Pseudo-gauge problem from the nucleon to the hydrodynamics

Kenji Fukushima

The University of Tokyo

— 10th Int. Conf. on Chirality, Vorticity and Magnetic Fields in Quantum Matter —

Pseudo-gauge Ambiguity

Conserved charge is unique, but the density isn't:

$$\partial_\mu j^\mu = 0 \longrightarrow \frac{dQ}{dt} = \frac{d}{dt}\int d^3x\, j^0(x) = 0$$

If no surface flow-in/out

$$\partial_\mu \Delta j^\mu = 0 \quad \rightarrow \quad \partial_\mu (j^\mu + \Delta j^\mu) = 0$$

Some other conserved charge or trivial identity

Modified current density

Which corresponds to the observable?

Pseudo-gauge Ambiguity

Origin of the Pseudo-gauge Ambiguity (PGA)

For example, the current is given with some I by

$$j^{\mu}(x) = \left.\frac{\delta I}{\delta A_{\mu}(x)}\right|_{A=0}$$ **where A_{μ} is a background field**

What if we have $I' = I + \Delta I$ **where** ΔI **satisfies:**

$$\Delta I\,|_{A=0} = 0 \quad \textbf{and} \quad \left.\frac{\delta \Delta I}{\delta A_{\mu}}\right|_{A=0} \neq 0$$

Pseudo-gauge Ambiguity

Trivial example of PGA

The electromagnetic theory is given by

$$I_M = -\frac{1}{4}\int d^4x f_{\mu\nu}f^{\mu\nu} \qquad f^{\mu\nu} = \partial^\mu a^\nu - \partial^\nu a^\mu$$

A modified current is $\boxed{j'^\mu = j^\mu + \kappa\partial_\nu f^{\nu\mu}}$

appearing from **Collins-Manohar-Wise (2005)**

$$\Delta I_M = -\frac{\kappa}{2}\int d^4x \mathcal{F}_{\mu\nu}f^{\mu\nu} \qquad \mathcal{F}^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$$

Pseudo-gauge Ambiguity

Trivial example of PGA continued

However, in this case,

$$j'^{\mu} = j^{\mu} + \kappa \partial_\nu f^{\nu\mu}$$

Maxwell eq.

$$\partial_\nu f^{\nu\mu} = q j^{\mu}$$

Therefore, the modified current is rescaled as

$$j'^{\mu} = (1 + \kappa q) j^{\mu}$$

But this changes the total charge: $\int d^3 j^0 \neq \int d^3 j'^0$

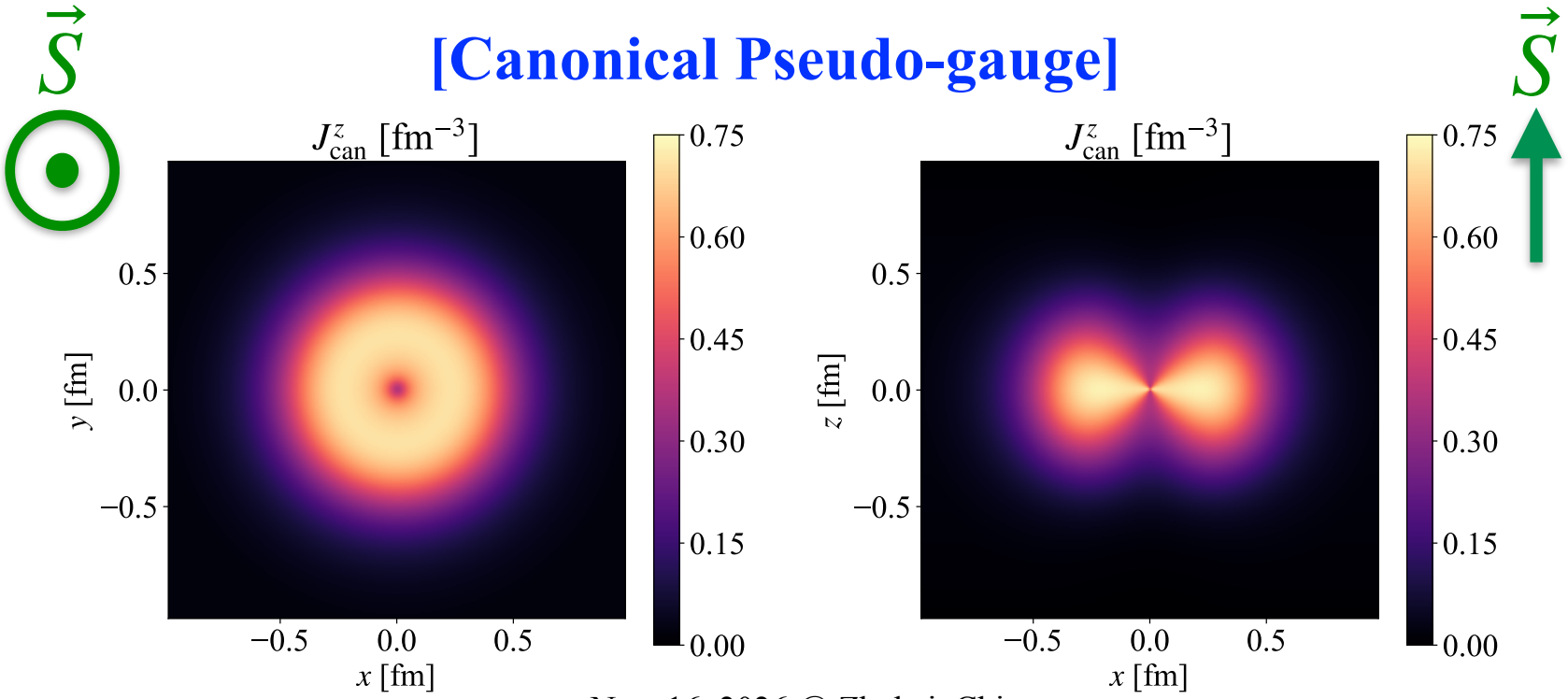
Not all PGA is problematic, but $\mathcal{J}^\mu$ (or S^μ) and $T^{\mu\nu}$ are…

