_1, x') G^<(x', x_2) - \int_{t_0}^{t_2} dx' \Sigma^<(x_1, x') G^>(x', x_2) \end{aligned} \quad (48)$$

Another form of I_{+-}

(continued)

$$\begin{aligned} I_{+-} &= \int_{t_0}^{t_1} dx' [\Sigma^>(x_1, x') - \Sigma^<(x_1, x')] G^<(x', x_2) \\ &\quad - \int_{t_0}^{t_2} dx' \Sigma^<(x_1, x') [G^>(x', x_2) - G^<(x', x_2)] \\ &\quad + \left[\int_{t_1}^{\infty} - \int_{t_2}^{\infty} + \int_{t_0}^{t_1} - \int_{t_0}^{t_2} \right] \Sigma^<(x_1, x') G^<(x', x_2) \\ &= \int_{t_0}^{t_1} dx' [\Sigma^>(x_1, x') - \Sigma^<(x_1, x')] G^<(x', x_2) \\ &\quad - \int_{t_0}^{t_2} dx' \Sigma^<(x_1, x') [G^>(x', x_2) - G^<(x', x_2)] \\ &= \int_{t_0}^{\infty} dx' \left[\Sigma^R(x_1, x') G^<(x', x_2) + \Sigma^<(x_1, x') G^A(x', x_2) \right] \end{aligned}$$

The same result is obtained by evaluating the CTP contour integral using the

Result for all components

Matrix element	Equivalent CTP expression
I_{++}	$\int_{t_0}^{\infty} dx' [\Sigma^F G^F - \Sigma^< G^>]$
I_{+-}	$\int_{t_0}^{\infty} dx' [\Sigma^R G^< + \Sigma^< G^A]$
I_{-+}	$\int_{t_0}^{\infty} dx' [\Sigma^R G^> + \Sigma^> G^A]$
I_{--}	$\int_{t_0}^{\infty} dx' [\Sigma^> G^< - \Sigma^{\bar{F}} G^{\bar{F}}]$

All four components match the CTP contour integral, proving equivalence.

Wigner coordinates

Introduce relative and center-of-mass coordinates:

$$y = x_1 - x_2, \quad X = \frac{1}{2}(x_1 + x_2) \quad (49)$$

with

$$y' = x'_1 - x_2, \quad X_1 = X + \frac{1}{2}y', \quad X_2 = X - \frac{1}{2}(y - y') \quad (50)$$

At leading order in the gradient expansion (zeroth order in ∂_X):

$$I_{+-}^{(0)}(y, X) = \int_{-\infty}^{\infty} dy' [\Sigma^R(y - y', X) G^<(y', X) + \Sigma^<(y - y', X) G^A(y', X)] \quad (51)$$

Fourier transform to momentum space

Convention:

$$\tilde{A}(p) = \int d^4x e^{ip \cdot x / \hbar} A(x), \quad A(x) = \int [dp] e^{-ip \cdot x / \hbar} \tilde{A}(p) \quad (52)$$

where $[dp] \equiv d^4p / (2\pi\hbar)^4$. **Zeroth-order transform:**

$$F[I_{+-}^{(0)}] = \Sigma^R(p, X) G^<(p, X) + \Sigma^<(p, X) G^A(p, X) \quad (53)$$

First-order (gradient) correction

The next-to-leading order gives Poisson-bracket structures:

$$F[I_{+-}^{(1)}] = -\frac{i\hbar}{2}\{\Sigma^R, G^<\}_{P.B.} - \frac{i\hbar}{2}\{\Sigma^<, G^A\}_{P.B.}$$
$$\{A, B\}_{P.B.} \equiv \partial_X^\mu A \partial_\mu^p B - \partial_\mu^p A \partial_X^\mu B \quad (54)$$

Physical Meaning

These terms encode **spatial inhomogeneity** and are suppressed by $\hbar$ relative to the local collision term.

For free fermions, $G^<$ and $G^>$ satisfy the Dirac equation:

$$\begin{aligned}(i\gamma_\mu\partial_{x_1}^\mu - m) G^<(x_1, x_2) &= 0 \\ G^<(x_1, x_2) (i\gamma_\mu\overleftarrow{\partial}_{x_2}^\mu + m) &= 0\end{aligned}\tag{55}$$

In Wigner variables:

$$\partial_{x_1}^\mu = \frac{1}{2}\partial_X^\mu + \partial_y^\mu, \quad \partial_{x_2}^\mu = \frac{1}{2}\partial_X^\mu - \partial_y^\mu\tag{56}$$

Fourier-transformed kinetic terms

After Fourier transformation $\int d^4y e^{ip \cdot y}$:

$$\begin{aligned} I_1 &= \left(i\gamma_\mu \frac{1}{2} \partial_X^\mu + \gamma_\mu p^\mu - m \right) G^<(p, X) \\ I_2 &= G^<(p, X) \left(i\gamma_\mu \frac{1}{2} \overleftarrow{\partial}_X^\mu - \gamma_\mu p^\mu + m \right) \end{aligned} \quad (57)$$

These are the **kinetic (streaming) terms** that will appear on the left-hand side of the Kadanoff–Baym equation.

