Thermodynamics of Dirac fermions at finite spin density in a boost-invariant fluid

Based on

JHEP 07 (2026) 180

In collaboration with D. Roselli

Andrea Palermo

INSTITUT DENIS POISSON
ORLÉANS • TOURS
Mathématiques & Physique Théorique





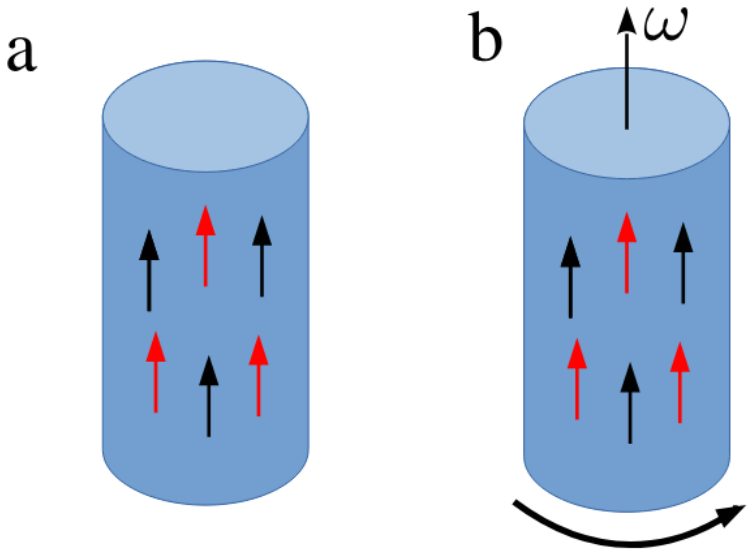
Funded by the European Union

Spin hydrodynamics

How to solve the polarization sign puzzle?

Maybe we should include the spin tensor and a spin potential in hydrodynamics?

$$\hat{\mathcal{J}}^{\lambda,\mu\nu} = x^{\mu}\hat{T}^{\lambda\nu} - x^{\nu}\hat{T}^{\lambda\mu} + \hat{\mathcal{S}}^{\lambda,\mu\nu}$$



Becattini, Florkowski, Speranza Phys.Lett.B 789 (2019) 419-425

Spin hydrodynamics:

$$\partial_{\mu} T^{\mu\nu} = 0$$
$$\partial_{\mu} \mathcal{S}^{\mu,\rho\sigma} = T^{\sigma\rho} - T^{\rho\sigma}$$

Talks at this conference:

Rischke – Tue 10:30	Panday – Wed 16:50
Florkowski – Fri 8:30	Singh – Thu 11:30

Many more contributors in the audience! Sheng, Yang, Fukushima, Hattori...

A new source of polarization was found: the **shear-induced polarization**

S.Liu & Y.Yin, JHEP 07 (2021) 188; F.Becattini, M.Buzzegoli & A.P., Phys.Lett.B 820 (2021) 136519; Fu, Liu, Pang, Song & Yin Phys.Rev.Lett. 127 (2021) 14, 142301

Can solve the polarization sign puzzle (assuming isothermal freeze-out)!

$$S^\mu_\xi(p) = -\frac{1}{8m}\epsilon^{\mu\nu\rho\sigma}p_\sigma \frac{\int \mathrm{d}\Sigma\cdot p\, n_F(1-n_F)2\hat{t}_\nu \xi_{\lambda\rho}\frac{p^\lambda}{\varepsilon}}{\int \mathrm{d}\Sigma\cdot p\, n_F}$$

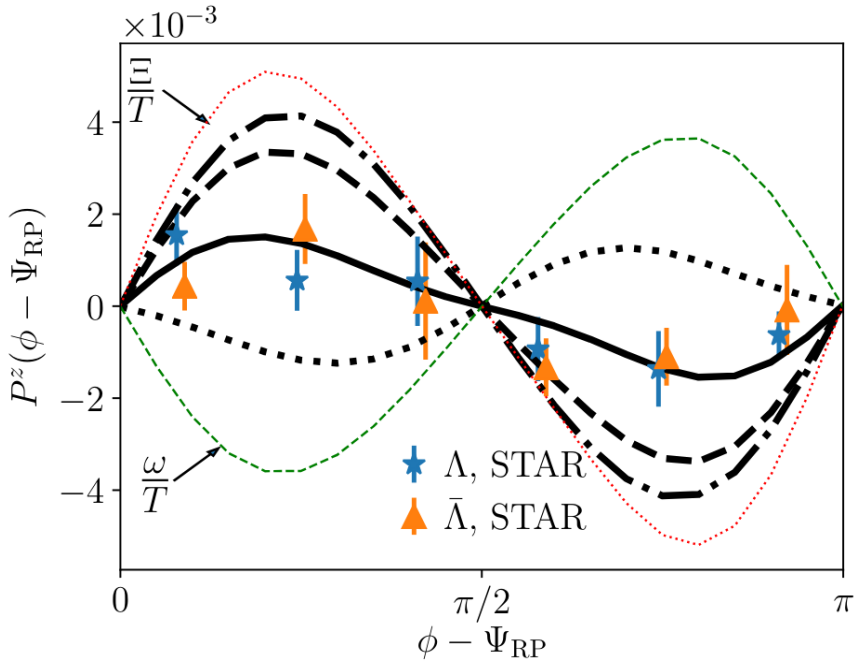
$$\xi_{\mu\nu} = \frac{1}{2}\left(\partial_\mu\beta_\nu + \partial_\nu\beta_\mu\right)$$

The isothermal approximation seems to be rather good even in low energy collisions

Palermo, Shokri Phys.Rev.D 112 (2025) 7, 076011

Further improvements recently!