How physics may be changed?

Fukushima-Uji (2026)

Distribution of the *total angular momentum* in a proton (calculated in a vectorized Skyrme model)

[Canonical Pseudo-gauge]

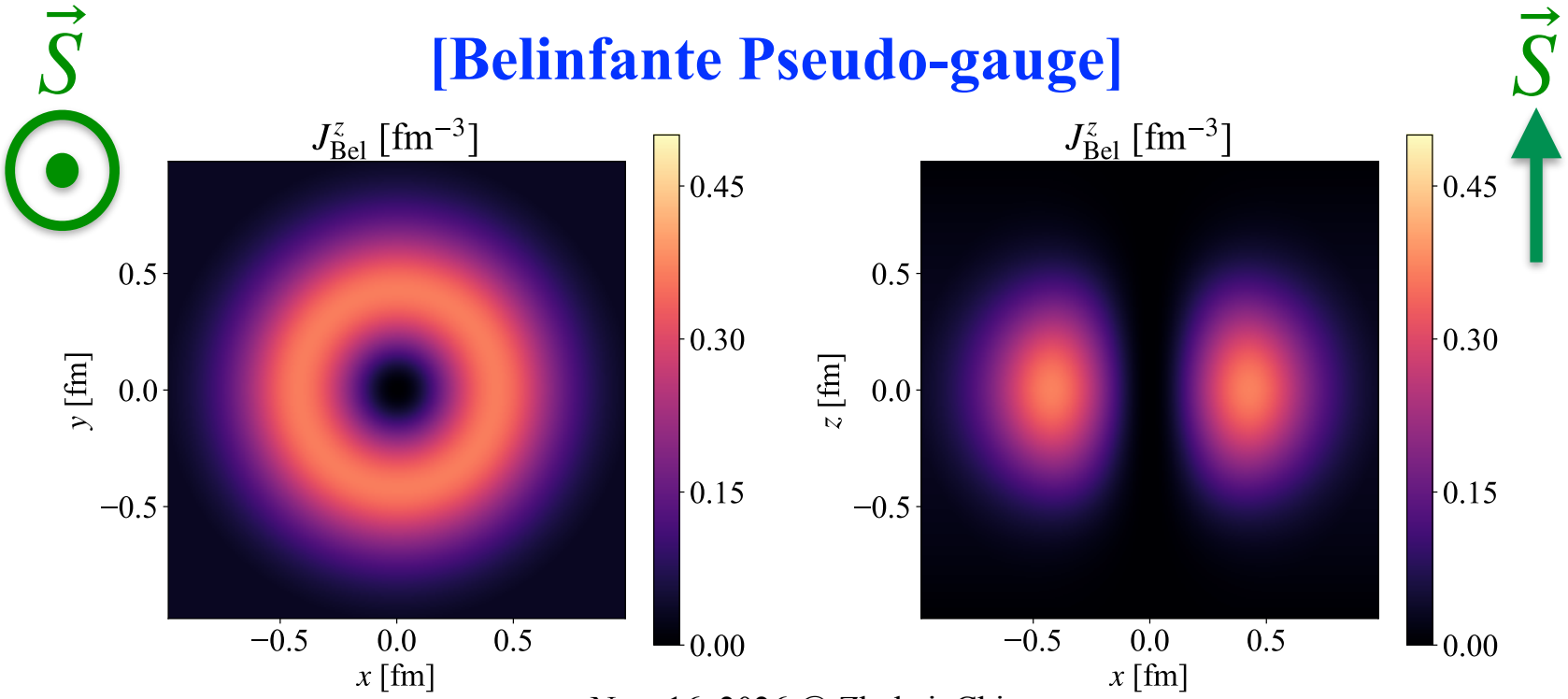


How physics may be changed?

Fukushima-Uji (2026)

Distribution of the *total angular momentum* in a proton (calculated in a vectorized Skyrme model)

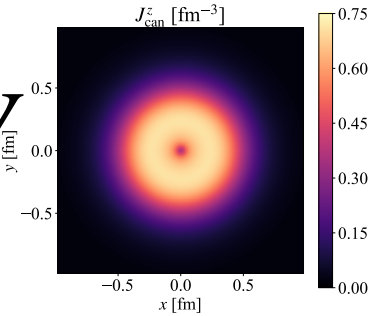
[Belinfante Pseudo-gauge]

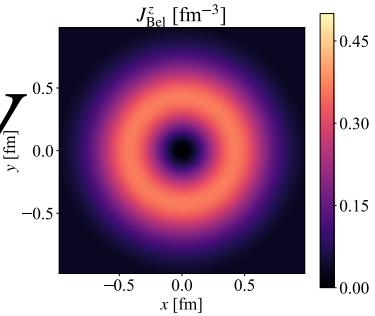


How physics may be changed?

Fukushima-Uji (2026)