KB equation for $G^<$ and $G^>$

For $(x_1, x_2) \in (t_+, t_-)$:

$$\begin{aligned} & -i(i\hbar\gamma_\mu\partial_{x_1}^\mu - m) G^<(x_1, x_2) \\ &= \hbar \int_{-\infty}^{\infty} dx' [\Sigma^R G^< + \Sigma^< G^A](x_1, x'; x', x_2) \end{aligned} \quad (58)$$

For $(x_1, x_2) \in (t_-, t_+)$:

$$\begin{aligned} & -i(i\gamma_\mu\partial_{x_1}^\mu - m) G^>(x_1, x_2) \\ &= \int_{-\infty}^{\infty} dx' [\Sigma^R G^> + \Sigma^> G^A](x_1, x'; x', x_2) \end{aligned} \quad (59)$$

KB equation in momentum space

Fourier transforming and keeping terms up to $O(\hbar^2)$:

$$\begin{aligned} & \left(i\gamma_\mu \frac{1}{2} \hbar \partial_X^\mu + \gamma_\mu p^\mu - m \right) G^<(p, X) \\ &= i\hbar [\Sigma^R G^< + \Sigma^< G^A] + \frac{\hbar^2}{2} [\{\Sigma^R, G^<\}_{PB} + \{\Sigma^<, G^A\}_{PB}] \end{aligned} \quad (60)$$

where $\{A, B\}_{PB} = \partial_x A \cdot \partial_p B - \partial_p A \cdot \partial_x B$ is Poisson bracket.

Counting powers of $\hbar$

Each propagator $\sim \hbar$, each vertex $\sim \hbar^{-1}$. The collision term is $O(\hbar)$; the Poisson-bracket term is $O(\hbar^2)$.

On-shell approximation

Taking only the imaginary (on-shell) part of the spectral functions:

$$\frac{1}{k_0 - p_0 \pm i\epsilon} \approx \mp i\pi\delta(k_0 - p_0) \quad (61)$$

Retarded and advanced functions simplify to

$$O^R \approx \frac{1}{2}(O^> - O^<), \quad O^A \approx -\frac{1}{2}(O^> - O^<) \quad (62)$$

The KB equation for $G^<$ becomes

$$\begin{aligned} & \left(i\gamma_\mu \frac{1}{2}\hbar\partial_X^\mu + \gamma_\mu p^\mu - m \right) G^<(p, X) \\ &= -\frac{i\hbar}{2} [\Sigma^< G^> - \Sigma^> G^<] - \frac{\hbar^2}{4} [\{\Sigma^<, G^>\}_{PB} - \{\Sigma^>, G^<\}_{PB}] \end{aligned} \quad (63)$$

The same collision term for $G^>$

The equation for $G^>$ has the **identical** right-hand side:

$$\begin{aligned} & \left(i\gamma_\mu \frac{1}{2} \hbar \partial_X^\mu + \gamma_\mu p^\mu - m \right) G^>(p, X) \\ &= -\frac{i\hbar}{2} [\Sigma^< G^> - \Sigma^> G^<] - \frac{\hbar^2}{4} [\{\Sigma^<, G^>\}_{PB} - \{\Sigma^>, G^<\}_{PB}] \end{aligned} \quad (64)$$

Subtracting the two equations yields the quantum Boltzmann equation for the distribution function encoded in $G^< - G^>$.

Clifford decomposition

The Wigner function $G^<(x, p)$ is a 4×4 matrix in Dirac space. Expand in the 16 independent generators of the Clifford algebra:

$$\Gamma_a \in \left\{ 1, \gamma^\mu, i\gamma^5, \gamma^5\gamma^\mu, \sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu] \right\} \quad (65)$$

$$G^<(x, p) = \frac{1}{4} \left(\mathcal{F} + i\gamma^5 \mathcal{P} + \gamma^\mu \mathcal{V}_\mu + \gamma^5 \gamma^\mu \mathcal{A}_\mu + \frac{1}{2} \sigma^{\mu\nu} \mathcal{S}_{\mu\nu} \right) \quad (66)$$

- $\mathcal{F}$: scalar, $\mathcal{P}$: pseudoscalar, $\mathcal{V}_\mu$: vector, $\mathcal{A}_\mu$: axial-vector, $\mathcal{S}_{\mu\nu}$: tensor components
- Real-valued coefficient functions $\Rightarrow$ obtained by tracing $G^<$ with the corresponding generator
- $\mathcal{F} \leftrightarrow$ phase-space distribution; $\mathcal{A}_\mu \leftrightarrow$ spin-polarization distribution

Conventions: $g_{\mu\nu} = \text{diag}(+, -, -, -)$, $\epsilon^{0123} = -\epsilon_{0123} = 1$, $\mathcal{A}^\mu = \text{Tr}(\gamma^\mu \gamma^5 G^<)$.