X.-L. Sheng, F. Becattini and D. Roselli, Phys.Rev.C 114 (2026) 3, 3



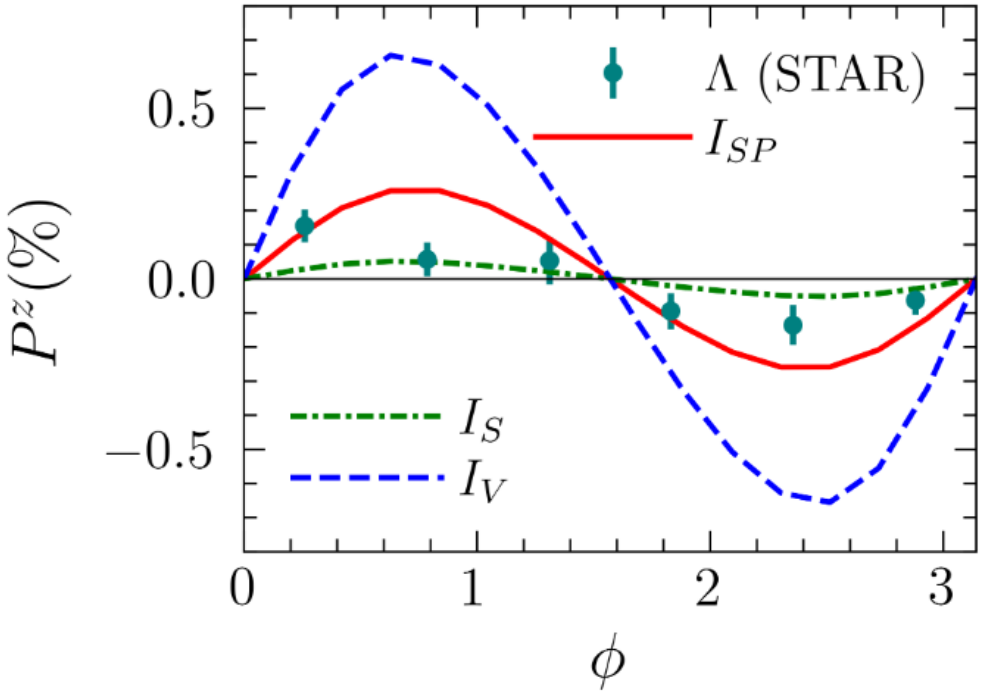
Becattini, Buzzegoli, Inghirami, Karpenko, Palermo, Phys.Rev.Lett. 127 (2021) 27, 272302

In the meanwhile, also spin hydrodynamics achieved a satisfactory agreement with the data

Dissipative spin hydrodynamics with the method of moments. Transport coefficients for spin are independent on coupling constants

Sapna, S.K. Singh, D.Wagner, Phys.Rev.C 112 (2025) 5, 054902

Different types of interactions are considered: scalar, scalar+pseudoscalar, vector couplings



Wagner – Wed 10:55

It’s not very clear how to **test the existence of a spin potential** in heavy ion collisions.

Thermodynamic relations are modified by the presence of the spin potential, but there is an ongoing debate about their appropriate form…

Becattini, Daher, Sheng, Phys.Lett.B 850 (2024) 138533; Becattini, Singh, Eur.Phys.J.C 85 (2025) 11, 1338; Ambrus, Gecic Phys.Lett.B 881 (2026) 140953; Armas, Jain 2601.14421

$$\mathcal{P} = TV^{-1}\log Z\ , \quad \mathcal{S} = \frac{\partial \mathcal{P}}{\partial \Omega}\ , \quad \mathcal{E} = -\mathcal{P} + \Omega\frac{\partial \mathcal{P}}{\partial \Omega} + T\frac{\partial \mathcal{P}}{\partial T}$$

Our goal: Compute exactly expectation values for a phenomenologically motivated yet simplified setup: a longitudinally boost-invariant fluid of Dirac fermions.

Results

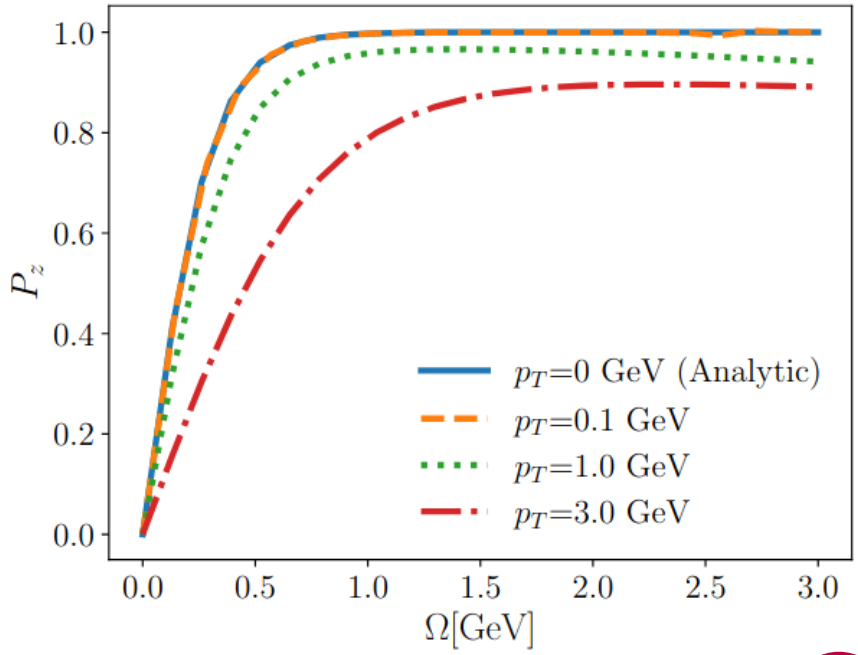
Exact expectation values out of equilibrium. We check that **thermodynamic relations hold in this specific setup**