$$\int dV \left[J^z_{\rm can}\ [\mathrm{fm}^{-3}] \right] = \int dV \left[J^z_{\rm Bel}\ [\mathrm{fm}^{-3}] \right] = \frac{1}{2}$$





Moments could be significantly changed.
Besides, $\mathcal{J}^z(r=0)$ (presumably pure spin) changes a lot!?

An interesting detour I:

A relation between rotation and confinement in a proton.

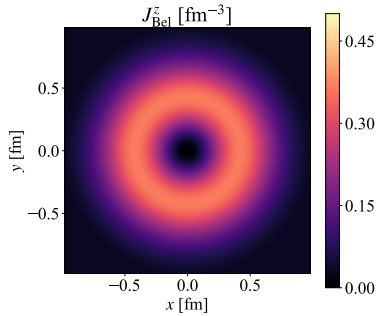
$$p_r'(r) = -\frac{2}{r} s(r)$$

Radial balance by the shear confining force.

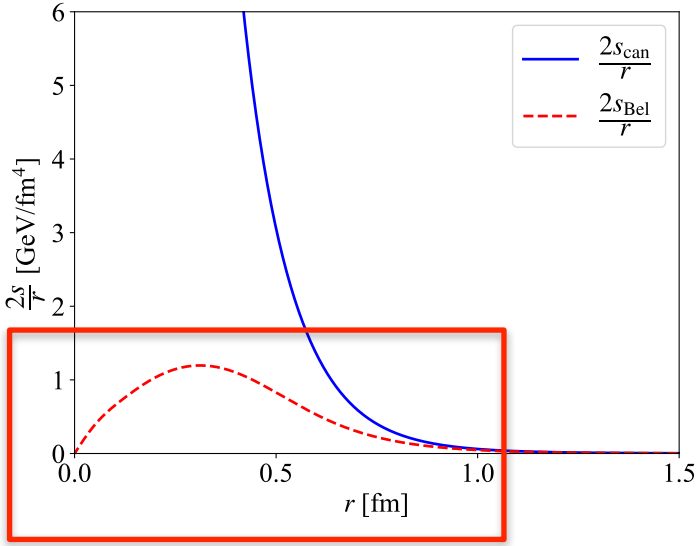
How physics may be changed?