Equations of motion: real parts

Insert the decomposition into the KB equation

$$(\gamma \cdot K - m)G^< = I_{\text{coll}}, \quad K^\mu \equiv p^\mu + \frac{i\hbar}{2}\partial_x^\mu \quad (67)$$

and take traces. Real parts are

$$\begin{aligned} p^\mu \mathcal{V}_\mu - m\mathcal{F} &= \text{ReTr}(I_{\text{coll}}), \\ m\mathcal{P} + \frac{\hbar}{2}\partial_x^\mu \mathcal{A}_\mu &= \text{ReTr}(i\gamma^5 I_{\text{coll}}), \\ p_\mu \mathcal{F} - m\mathcal{V}_\mu + \frac{\hbar}{2}\partial_x^\nu \mathcal{S}_{\mu\nu} &= \text{ReTr}(\gamma_\mu I_{\text{coll}}), \\ \frac{1}{2}\epsilon_{\mu\nu\alpha\beta} p^\nu \mathcal{S}^{\alpha\beta} + m\mathcal{A}_\mu - \frac{\hbar}{2}\partial_{x,\mu} \mathcal{P} &= \text{ReTr}(\gamma^5 \gamma_\mu I_{\text{coll}}), \\ \epsilon_{\mu\nu\alpha\beta} p^\alpha \mathcal{A}^\beta + m\mathcal{S}_{\mu\nu} - \frac{\hbar}{2}\partial_{x,[\mu} \mathcal{V}_{\nu]} &= -\text{ReTr}(\sigma_{\mu\nu} I_{\text{coll}}), \end{aligned} \quad (68)$$

Collision term (up to $\mathcal{O}(\hbar^2)$):

$$I_{\text{coll}} = -\frac{i\hbar}{2} (\Sigma^< G^> - \Sigma^> G^<) - \frac{\hbar^2}{4} (\{\Sigma^<, G^>\}_{PB} - \{\Sigma^>, G^<\}_{PB}). \quad (69)$$

Equations of motion: imaginary parts

$$\begin{aligned}\frac{\hbar}{2}\partial_x^\mu \mathcal{V}_\mu &= \text{ImTr} (I_{\text{coll}}) , \\ p^\mu \mathcal{A}_\mu &= \text{ImTr} (-i\gamma^5 I_{\text{coll}}) , \\ p^\nu \mathcal{S}_{\nu\mu} + \frac{\hbar}{2}\partial_{x,\mu} \mathcal{F} &= \text{ImTr} (\gamma_\mu I_{\text{coll}}) , \\ p_\mu \mathcal{P} + \frac{\hbar}{4}\epsilon_{\mu\nu\alpha\beta}\partial_x^\nu \mathcal{S}^{\alpha\beta} &= \text{ImTr} (\gamma^5 \gamma_\mu I_{\text{coll}}) , \\ p_{[\mu} \mathcal{V}_{\nu]} + \frac{\hbar}{2}\epsilon_{\mu\nu\alpha\beta}\partial_x^\alpha \mathcal{A}^\beta &= -\text{ImTr} (\sigma_{\mu\nu} I_{\text{coll}}) .\end{aligned}\tag{70}$$

- Without collisions: equivalent to kinetic theory for massive fermions
- Collision terms from a *different derivation* in earlier work reproduce these equations
- Written perturbatively, order by order in $\hbar$

Mass-shell and Boltzmann-type equations

Apply $\gamma \cdot K + m$ to the KB equation and combine with its Hermitian conjugate (using $\gamma^0(G^<)^\dagger \gamma^0 = G^<)$:

On-shell (Klein–Gordon type)

$$\left(p^2 - \frac{\hbar^2}{4}\partial_x^2 - m^2\right) \text{Tr}(\Gamma_a G^<) = \text{ReTr}[\Gamma_a(\gamma \cdot K + m)I_{\text{coll}}] \quad (71)$$

$$\hbar p \cdot \partial_x \text{Tr}(\Gamma_a G^<) = \text{ImTr}[\Gamma_a(\gamma \cdot K + m)I_{\text{coll}}] \quad (72)$$

Equivalent to the ten component equations of the previous slides.