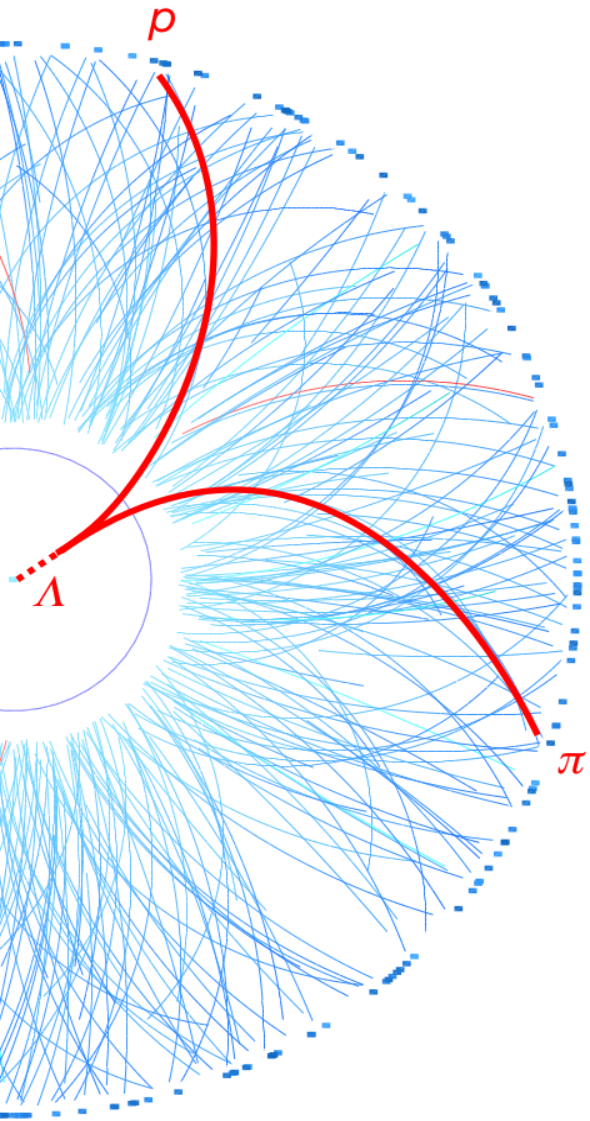
$$\log Z = \frac{V}{(2\pi)^3 \tau} \int \mathrm{d}^3\mathfrak{p} \sum_{s=\pm} \left[\log\left(1 + e^{-\frac{\omega_s}{T}+\zeta}\right) + \log\left(1 + e^{-\frac{\omega_s}{T}-\zeta}\right) \right]$$

No shear induced polarization!

Exact spin polarization for any spin potential

Check our paper!





Outline

- Non equilibrium density operator formalism
- Boost invariance
- Diagonalization of the density operator
- Results and conclusion

Density operator

Given a space-time foliation in hypersurfaces $\Sigma(\tau)$, the entropy $S = -\mathrm{Tr}\left(\widehat{\rho}\log\widehat{\rho}\right)$ is maximized constraining the mean values of the operators to be the actual ones

Becattini, Buzzegoli, Grossi Particles2(2019)2,197-207

$$n_\mu \langle \widehat{T}^{\mu\nu} \rangle = n_\mu \mathrm{Tr}\left(\widehat{\rho}\widehat{T}^{\mu\nu}\right) = n_\mu T^{\mu\nu}{}_{\mathrm{true}}$$

$$\widehat{\mathcal{J}}^{\lambda,\mu\nu} = x^\mu \widehat{T}^{\lambda\nu} - x^\nu \widehat{T}^{\lambda\mu} + \widehat{\mathcal{S}}^{\lambda,\mu\nu}$$

$$n_\mu \langle \widehat{\mathcal{S}}^{\mu,\nu\rho} \rangle = n_\mu \mathrm{Tr}\left(\widehat{\rho}\widehat{\mathcal{S}}^{\mu,\nu\rho}\right) = n_\mu \mathcal{S}^{\mu,\nu\rho}_{\mathrm{true}}$$

"$\mathbf{n}$" is the normal vector to the hypersurface $\Sigma(\tau)$. The density operator reads:

$$\widehat{\rho} = \frac{1}{Z}\exp\left(-\int_{\Sigma(\tau)} \mathrm{d}\Sigma n_\mu \widehat{T}^{\mu\nu}\beta_\nu - n_\mu \widehat{\mathcal{S}}^{\mu,\nu\rho}\mathfrak{S}_{\nu\rho}\right)$$

$$Z = \mathrm{Tr}\left[\exp\left(-\int_{\Sigma(\tau)} \mathrm{d}\Sigma n_\mu \widehat{T}^{\mu\nu}\beta_\nu - n_\mu \widehat{\mathcal{S}}^{\mu,\nu\rho}\mathfrak{S}_{\nu\rho}\right)\right]$$

Boost invariant density operator

In general, local equilibrium expectation values cannot be computed exactly, unless there is some symmetry. We impose the density operator to be invariant under longitudinal boosts:

$$\widehat{\rho} = \frac{1}{Z} \exp\left(- \int_{\Sigma(\tau)} \mathrm{d}\Sigma n_\mu \widehat{T}^{\mu\nu} \beta_\nu - n_\mu \widehat{\mathcal{S}}^{\mu,\nu\rho} \mathfrak{S}_{\nu\rho} \right)$$

$$U(\Lambda_z) \widehat{\rho}[\Sigma, \beta, \mathfrak{S}] U(\Lambda_z)^\dagger = \widehat{\rho}[\Sigma, \beta, \mathfrak{S}]$$

Longitudinal boost invariance implies:

$$\Sigma : \ \tau \equiv \sqrt{t^2 - z^2} = \text{const.} \,, \qquad \beta^\mu = \frac{u^\mu}{T(\tau)} \,, \qquad u^\mu = \frac{1}{\tau}(t, 0, 0, z) \,,$$

$$\mathfrak{S}_{\rho\sigma} = \frac{1}{2} \mathfrak{S}(\tau) \left(\delta^1_\rho \delta^2_\sigma - \delta^2_\rho \delta^1_\sigma \right)$$

It is most convenient to switch to Milne coordinates $\mathrm{d}s^2 = \mathrm{d}\tau^2 - \mathrm{d}\mathbf{x}_\mathrm{T}^2 - \tau^2\mathrm{d}\eta^2$

$$t = \tau\cosh\eta\ , \qquad z = \tau\sinh\eta$$

The density operator reads $\mathfrak{S} = \dfrac{\Omega}{T}$

$$\widehat{\rho}_{\mathrm{LE}}(\tau) = \frac{1}{Z_{\mathrm{LE}}(\tau)} e^{-\frac{\Pi(\tau)}{T(\tau)}} = \frac{1}{Z_{\mathrm{LE}}(\tau)} \exp\left\{-\frac{1}{T(\tau)}\int \mathrm{d}^2\mathrm{x}_\mathrm{T}\mathrm{d}\eta\tau\ \left[\widehat{T}^{\tau\tau} - \Omega(\tau)\widehat{S}^{\tau,xy}\right]\right\}$$