Fukushima-Uji (2026)

Rotation distribution

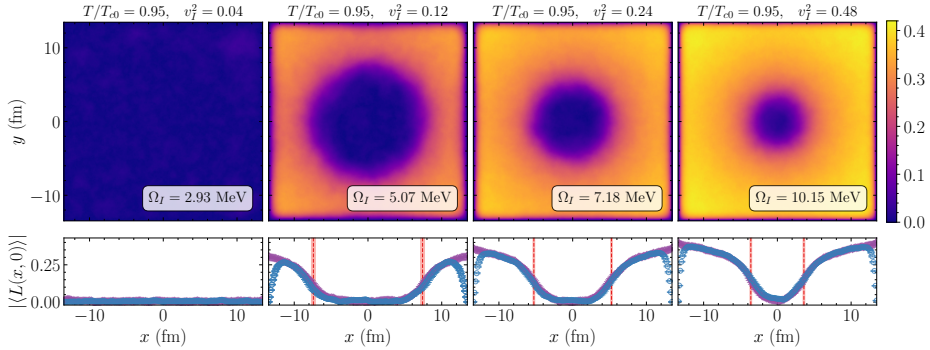


Shear force distribution



Peaked at the same region?

cf.



Braguta-Chernodub-Roenko (2023)

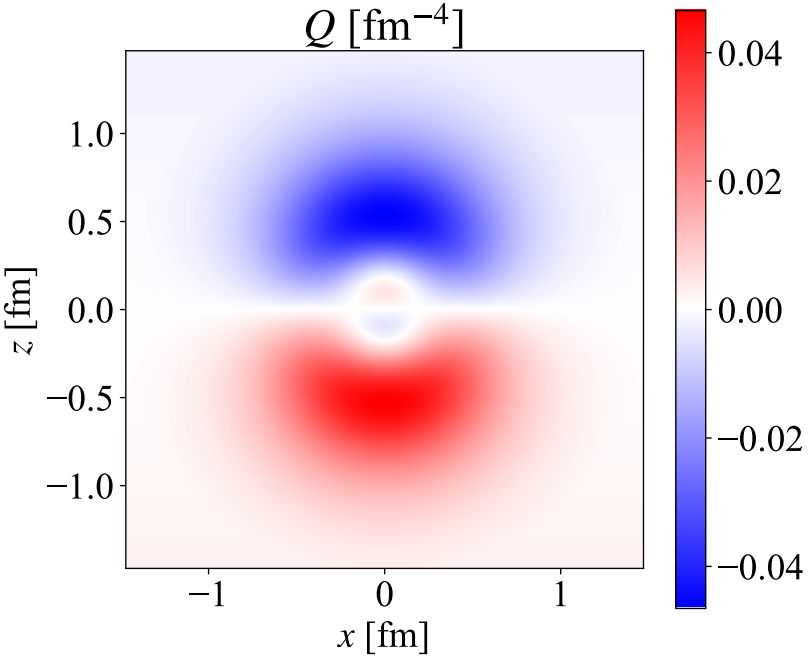
Lattice implies rotation favors confinement…

How physics may be changed?

Fukushima-Uji (2026)

An interesting detour II:

The same machinery can quantify the topological charge distribution in a proton.



Spin ~ Chern-Simons Current

Jaffe-Manohar (1990)

If QCD matter / or proton is globally polarized, the system must have a nonzero topological dipole.

Thermal Descriptions

Energy-momentum and spin tensors and PGA are introduced by external curved geometry

Remember: $j^{\mu}(x) = \left.\frac{\delta I}{\delta A_{\mu}(x)}\right|_{A=0}$ generalized into

$$T^{\mu}{}_{a} = \frac{1}{e}\frac{\delta I}{\delta e^{a}{}_{\mu}}\bigg|_{\text{flat}}, \quad S^{\mu}{}_{ab} = -\frac{2}{e}\frac{\delta I}{\delta \omega_{\mu}{}^{ab}}\bigg|_{\text{flat}}, \quad j^{\mu} = \frac{1}{e}\frac{\delta I}{\delta A_{\mu}}\bigg|_{\text{flat}}$$

$I' = I + \Delta I$ with

$$\Delta I = \int d^4x\, e\left[\frac{1}{2}\mathcal{T}_{\mu\nu}{}^{a}Y^{\mu\nu}{}_{a} + \frac{1}{4}\mathcal{R}_{\sigma\mu}{}^{ab}Z^{\sigma\mu}{}_{ab} - \frac{1}{2}\mathcal{F}_{\mu\nu}M^{\mu\nu}\right]$$