Off-shell terms

Decompose into on-shell and off-shell part,

$$G^{<} = G_{\text{on}}^{<} + G_{\text{off}}^{<}$$

with $(p^2 - m^2)G_{\text{on}}^{<} \equiv 0$. The on-shell condition (71) gives

$$\text{Tr}(\Gamma_a G_{\text{off}}^{<}) = \frac{1}{p^2 - m^2} \text{ReTr}[\Gamma_a(\gamma \cdot K + m)l_{\text{coll}}] + \mathcal{O}(\hbar^2).$$

The adjoint KB equation implies (∂_x acts only on $G^{\lessgtr}$ not on $\Sigma^{\lessgtr}$, $l_{\text{coll}} \sim g^2$)

$$l_{\text{coll}} \left(\gamma \cdot p - m - \frac{i\hbar}{2} \gamma \cdot \overleftarrow{\partial}_x \right) = \mathcal{O}(\hbar^2 g^4)$$

$$\begin{aligned} l_{\text{coll}} \left(\gamma \cdot p - m - \frac{i\hbar}{2} \gamma \cdot \overleftarrow{\partial}_x \right) \left(\gamma \cdot p + m - \frac{i\hbar}{2} \gamma \cdot \overleftarrow{\partial}_x \right) &= l_{\text{coll}} \left(p^2 - m^2 - \frac{\hbar^2}{4} \overleftarrow{\partial}_x^2 - \frac{i\hbar}{2} \gamma \cdot \overleftarrow{\partial}_x \right) \\ &= \mathcal{O}(\hbar^2 g^4) \end{aligned}$$

We obtain

$$I_{\text{coll}}^{(\text{off})} = i \frac{\hbar}{p^2 - m^2} p \cdot \partial_x I_{\text{coll}} + \mathcal{O}(\hbar^2 g^4) + \mathcal{O}(\hbar^4 g^2) \quad (73)$$

where ∂_x acts only on $G^{\lessgtr}$ not on $\Sigma^{\lessgtr}$. From Eq. (72) we obtain

$$\begin{aligned} \hbar p \cdot \partial_x \text{Tr} (\Gamma_a G_{\text{on}}^{\leq}) &= \text{ImTr} [\Gamma_a (\gamma \cdot K + m) I_{\text{coll}}] - \hbar p \cdot \partial_x \text{Tr} (\Gamma_a G_{\text{off}}^{\leq}) \\ &= \text{ImTr} \left[\Gamma_a (\gamma \cdot K + m) I_{\text{coll}}^{(\text{on})} \right] \\ &\quad + \frac{\hbar}{p^2 - m^2} p \cdot \partial_x \text{ImTr} [i \Gamma_a (\gamma \cdot K + m) I_{\text{coll}}] \\ &\quad - \frac{\hbar}{p^2 - m^2} p \cdot \partial_x \text{ReTr} [\Gamma_a (\gamma \cdot K + m) I_{\text{coll}}] \\ &\quad + \mathcal{O}(\hbar^3) + \mathcal{O}(\hbar^2 g^4) + \mathcal{O}(\hbar^4 g^2) \end{aligned} \quad (74)$$

Off-shell pieces cancel in the Boltzmann equation to lowest order in g :

$$\begin{aligned} \hbar p \cdot \partial_x \text{Tr} (\Gamma_a G_{\text{on}}^<) &= \text{ImTr} \left[\Gamma_a (\gamma \cdot K + m) I_{\text{coll}}^{(\text{on})} \right] \\ &+ \mathcal{O}(\hbar^3) + \mathcal{O}(\hbar^2 g^4) + \mathcal{O}(\hbar^4 g^2) \end{aligned} \quad (75)$$

Expansion scheme

Expand every function and operator:

$$F = \sum_{n=0}^{\infty} \hbar^n F^{(n)} \quad (76)$$

- I_{coll} starts at $\mathcal{O}(\hbar)$: $I_{\text{coll}}^{(0)} = 0$
- Zeroth order: free, on-shell kinetic theory
- First order: local collision term $\rightarrow$ leading-order Boltzmann equations
- Second order: nonlocal (Poisson-bracket) collision terms

Independent components: $\mathcal{F}$ (scalar) and $\mathcal{A}_\mu$ (axial-vector); all others are expressed through them.

Zeroth order $\mathcal{O}(\hbar^0)$

With $I_{\text{coll}}^{(0)} = 0$, all component at $\mathcal{O}(\hbar^0)$ are on-shell,

$$\begin{aligned} p^\mu \mathcal{V}_\mu^{(0)} - m \mathcal{F}^{(0)} &= 0, & \mathcal{P}^{(0)} &= 0, & p^\mu \mathcal{A}_\mu^{(0)} &= 0, \\ p_\mu \mathcal{F}^{(0)} - m \mathcal{V}_\mu^{(0)} &= 0, & p^\nu \mathcal{S}_{\mu\nu}^{(0)} &= 0, & p_{[\mu} \mathcal{V}_{\nu]}^{(0)} &= 0, \end{aligned} \quad (77)$$

$$\frac{1}{2} \epsilon_{\mu\nu\alpha\beta} p^\nu \mathcal{S}^{(0)\alpha\beta} + m \mathcal{A}_\mu^{(0)} = 0, \quad \epsilon_{\mu\nu\alpha\beta} p^\alpha \mathcal{A}^{(0)\beta} + m \mathcal{S}_{\mu\nu}^{(0)} = 0. \quad (78)$$