Expectation values are also constrained

$$\langle\widehat{T}^{\mu\nu}(x)\rangle = \mathcal{E}(\tau)\hat{\tau}^\mu\hat{\tau}^\nu + \mathcal{P}_\mathrm{T}(\tau)\left(\hat{x}^\mu\hat{x}^\nu + \hat{y}^\mu\hat{y}^\nu\right) + \mathcal{P}_\mathrm{L}(\tau)\hat{\eta}^\mu\hat{\eta}^\nu + \frac{\mathcal{T}(\tau)}{2}\left(\hat{y}^\mu\hat{x}^\nu - \hat{x}^\mu\hat{y}^\nu\right)$$

$$\langle\widehat{S}^{\mu,\lambda\nu}(x)\rangle = \epsilon^{\mu\lambda\nu\rho}\mathcal{S}(\tau)\hat{\eta}_\rho$$

$$\frac{1}{\tau}\frac{\partial}{\partial\tau}\left(\tau\mathcal{S}(\tau)\right) = \mathcal{T}(\tau)$$

Free Dirac field in Milne coordinates

Dirac equation in Milne space

$$\left[i\gamma^{\underline{0}}\,\partial_\tau + i\boldsymbol{\gamma}_{\mathrm{T}}\cdot\boldsymbol{\nabla}_{\mathrm{T}} + \frac{i\gamma^{\underline{3}}}{\tau}\left(\partial_\eta - \frac{\gamma^{\underline{3}}\gamma^{\underline{0}}}{2}\right) - m\right]\psi\left(\tau, \mathbf{x}_{\mathrm{T}}, \eta\right) = 0$$

To solve it, we use the ansatz

Akkelin, Phys.Rev.D 103 (2021) 11, 116014

$$\psi(\tau, \mathbf{x}_{\mathrm{T}}, \eta) \equiv e^{i\eta\Sigma^{\underline{03}}}\widetilde{\psi}(\tau, \mathbf{x}_{\mathrm{T}}, \eta) \qquad \widetilde{\psi}^{(+)} \propto \int \mathrm{d}^3p\, e^{-ip\cdot x}u(p)$$

$$(\not{p} - m)u(p) = 0$$

The solution of the Dirac equation can be written $\mathfrak{p} = (\boldsymbol{p_T}, \mu)$

$$\psi\left(\tau, \mathbf{x}_{\mathrm{T}}, \eta\right) = \frac{1}{(2\pi)^{3/2}} \sum_{r=\pm} \int \mathrm{d}^3\mathfrak{p} \left[e^{i\mathfrak{p}\cdot\mathfrak{x}} U_r(\tau, \mathfrak{p}) \widehat{A}_r(\mathfrak{p}) + e^{-i\mathfrak{p}\cdot\mathfrak{x}} V_r(\tau, \mathfrak{p}) \widehat{B}_r^{\dagger}(\mathfrak{p}) \right]$$

The spinors involve matrix valued Hankel functions $\nu = \frac{1}{2}\gamma^{\underline{3}}\gamma^{\underline{0}} - i\mu\mathbb{I}$

$$U_r\left(\tau, \mathfrak{p}\right) = -\frac{i\sqrt{\pi}}{2} e^{-i\pi\nu/2} \mathrm{H}_{\nu}^{(2)}(m_{\mathrm{T}}\tau)\, u_r(\mathbf{p}_{\mathrm{T}})$$

$$u_r(\mathbf{p}_{\mathrm{T}}) = \frac{m + m_{\mathrm{T}}\gamma^{\underline{0}} + \mathbf{p}_{\mathrm{T}} \cdot \boldsymbol{\gamma}_{\mathrm{T}}}{\sqrt{2\left(m_{\mathrm{T}} + m\right)}} \xi_r \qquad \mathrm{H}_{\nu}^{(1,2)}(z) \equiv \begin{pmatrix} \mathrm{H}^{(1,2)}_{\frac{1}{2}-i\mu}(z) & 0 & 0 & 0 \\ 0 & \mathrm{H}^{(1,2)}_{-\frac{1}{2}-i\mu}(z) & 0 & 0 \\ 0 & 0 & \mathrm{H}^{(1,2)}_{-\frac{1}{2}-i\mu}(z) & 0 \\ 0 & 0 & 0 & \mathrm{H}^{(1,2)}_{\frac{1}{2}-i\mu}(z) \end{pmatrix}$$

We must make a choice about the energy momentum and spin tensors. We will focus on two possible choices.

Canonical pseudogauge

Buzzegoli, AP, Phys.Rev.Lett. 133 (2024) 26, 262301
Dey, Flowrkowski, Jaiswal, Ryblewski, Phys.Lett.B 843 (2023) 137994

$$\widehat{T}^{\mu\nu}_{\rm C}(x) = \frac{i}{2}\bar{\psi}(x)\gamma^{\mu}(x)D^{\nu}\psi(x) - \frac{i}{2}D^{\nu}\bar{\psi}(x)\gamma^{\mu}(x)\psi(x)$$

$$\widehat{\mathcal{S}}^{\mu,\nu\lambda}_{\rm C}(x) = \frac{1}{2}\epsilon^{\rho\mu\nu\lambda}(x)\bar{\psi}(x)\gamma_{\rho}(x)\gamma^{5}\psi(x)$$

Belinfante pseudogauge

Becattini, Hoyos,Phys.Rev.Lett. 137 (2026) 4, 041602

$$\widehat{T}^{\mu\nu}_{\rm B}(x) = \frac{\widehat{T}^{\mu\nu}_{C} + \widehat{T}^{\nu\mu}_{C}}{2}$$

$$\widehat{\mathcal{S}}^{\mu,\nu\lambda}_{\rm B}(x) = 0$$

$$\widehat{\Pi}_C = \tau \int \mathrm{d}^3\mathfrak{r}\, \left\{ \mathrm{Im}\left[\dot{\psi}^\dagger(\tau,\mathfrak{r})\, \psi(\tau,\mathfrak{r}) \right] - i\Omega(\tau) \psi^\dagger(\tau,\mathfrak{r})\, \gamma^{\underline{1}}\gamma^{\underline{2}} \psi(\tau,\mathfrak{r}) \right\}$$