Torsion $[e, \omega]$ **Curvature $[e, \omega]$** **Field Strength $[A]$**

Thermal Descriptions

Torsion, curvature, and field strength are zero in the flat geometry limit, but their derivatives remain nonzero:

$$\Delta I = \int d^4x\, e \left[\frac{1}{2} \mathcal{T}_{\mu\nu}{}^{a} Y^{\mu\nu}{}_{a} + \frac{1}{4} \mathcal{R}_{\sigma\mu}{}^{ab} Z^{\sigma\mu}{}_{ab} - \frac{1}{2} \mathcal{F}_{\mu\nu} M^{\mu\nu} \right]$$

leading to the pseudo-gauge transformation:

$$T'^{\mu}{}_{a} = T^{\mu}{}_{a} + \partial_\nu Y^{\mu\nu}{}_{a}$$

$$S'^{\mu}{}_{ab} = S^{\mu}{}_{ab} + \eta_{a\nu} Y^{\mu\nu}{}_{b} - \eta_{b\nu} Y^{\mu\nu}{}_{a} + \partial_\sigma Z^{\sigma\mu}{}_{ab}$$

$$j'^{\mu} = j^{\mu} + \partial_\nu M^{\nu\mu}$$

Iosifidis et al. (2025)
Matthaiakakis (2026)

Becattini and Hoyos (2025) dropped Z and M for no reason.

Thermal Descriptions

However, not all is allowed, as seen for non-conserved Q.

Restricting only to Dirac QED terms, only allowed are:

$$T^{\mu}_{\alpha\, a} = T^{\mu}{}_{a} + \alpha \partial_{\nu} Y^{\mu\nu}_{\mathrm{Bel}\, a}$$

$$S^{\mu}_{\alpha\, ab} = (1-\alpha) S^{\mu}{}_{ab}$$

Interpolate the Belinfante ($\alpha = 1$) and the canonical ($\alpha \neq 1$) PGs smoothly.

Fukushima-Ono in prep.

Local Gibbs states: $\rho^* \sim e^{-\Upsilon}$

$$\Upsilon[\beta, \Omega, \zeta] = \int_{\Sigma} d\Sigma_{\mu} (\beta^{a} T^{\mu}_{a} - \frac{1}{2} \Omega^{ab} S^{\mu}_{ab} + \zeta j^{\mu}) \qquad \varpi_{ab} = -\partial_{[a} \beta_{b]}$$

α-transformation: $\Upsilon_{\alpha=0}[\beta, \Omega, \zeta] = \Upsilon_{\alpha}[\beta, \frac{\Omega - \alpha\varpi}{1-\alpha}, \zeta]$

Thermal Descriptions

Fukushima-Ono in prep.

$$\Upsilon_{\alpha=0}[\beta, \Omega, \zeta] = \Upsilon_{\alpha}[\beta, \frac{\Omega - \alpha\varpi}{1-\alpha}, \zeta]$$

In this α-family, the PG changes the multipliers, but the states are invariant. Thus, PG does not affect physics.

cf. Becattini-Hoyos (2025)

Belinfante limit ($\alpha \to 1$):

$$\Upsilon_0[\beta, \Omega, \zeta] - \Upsilon_1[\beta, \zeta] = -\frac{1}{2}\int_\Sigma d\Sigma_\mu (\Omega^{ab} - \varpi^{ab}) S^\mu{}_{ab}$$

Speranza-Weickgenannt (2020)

Thermal Descriptions

Belinfante limit ($\alpha \to 1$):

$$\Upsilon_0[\beta, \Omega, \zeta] - \Upsilon_1[\beta, \zeta] = -\frac{1}{2}\int_\Sigma d\Sigma_\mu (\Omega^{ab} - \varpi^{ab}) S^\mu{}_{ab}$$

Speranza-Weickgenannt (2020)

This is a very singular limit because the spin drops off.

If we impose the invariance $\to$ $\boxed{\Omega = \varpi}$ **(global equil.)**

required not from the Killing condition

Therefore, only off-equilibrium allows for changing physics with different PG choices (unless the Belinfante limit is only discontinuously connected).

Thermal Descriptions

Complex scalar theory

Unlike Dirac QED, the thermal shear ξ appears:

$$T'^{0}{}_{0} = T^{0}{}_{0} + c\,\nabla^2(\phi^\dagger\phi)$$

$$T'^{0}{}_{i} = T^{0}{}_{i} + c\partial_i\partial^0(\phi^\dagger\phi)$$

$$S'^{0}{}_{0i} = -(c + 2\xi)\partial_i(\phi^\dagger\phi)$$

$$S'^{0}{}_{ij} = 0$$

In this case, the state invariance requires the Killing condition…

Summary

- **Geometrical interpretation of the pseudo-gauge ambiguity has developed recently**
  - □ Derivative of "zero" may not be zero.
  - □ Curved spacetime should be a better starting point.
- **IF the operators are selected by discrete symmetries and renormalizable dimensions…**
  - □ Dirac QED → PGA is gone except the Belinfante limit.
  - □ Global equilibrium condition from the Belinfante limit.
  - □ PG inv. in complex scalar needs the Killing cond.
- **Phenomenological implications?**