Solution in terms of the independent components:

$$\mathcal{P}^{(0)} = 0, \quad \mathcal{V}_\mu^{(0)} = \frac{p_\mu}{m} \mathcal{F}^{(0)}, \quad \mathcal{S}_{\mu\nu}^{(0)} = -\frac{1}{m} \epsilon_{\mu\nu\alpha\beta} p^\alpha \mathcal{A}^{(0)\beta}. \quad (79)$$

All components satisfy on-shell conditions and kinetic equations.

First order $\mathcal{O}(\hbar)$: collision term

$$I_{\text{coll}}^{(1)} = -\frac{i}{2} \left[\Sigma^{<(0)} G^{>(0)} - \Sigma^{>(0)} G^{<(0)} \right] \quad (80)$$

- Only *zeroth-order* Wigner functions appear $\Rightarrow$ on-shell
- Local in space-time: collisions at a single space-time point
- Conserves energy-momentum at the four-fermion vertex

Equations of motion at this order couple gradients of $\mathcal{O}(\hbar^0)$ components to $\mathcal{O}(\hbar)$ components.

First order: Boltzmann equations for $\mathcal{F}^{(0)}$ and $\mathcal{A}^{(0)\mu}$

From $\partial_x \cdot$ Eq. (vector, real) + Eq. (scalar, imaginary):

$$p \cdot \partial_x \mathcal{F}^{(0)} = 2m \text{ImTr} \left(I_{\text{coll}}^{(1)} \right) \quad (81)$$

From $\epsilon^{\rho\sigma\mu\nu} p_\sigma \times$ Eq. (tensor, imaginary):

$$\begin{aligned} p \cdot \partial_x \mathcal{A}^{(0)\mu} &= -\epsilon^{\mu\nu\alpha\beta} p_\nu \text{ImTr} \left(\sigma_{\alpha\beta} I_{\text{coll}}^{(1)} \right) \\ &= -2p^\mu \text{ImTr} \left(\gamma_5 I_{\text{coll}}^{(1)} \right) - 2m \text{ImTr} \left(\gamma_5 \gamma^\mu I_{\text{coll}}^{(1)} \right) \end{aligned} \quad (82)$$

- On-shell property of $I_{\text{coll}}^{(1)}$ allows $p \cdot \gamma \rightarrow m$ under traces
- These drive the *leading-order* Boltzmann equations for the MVSDs

First order: mass-shell constraints

$$(p^2 - m^2) \mathcal{F}^{(1)} = 2m \text{ReTr} \left(I_{\text{coll}}^{(1)} \right),$$

$$(p^2 - m^2) \mathcal{P}^{(1)} = 0,$$

$$(p^2 - m^2) \mathcal{V}_{\mu}^{(1)} = 2p_{\mu} \text{ReTr} \left(I_{\text{coll}}^{(1)} \right),$$

$$(p^2 - m^2) \mathcal{A}_{\mu}^{(1)} = -\epsilon_{\mu\nu\alpha\beta} p^{\nu} \text{ReTr} \left(\sigma^{\alpha\beta} I_{\text{coll}}^{(1)} \right),$$

$$(p^2 - m^2) \mathcal{S}_{\mu\nu}^{(1)} = 2p_{[\mu} \text{ImTr} \left(\gamma_{\nu]} I_{\text{coll}}^{(1)} \right) + 2m \text{ReTr} \left(\sigma_{\mu\nu} I_{\text{coll}}^{(1)} \right),$$

- $\mathcal{F}^{(1)}, \mathcal{V}^{(1)}, \mathcal{A}^{(1)}, \mathcal{S}^{(1)}$ can have off-shell parts at $\mathcal{O}(\hbar)$
- $\mathcal{P}^{(1)}$ stays on-shell $\Rightarrow$ off-shell effects enter at most at $\mathcal{O}(\hbar^2)$