Introducing $\Phi^T(\mathfrak{p}) \equiv \left(\widehat{A}_+(\mathfrak{p}), \widehat{A}_-(\mathfrak{p}), \widehat{B}^\dagger_+(-\mathfrak{p}), \widehat{B}^\dagger_-(-\mathfrak{p}) \right)$

$$\widehat{\Pi}_C(\tau) \equiv \int \mathrm{d}^3\mathfrak{p}\, \tau\, \Phi^\dagger(\mathfrak{p}) \mathcal{H}_{\mathrm{tot}}(\tau,\mathfrak{p}) \Phi(\mathfrak{p})$$

With $\mathcal{H}_{tot} = \mathcal{H} - \dfrac{\tau\Omega}{2}\mathcal{S}$

$$\mathcal{H} = m_{\mathrm{T}}^2 \tau \begin{pmatrix} \mathfrak{h}(\tau,\mathfrak{p})\, \mathbb{I}_{2\times 2} & \mathrm{j}^*(\tau,\mathfrak{p})\, \sigma^x \\ \mathrm{j}(\tau,\mathfrak{p})\, \sigma^x & -\mathfrak{h}(\tau,\mathfrak{p})\, \mathbb{I}_{2\times 2} \end{pmatrix}$$

$$\mathcal{S} = \begin{pmatrix} \frac{m}{m_{\mathrm{T}}\tau}\sigma^z - \mathrm{s}(\tau,\mathfrak{p})\,(p^x\sigma^x + p^y\sigma^y) & \mathrm{t}^*(\tau,\mathfrak{p})\,(p^y\sigma^z + ip^x\mathbb{I}_{2\times 2}) \\ \mathrm{t}(\tau,\mathfrak{p})\,(p^y\sigma^z - ip^x\mathbb{I}_{2\times 2}) & -\frac{m}{m_{\mathrm{T}}\tau}\sigma^z + \mathrm{s}(\tau,\mathfrak{p})\,(p^x\sigma^x - p^y\sigma^y) \end{pmatrix}$$

We need to diagonalize this matrix with an appropriate Bogoliubov transformation

Belinfante pseudogauge

The Bogolyubov transformation can be found analytically. Expectation values are the same as in classical relativistic kinetic theory

$$\mathcal{E}^{(ren,\tau)}_{\mathrm{LE}}(\tau) = \frac{2}{(2\pi)^3\,\tau}\int \mathrm{d}p_x \mathrm{d}p_y \mathrm{d}\mu \sqrt{m_{\mathrm{T}}^2 + \frac{\mu^2}{\tau^2}}\left[n^-_{\mathrm{F}}(\varepsilon(\tau)) + n^+_{\mathrm{F}}(\varepsilon(\tau))\right]$$

To study spin polarization, we compute the *spin density matrix*

Becattini, Lect.Notes Phys. 987 (2021) 15-52

$$\Theta_{rs}(p) = \frac{\langle \widehat{a}^\dagger_s(p)\widehat{a}_r(p)\rangle}{\sum_t \langle \widehat{a}^\dagger_t(p)\widehat{a}_t(p)\rangle} = \frac{\int_{-\infty}^{\infty}\mathrm{d}\mu\,\langle \widehat{A}^\dagger_s(\mathfrak{p})\widehat{A}_r(\mathfrak{p}')\rangle_{\mathrm{LE}}}{\int_{-\infty}^{\infty}\mathrm{d}\mu\,\mathrm{tr}(\langle \widehat{A}^\dagger(\mathfrak{p})\widehat{A}(\mathfrak{p}')\rangle_{\mathrm{LE}})}$$

In the Belinfante pseudogauge, we find: $\Theta_{rs} = \frac{1}{2}\delta_{rs}$

No shear-induced polarization!

consistent with Liu, Yin JHEP 07(2021) 188, X.-L. Sheng, F. Becattini and D. Roselli, Phys.Rev.C 114 (2026) 3, 3

Canonical pseudogauge

Eigenvalue degeneracy is broken by spin

$$\omega_s^2(\tau,\mathfrak{p}) \equiv \varepsilon^2(\tau,\mathfrak{p}) + \frac{\Omega^2(\tau)}{4} + s|\Omega(\tau)|m_{\rm L}(\mu)\ ; \quad s=\pm 1$$

This is the **same spectum as found in mean field interacting theory**, as expected

Kiefer, Dash, Rischke, Phys.Rev.D 113 (2026) 9, 094032
Buzzegoli, AP, Phys.Rev.Lett. 133 (2024) 26, 262301

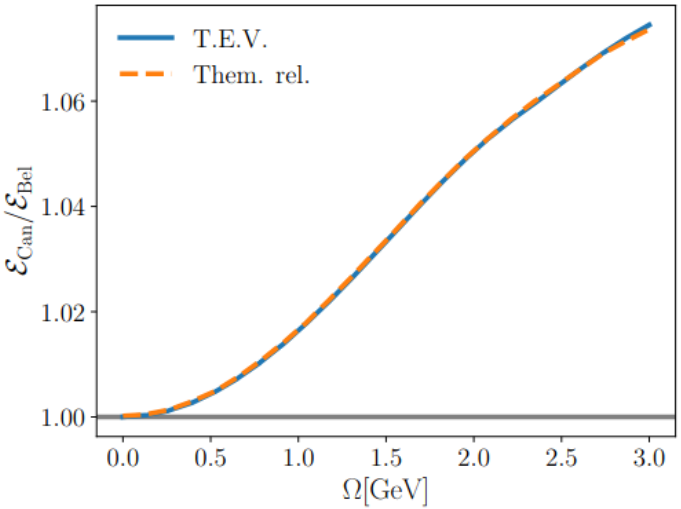
The density operator is diagonalized, and the partition function can be computed