Other components at $\mathcal{O}(\hbar)$

Once $\mathcal{F}^{(1)}$ and $\mathcal{A}_\mu^{(1)}$ are known, We obtain all other components at order $\mathcal{O}(\hbar)$,

$$\begin{aligned}\mathcal{P}^{(1)} &= -\frac{1}{2m}\partial_x^\mu \mathcal{A}_\mu^{(0)} + \frac{1}{m}\text{Re Tr} \left(i\gamma^5 I_{\text{coll}}^{(1)} \right) \\ \mathcal{V}_\mu^{(1)} &= \frac{1}{m}p_\mu \mathcal{F}^{(1)} - \frac{1}{2m}\partial_x^\nu \mathcal{S}_{\nu\mu}^{(0)} - \frac{1}{m}\text{Re Tr} \left(\gamma_\mu I_{\text{coll}}^{(1)} \right) \\ \mathcal{S}_{\mu\nu}^{(1)} &= -\frac{1}{m}\epsilon_{\mu\nu\alpha\beta}p^\alpha \mathcal{A}^{(1)\beta} + \frac{1}{2m}\partial_{x[\mu} \mathcal{V}_{\nu]}^{(0)} - \frac{1}{m}\text{Re Tr} \left(\sigma_{\mu\nu} I_{\text{coll}}^{(1)} \right)\end{aligned}\tag{83}$$

Second order: collision terms

The second order collision terms

$$I_{\text{coll}}^{(2)} = \Delta I_{\text{coll}}^{(1)} + I_{\text{coll,PB}}^{(0)} \quad (84)$$

Local part (first-order expansion of $\Sigma \lesseqgtr G \gtrless$):

$$\Delta I_{\text{coll}}^{(1)} = -\frac{i}{2} \left[\Sigma^{<(1)} G^{>(0)} - \Sigma^{>(1)} G^{<(0)} + \Sigma^{<(0)} G^{>(1)} - \Sigma^{>(0)} G^{<(1)} \right] \quad (85)$$

Nonlocal part (Poisson brackets $\rightarrow$ collisions at different space-time points):

$$I_{\text{coll,PB}}^{(0)} = -\frac{1}{4} \left[\left\{ \Sigma^{<(0)}, G^{>(0)} \right\}_{PB} - \left\{ \Sigma^{>(0)}, G^{<(0)} \right\}_{PB} \right],$$
$$\{A, B\}_{PB} \equiv (\partial_x A) \cdot (\partial_p B) - (\partial_p A) \cdot (\partial_x B). \quad (86)$$

The nonlocal term is the seed of spin–vorticity coupling: it converts orbital into spin angular momentum.

Second order: Boltzmann equations

Evolution equations for scalar and axial vector components

$$p \cdot \partial_x \mathcal{F}^{(1)} = 2m \text{ImTr} \left(I_{\text{coll}}^{(2)} \right) + \text{ReTr} \left(\gamma \cdot \partial_x I_{\text{coll}}^{(1)} \right) \quad (87)$$

$$\begin{aligned} p \cdot \partial_x \mathcal{A}^{(1)\mu} &= -\epsilon^{\mu\nu\alpha\beta} p_\nu \text{ImTr} \left(\sigma_{\alpha\beta} I_{\text{coll}}^{(2)} \right) - \text{ReTr} \left(\gamma^5 \partial_x^\mu I_{\text{coll}}^{(1)} \right) \\ &= -2p^\mu \text{ImTr} \left(\gamma^5 I_{\text{coll}}^{(2)} \right) - 2 \text{ImTr} \left(\gamma \cdot p \gamma^5 \gamma^\mu I_{\text{coll}}^{(2)} \right) \\ &\quad - \text{ReTr} \left(\gamma^5 \partial_x^\mu I_{\text{coll}}^{(1)} \right) \end{aligned} \quad (88)$$

Consistency between the two derivations (divergence vs. momentum contraction) yields constraint equations,

$$\text{ReTr} \left(\gamma \cdot \partial_x I_{\text{coll}}^{(1)} \right) = 2 \text{ImTr} \left[(\gamma \cdot p - m) I_{\text{coll}}^{(2)} \right], \quad (89)$$

satisfied up to $\mathcal{O}(\hbar g^4)$ using the adjoint KB equation.

Decomposition: on-shell and off-shell part

In general, the Wigner function at $\mathcal{O}(\hbar)$ can be decomposed into

$$\begin{aligned} G^{\leq(1)} &= G_{(\text{on})}^{\leq(1)} + G_{(\text{off})}^{\leq(1)} \\ I_{\text{coll}}^{(2)} &= I_{\text{coll}}^{(2,\text{on})} + I_{\text{coll}}^{(2,\text{off})} \\ &= \Delta I_{\text{coll}}^{(1,\text{on})} + I_{\text{coll,PB}}^{(0,\text{on})} + \Delta I_{\text{coll}}^{(1,\text{off})} + I_{\text{coll,PB}}^{(0,\text{off})} \end{aligned} \quad (90)$$

The evolution equations (87) and (88) can be decomposed into on-shell and off-shell parts. Equation (83) can also be decomposed into on-shell and off-shell parts as