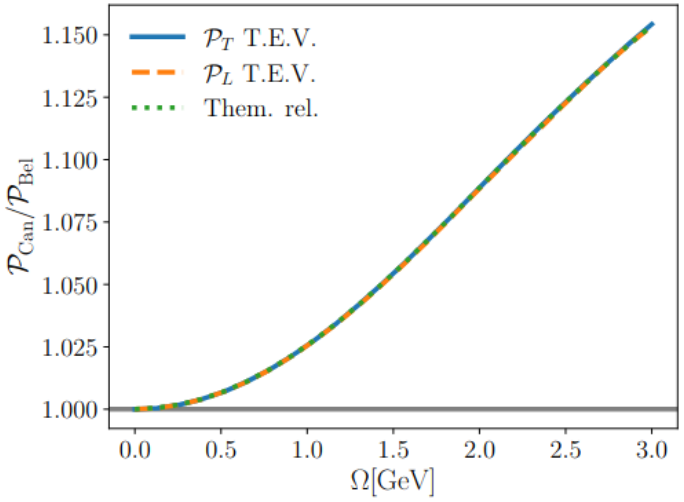
$$\log Z = \frac{V}{(2\pi)^3\tau}\int \mathrm{d}^3\mathfrak{p}\sum_{s=\pm}\left[\log\left(1+e^{-\frac{\omega_s}{T}+\zeta}\right)+\log\left(1+e^{-\frac{\omega_s}{T}-\zeta}\right)\right]$$

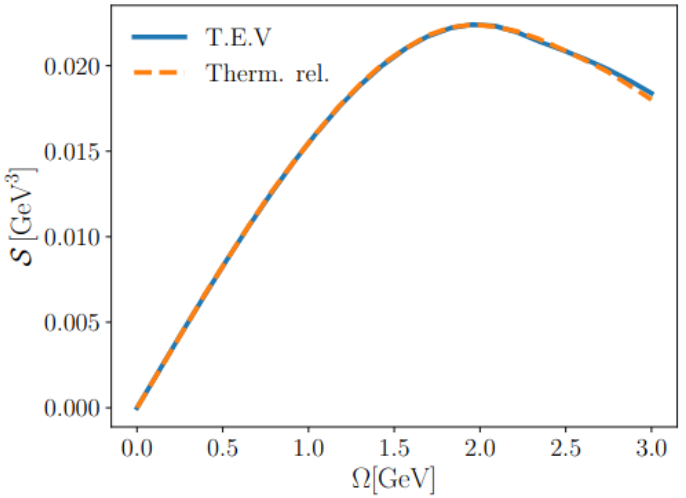
We can compute expectation values using field theory methods, and compare them with thermodynamic relations with spin potential.

$$\mathcal{P}_L = \langle \widehat{T}^{\eta\eta} \rangle, \qquad \mathcal{P}_T = \frac{\langle \widehat{T}^{xx} \rangle + \langle \widehat{T}^{yy} \rangle}{2}, \qquad \mathcal{S} = \langle \widehat{\mathcal{S}} \rangle, \qquad \mathcal{E} = \langle \widehat{T}^{00} \rangle$$

$$\mathcal{P} = TV^{-1} \log Z \ , \quad \mathcal{S} = \frac{\partial \mathcal{P}}{\partial \Omega} \ , \quad \mathcal{E} = -\mathcal{P} + \Omega \frac{\partial \mathcal{P}}{\partial \Omega} + T \frac{\partial \mathcal{P}}{\partial T}$$







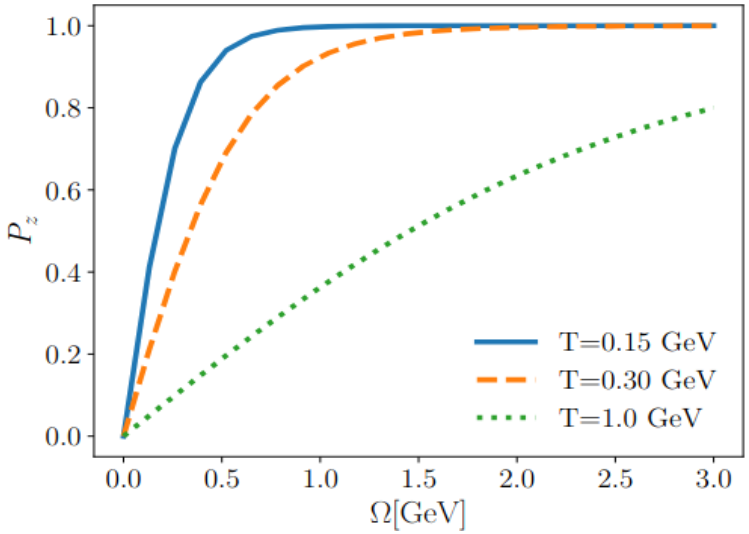
We find that the two methods coincide up to numerical precision.

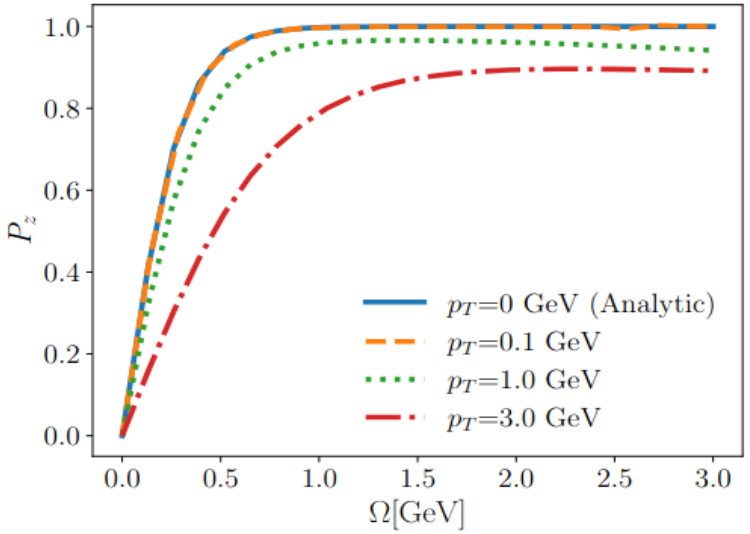
Polarization

It is possible to find an analytic result for particles with vanishing transverse momentum

$$P_z(T,\Omega) = \frac{\sinh\left(\frac{\Omega}{2T}\right)}{\tau m^2} \frac{\int_0^\infty \mathrm{d}\mu \left[\cosh\left(\frac{m_{\mathrm{L}}(\mu)}{T}\right) + \cosh\left(\frac{\Omega}{2T}\right)\right]^{-1}}{\int_0^\infty \mathrm{d}\mu \, \frac{\mathfrak{h}(\tau,\mu)}{m_{\mathrm{L}}(\mu)} \left[e^{-\frac{m_{\mathrm{L}}(\mu)}{T}} + \cosh\left(\frac{\Omega}{2T}\right)\right] \left[\cosh\left(\frac{m_{\mathrm{L}}(\mu)}{T}\right) + \cosh\left(\frac{\Omega}{2T}\right)\right]^{-1}}$$