Decomposition: on-shell and off-shell part

On-shell part:

$$\begin{aligned}\mathcal{P}^{(1,\text{on})} &= -\frac{1}{2m}\partial_x^\mu \mathcal{A}_\mu^{(0)} + \frac{1}{m}\text{Re Tr} \left(i\gamma^5 I_{\text{coll}}^{(1)} \right) \\ \mathcal{V}_\mu^{(1,\text{on})} &= \frac{1}{m}p_\mu \mathcal{F}_{(\text{on})}^{(1)} - \frac{1}{2m}\partial_x^\nu \mathcal{S}_{\nu\mu}^{(0)} - \frac{1}{m}\text{Re Tr} \left(\gamma_\mu I_{\text{coll}}^{(1)} \right) \\ \mathcal{S}_{\mu\nu}^{(1,\text{on})} &= -\frac{1}{m}\epsilon_{\mu\nu\alpha\beta}p^\alpha \mathcal{A}_{(\text{on})}^{(1)\beta} + \frac{1}{2m}\partial_{x[\mu} \mathcal{V}_{\nu]}^{(0)} - \frac{1}{m}\text{Re Tr} \left(\sigma_{\mu\nu} I_{\text{coll}}^{(1)} \right)\end{aligned}\quad (91)$$

Off-shell part:

$$\begin{aligned}\mathcal{P}^{(1,\text{off})} &= 0, \quad \mathcal{V}_\mu^{(1,\text{off})} = \frac{1}{m}p_\mu \mathcal{F}_{(\text{off})}^{(1)} \\ \mathcal{S}_{\mu\nu}^{(1,\text{off})} &= -\frac{1}{m}\epsilon_{\mu\nu\alpha\beta}p^\alpha \mathcal{A}_{(\text{off})}^{(1)\beta}\end{aligned}\quad (92)$$

On-shell evolution equations

We only need to solve on-shell evolution equations. The zeroth order evolution equations

$$\begin{aligned} p \cdot \partial_x \mathcal{F}^{(0)} &= 2m \text{ImTr} \left(I_{\text{coll}}^{(1)} \right) \\ p \cdot \partial_x \mathcal{A}^{(0)\mu} &= -\epsilon^{\mu\nu\alpha\beta} p_\nu \text{ImTr} \left(\sigma_{\alpha\beta} I_{\text{coll}}^{(1)} \right) \end{aligned} \quad (93)$$

The first order evolution equations

$$\begin{aligned} p \cdot \partial_x \mathcal{F}_{(\text{on})}^{(1)} &= 2m \text{ImTr} \left(I_{\text{coll}}^{(2,\text{on})} \right) + \text{ReTr} \left(\gamma \cdot \partial_x I_{\text{coll}}^{(1)} \right) \\ p \cdot \partial_x \mathcal{A}_{(\text{on})}^{(1)\mu} &= -\epsilon^{\mu\nu\alpha\beta} p_\nu \text{ImTr} \left(\sigma_{\alpha\beta} I_{\text{coll}}^{(2,\text{on})} \right) - \text{ReTr} \left(\gamma^5 \partial_x^\mu I_{\text{coll}}^{(1)} \right) \end{aligned} \quad (94)$$

Matrix valued spin distributions

The MVSD as 2×2 matrices can be parametrized as

$$\begin{aligned} f_{rs}(x, \mathbf{p}) &= \frac{1}{2} \left[\delta_{rs} f_{(\text{unpol})}(x, \mathbf{p}) + \boldsymbol{\tau}_{rs} \cdot \mathbf{f}_{(\text{pol})}(x, \mathbf{p}) \right], \\ \bar{f}_{rs}(x, \mathbf{p}) &= \frac{1}{2} \left[\delta_{rs} \bar{f}_{(\text{unpol})}(x, \mathbf{p}) + \boldsymbol{\tau}_{rs} \cdot \bar{\mathbf{f}}_{(\text{pol})}(x, \mathbf{p}) \right], \end{aligned} \quad (95)$$

where $\boldsymbol{\tau} = (\tau_1, \tau_2, \tau_3)$ are Pauli matrices in spin space, $f_{(\text{unpol})}$ and $\mathbf{f}_{(\text{pol})}$ are unpolarized and polarized distributions for particles respectively, and $\bar{f}_{(\text{unpol})}$ and $\bar{\mathbf{f}}_{(\text{pol})}$ are those for antiparticles respectively.

Matrix valued spin distributions

They are given by

$$\begin{aligned}f_{(\text{unpol})} &= \text{Tr} f, \quad \bar{f}_{(\text{unpol})} = \text{Tr} \bar{f}, \\ \mathbf{f}_{(\text{pol})} &= (f_{(1,\text{pol})}, f_{(2,\text{pol})}, f_{(3,\text{pol})}) = (\text{Tr}(\tau_1 f), \text{Tr}(\tau_2 f), \text{Tr}(\tau_3 f)), \\ \bar{\mathbf{f}}_{(\text{pol})} &= (\bar{f}_{(1,\text{pol})}, \bar{f}_{(2,\text{pol})}, \bar{f}_{(3,\text{pol})}) = (\text{Tr}(\tau_1 \bar{f}), \text{Tr}(\tau_2 \bar{f}), \text{Tr}(\tau_3 \bar{f})).\end{aligned}\tag{96}$$

We see that polarized distributions are three-vector that gives the polarization vector.

Two-point Green's function at $\mathcal{O}(\hbar^0)$

The zeroth order (on-shell) function $G_{(0)}^<(x, p)$ can be expressed in terms of the zeroth order MVSDs as

$$G_{(0)}^<(x, p) \approx - (2\pi\hbar)\theta(p_0)\delta(p^2 - m^2) u(r, \mathbf{p}) \bar{u}(s, \mathbf{p}) f_{rs}^{(0)}(x, \mathbf{p}) \\ - (2\pi\hbar)\theta(-p_0)\delta(p^2 - m^2) v(r, -\mathbf{p}) \bar{v}(s, -\mathbf{p}) \left[\delta_{sr} - \bar{f}_{sr}^{(0)}(x, -\mathbf{p}) \right]. \quad (97)$$

Note that $G_{(0)}^>(x, p)$ can be obtained from (97) by replacing $f^{(0)} \rightarrow -(1 - f^{(0)})$ and $1 - \bar{f}^{(0)} \rightarrow -\bar{f}^{(0)}$.

Two-point Green's function at $\mathcal{O}(\hbar)$

In terms of the first order MVSDs $f_{rs}^{(1)}$ and $\bar{f}_{sr}^{(1)}$, The first order on-shell function $G_{(0)}^{<}(x, p)$ can be expressed as

$$\begin{aligned} G_{(1,\text{on})}^{<}(x, p) \approx & - (2\pi\hbar)\theta(p_0)\delta(p^2 - m^2) u(r, \mathbf{p}) \bar{u}(s, \mathbf{p}) f_{rs}^{(1)}(x, \mathbf{p}) \\ & + (2\pi\hbar)\theta(-p_0)\delta(p^2 - m^2) v(r, -\mathbf{p}) \bar{v}(s, -\mathbf{p}) \bar{f}_{sr}^{(1)}(x, -\mathbf{p}) \end{aligned} \quad (98)$$

Extracting scalar and axial vector components

We can extract the scalar and axial vector components of $G_{(0)}^{<}(x, p)$ in Eq. (97) at $\mathcal{O}(\hbar^0)$ according to the decomposition in Eq. (66)

$$\begin{aligned}\mathcal{F}_{(0)} &= -(2\pi\hbar)(2m)\delta(p^2 - m^2) \\ &\quad \times \left\{ \theta(p_0) f_{(\text{unpol})}^{(0)}(x, \mathbf{p}) - \theta(-p_0) \left[2 - \bar{f}_{(\text{unpol})}^{(0)}(x, -\mathbf{p}) \right] \right\}, \\ \mathcal{A}_{(0)}^\mu &= -(2\pi\hbar)(2m)\delta(p^2 - m^2) \\ &\quad \times \left\{ \theta(p_0) n^{(j)\mu}(\mathbf{p}) f_{(j,\text{pol})}^{(0)}(x, \mathbf{p}) + \theta(-p_0) n^{(j)\mu}(-\mathbf{p}) \bar{f}_{(j,\text{pol})}^{(0)}(x, -\mathbf{p}) \right\},\end{aligned}\quad (99)$$

where $n^{(j)\mu}(\mathbf{p})$ with $j = 1, 2, 3$ are Lorentz boost of three orthorgonal vectors $(\hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\phi}}, \hat{\mathbf{n}})$ in the particle's rest frame.

Evolution equations for MVSDs

From evolution equations (93) and (94) we can derive evolution equations for MVSDs at $\mathcal{O}(\hbar^0)$

$$\begin{aligned} p \cdot \partial_x f_{(\text{unpol})}^{(0)} &= \mathcal{C}_{\text{scalar}}(f^{(0)}) \\ p \cdot \partial_x \left[n^{(j)\mu} f_{(j,\text{pol})}^{(0)} \right] &= \mathcal{C}_{\text{av}}^\mu(f^{(0)}) \end{aligned} \quad (100)$$

and at $\mathcal{O}(\hbar^1)$

$$\begin{aligned} p \cdot \partial_x f_{(\text{unpol})}^{(1)} &= \mathcal{C}_{\text{scalar}}(f^{(0)}, \partial_x f^{(0)}, f^{(1)}) \\ p \cdot \partial_x \left[n^{(j)\mu} f_{(j,\text{pol})}^{(0)} \right] &= \mathcal{C}_{\text{av}}^\mu(f^{(0)}, \partial_x f^{(0)}, f^{(1)}) \end{aligned} \quad (101)$$