Otherwise, the calculation can be done numerically





Conclusions

We have studied a gas of free fermions in a boost invariant fluid at finite spin density

Solving the Dirac equation in Milne space we can exactly diagonalize the density operator

If no spin potential is present no spin polarization is produced – **no spin polarization in high-energy central heavy ion collisions**

We compute exactly the partition function, and **verify explicitly thermodynamic relations** in the Canonical pseudogauge

We compute exact spin polarization as a function of spin potential, which can be a useful banchmark for future studies

THANK YOU FOR THE ATTENTION!

Backup

The definition of energy momentum tensor and spin tensor is not unique.

Theoretical ambiguity!

$$\widehat{T}'^{\mu\nu} = \widehat{T}^{\mu\nu} + \frac{1}{2}\nabla_\lambda\left(\widehat{\Phi}^{\lambda,\mu\nu} - \widehat{\Phi}^{\mu,\lambda\nu} - \widehat{\Phi}^{\nu,\lambda\mu}\right)$$
$$\widehat{\mathcal{S}}'^{\lambda,\mu\nu} = \widehat{\mathcal{S}}^{\lambda,\mu\nu} - \widehat{\Phi}^{\lambda,\mu\nu}$$

The new operators describe the same global charges (four-momentum and angular momentum boost operators).

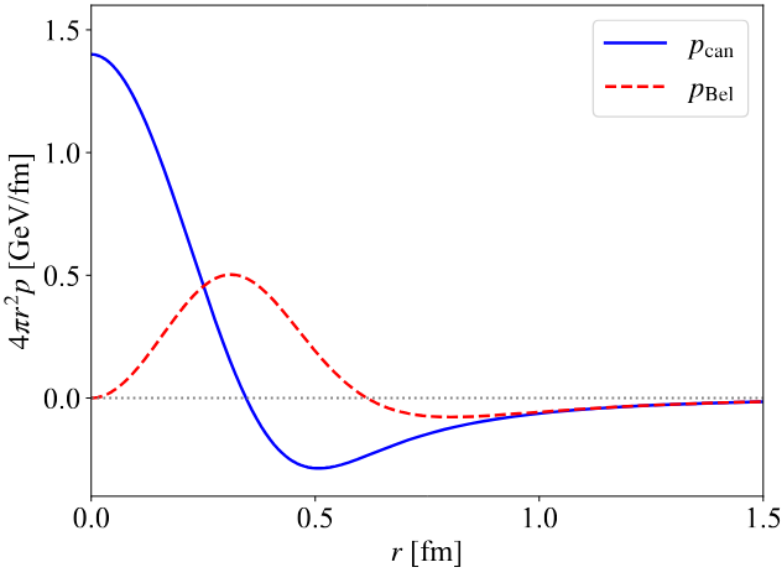
$$\widehat{\mathcal{J}}^{\lambda,\mu\nu} = x^\mu \widehat{T}^{\lambda\nu} - x^\nu \widehat{T}^{\lambda\mu} + \widehat{\mathcal{S}}^{\lambda,\mu\nu}$$
$$\widehat{P'}^\mu = \widehat{P}^\mu = \int \mathrm{d}\Sigma_\rho \widehat{T^{(\prime)}}^{\rho\mu} \qquad \widehat{J}'^{\mu\nu} = \widehat{J}^{\mu\nu} = \int \mathrm{d}\Sigma_\lambda \widehat{\mathcal{J}^{(\prime)}}^{\lambda,\mu\nu}$$

$$\nabla_\mu \widehat{T}'^{\mu\nu} = \nabla_\mu \widehat{T}^{\mu\nu} = 0 \qquad \nabla_\lambda \widehat{\mathcal{J}'}^{\lambda,\mu\nu} = \nabla_\lambda \widehat{\mathcal{J}}^{\lambda,\mu\nu} = 0$$

Many things depend on the pseudogauge choice:

Pressure distribution inside a nucleon

Fukushima, Uji, Phys.Rev.D 113 (2026) 1, 016025



Phase transition temperature (with a spin density)

Singha, Ambrus, Busuioc, Bandyopadhyay, Chernodub
2508.20237, Braguta, Chernodub, Roenko
Phys.Rev.D 111 (2025) 11, 114508

Kubo formulae:

Becattini, Tinti, Phys.Rev.D 87 (2013) 2, 025029

$$\eta = \lim_{q\to 0} \frac{\mathrm{d}}{\mathrm{d}\omega} \mathrm{Im} \left\{ i \int \mathrm{d}^4 x e^{i\omega t - qx} \langle \left[\widehat{T}^{12}(x), \widehat{T}^{12}(0) \right] \rangle \theta(t) \right\} = \lim_{q\to 0} \frac{\mathrm{d}}{\mathrm{d}\omega} G_R^{T^{12}T^{12}}(q)$$

Local equilibrium density operator and expectation values in local equilibrium:

$$\widehat{\rho} = \frac{1}{Z} \exp\left(- \int_{\Sigma(\tau)} \mathrm{d}\Sigma n_\mu \widehat{T}^{\mu\nu} \beta_\nu - n_\mu \mathfrak{S}_{\rho\sigma} \widehat{\mathcal{S}}^{\mu,\rho\sigma} \right)$$

Renormalization

Even in a free theory, vacuum terms should be subtracted to obtain finite thermal expectation values. However in curved space this is not trivial.

$$H(\tau)|0_\tau\rangle = 0 \Rightarrow H(\widetilde{\tau})|0_\tau\rangle \neq 0$$

Vacuum state is proper-time dependent. This implies $\nabla_\mu \left(\langle \widehat{T}^{\mu\nu} \rangle - \langle 0_\tau | \widehat{T}^{\mu\nu} | 0_\tau \rangle \right) \neq 0.$

Proper normalization should be done at an initial time and kept fixed
To identify the vacuum state, we compute the limits

$$\lim_{T\to 0} \left(\lim_{\zeta \to 0} \lim_{\mathfrak{S}\to 0} \widehat{\rho} \right) = \lim_{T\to 0} \frac{1}{Z_{\zeta,\mathfrak{S}=0}} \exp\left(-\frac{1}{T(\tau_0)} \int \mathrm{d}^3\mathfrak{x}\, \widehat{T}^{00} \right) = |0_{\tau_0}\rangle\langle 0_{\tau_0}| \,.$$

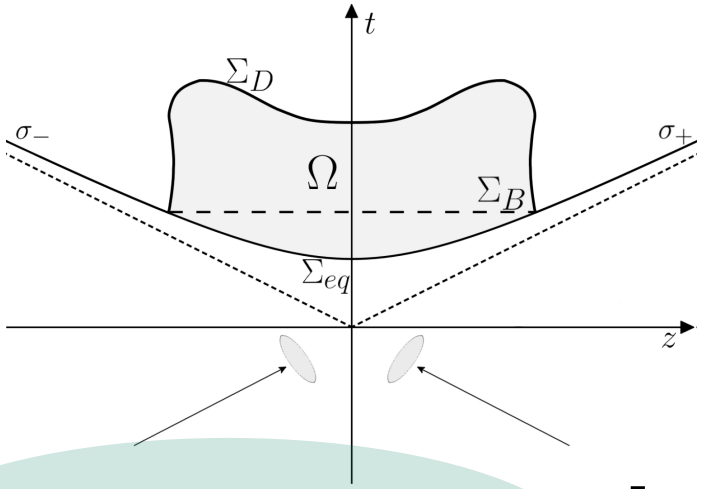
$$\langle : \widehat{O} : \rangle = \langle \widehat{O} \rangle - \lim_{T\to 0} \langle \widehat{O} \rangle_{\mathfrak{S}=\mu=0}$$

We do not study time dependence of observables, so in practice we impose $\tau_0 = \tau$

β is the four temperature and is the spin potential: $\beta^\mu = \dfrac{u^\mu}{T}$

Warning: The true density operator should be constant in the Heisenberg representation! *Zubarev, Prozorkevich, & Smolyanskii, 1979*

$$\widehat{\rho}_{true} = \frac{1}{Z}\exp\left(-\int_{\Sigma(\tau_0)} \mathrm{d}\Sigma n_\mu \widehat{T}^{\mu\nu}\beta_\nu - n_\mu \widehat{\mathcal{S}}^{\mu,\nu\rho}\mathfrak{S}_{\nu\rho}\right)$$



We use Gauss theorem to connect with present time.

$$\widehat{\rho}_{true} = \frac{1}{Z}\exp\left[-\int_{\Sigma(\tau)} \mathrm{d}\Sigma n_\mu \left(\widehat{T}^{\mu\nu}\beta_\nu - n_\mu \widehat{\mathcal{S}}^{\mu,\nu\rho}\mathfrak{S}_{\nu\rho}\right) + \int_\Omega \mathrm{d}\Omega \left(\widehat{T}^{\mu\nu}\nabla_\mu\beta_\nu - \nabla_\mu\left(\widehat{\mathcal{S}}^{\mu,\nu\rho}\mathfrak{S}_{\nu\rho}\right)\right)\right]$$

Local equilibrium **Dissipation**

Global equilibrium

The global equilibrium is achieved if there is no dependence on the hypersurface:

$$\partial_\mu \beta_\nu + \partial_\nu \beta_\mu = 0$$

$$\frac{\mu}{T} \equiv \zeta = \text{const.}$$

$$\mathfrak{S} = \varpi = \text{const.}$$

$$\beta^\mu = b^\mu + \varpi^{\mu\nu} x_\nu$$

ϖ Constant thermal vorticity

The statistical operator in this case reads:

$$\widehat{\rho}_{GE} = \frac{1}{Z} \exp\left(-\widehat{P}^\mu b_\mu + \frac{\varpi_{\mu\nu}}{2} \widehat{J}^{\mu\nu} + \zeta \widehat{Q}\right)$$

P and J are the generators of the Poincaré group.
Q conserved charge.

ϖ describes rotation and acceleration

$$\varpi^{\mu\nu} = \epsilon^{\mu\nu\alpha\beta} w_\alpha u_\beta + \alpha^\mu u^\nu - \alpha^\nu u^\mu \qquad w^\mu = \frac{\omega^\mu}{T} \qquad \alpha^\mu = \frac{A^\mu}{T}$$

Entropy production

The local thermodynamic equilibrium has vanishing entropy production

Zubarev, Prozorkevich, & Smolyanskii, 1979; Becattini, Buzzegoli, & Grossi, Particles, 2019, 1902.01089; Becattini, Daher, Sheng Phys.Lett.B 850 (2024)

$$S = -\mathrm{Tr}\left(\widehat{\rho}\,\log\widehat{\rho}\right) = \int_{\Sigma} \mathrm{d}\Sigma_{\mu} s^{\mu}$$

$$\nabla_{\mu} s^{\mu} = \left(T_{S}^{\mu\nu} - T_{S,LE}^{\mu\nu}\right)\partial_{\mu}\beta_{\nu} - \left(j^{\mu} - j_{LE}^{\mu}\right)\partial_{\mu}\frac{\mu}{T} + \left(T_{A}^{\mu\nu} - T_{A,LE}^{\mu\nu}\right)\left(\varpi - \mathfrak{S}\right)_{\mu\nu}$$
$$- \frac{1}{2}(\mathcal{S}^{\mu,\nu\rho} - \mathcal{S}_{LE}^{\mu,\nu\rho})\partial_{\mu}\mathfrak{S}_{\nu\rho}$$

Deviations of the actual values of the conserved current from the local equilibrium value are responsible for entropy production.

The local equilibrium operator describes an ideal fluid.