Non-equilibrium corrections to the Wigner function at freeze-out

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Introduction

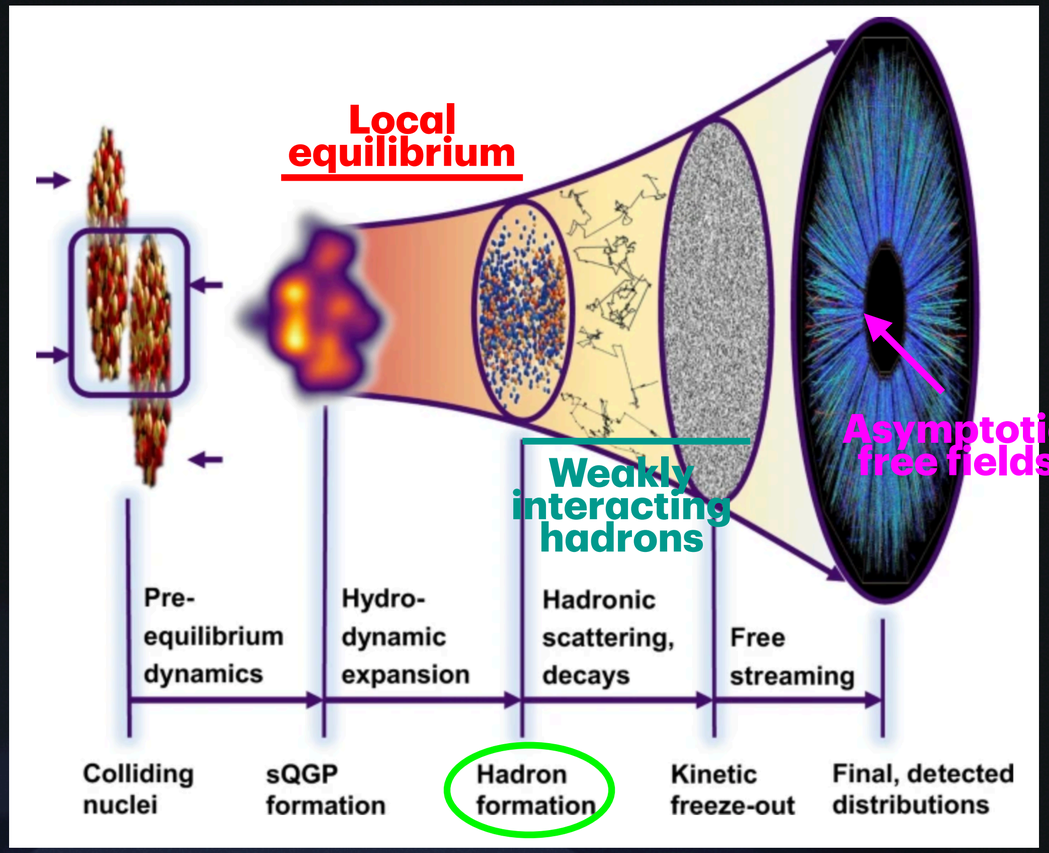
The Cooper-Frye formula

QGP evolves as a fluid from equilibrium to decoupling

Spectra of hadrons emitted from QGP:

$$E_k \frac{\mathrm{d}N}{\mathrm{d}^3 k} = \int_{\Sigma_\mathrm{D}} \mathrm{d}\Sigma(x) \cdot p \, \frac{1}{\mathrm{e}^{\beta(x)\cdot k} \pm 1}$$

F. Cooper, G. Frye, *Phys.Rev.D* 10 (1974) 186



~ decoupling Σ_D

Is spectrum completely determined by the local state at decoupling?

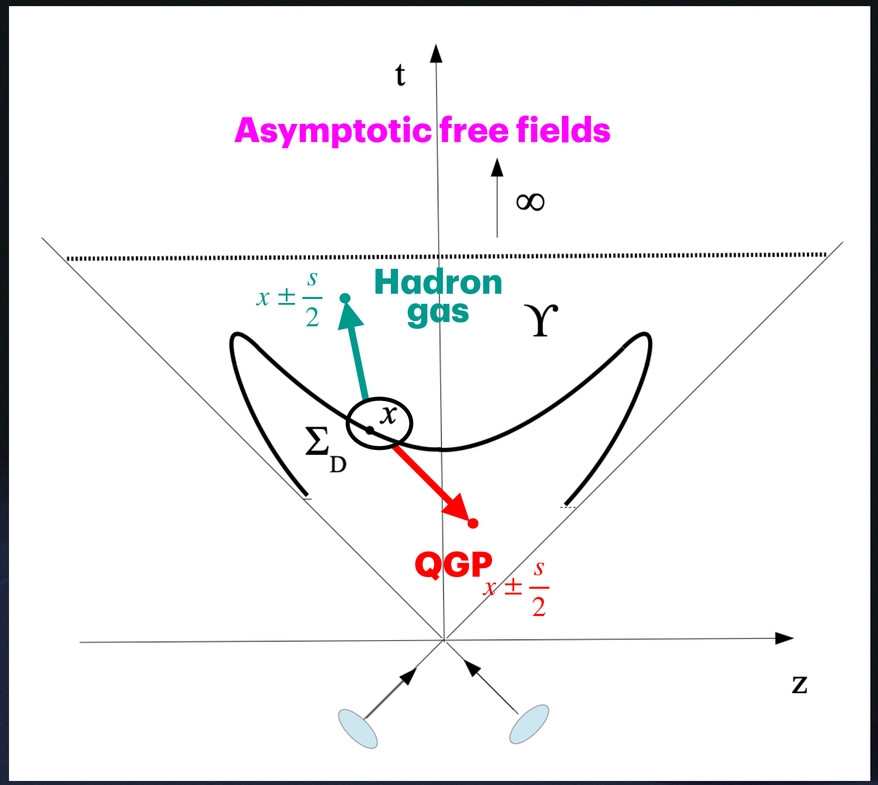
The covariant Wigner function

Bridge QFT $\widehat{\phi}$ with experimental hadronic observables

$$\widehat{W}(x,k) = \frac{2}{(2\pi)^4}\int \mathrm{d}^4 s\, \mathrm{e}^{-\mathrm{i}k\cdot s}\, \widehat{\phi}^{\dagger}(x+\tfrac{s}{2})\, \widehat{\phi}\,(x-\tfrac{s}{2})$$

$$W(x,k) = \mathrm{Tr}(\hat{\rho}\, \widehat{W}(x,k))$$

$\widehat{W}(x,k)$ **Is non-local!**



Spectra of emitted hadrons from Σ_D

$$\frac{\mathrm{d}N}{\mathrm{d}^3 k} = \int \mathrm{d}k^0 \int_{\Sigma_D} \mathrm{d}\Sigma(x)\cdot k\, W(x,k) + \int \mathrm{d}k^0 \int_{\Upsilon} \mathrm{d}\Upsilon(y)\, k\cdot \partial^y W(y,k)$$

Dominant term

Kinetic theory

Quantum statistical mechanics and linear response

The quantum state of the QGP

F.Becattini, M.Buzzegoli, E.Grossi, *Particles* 2 (2019) 2, 197-207

Non-equilibrium statistical operator

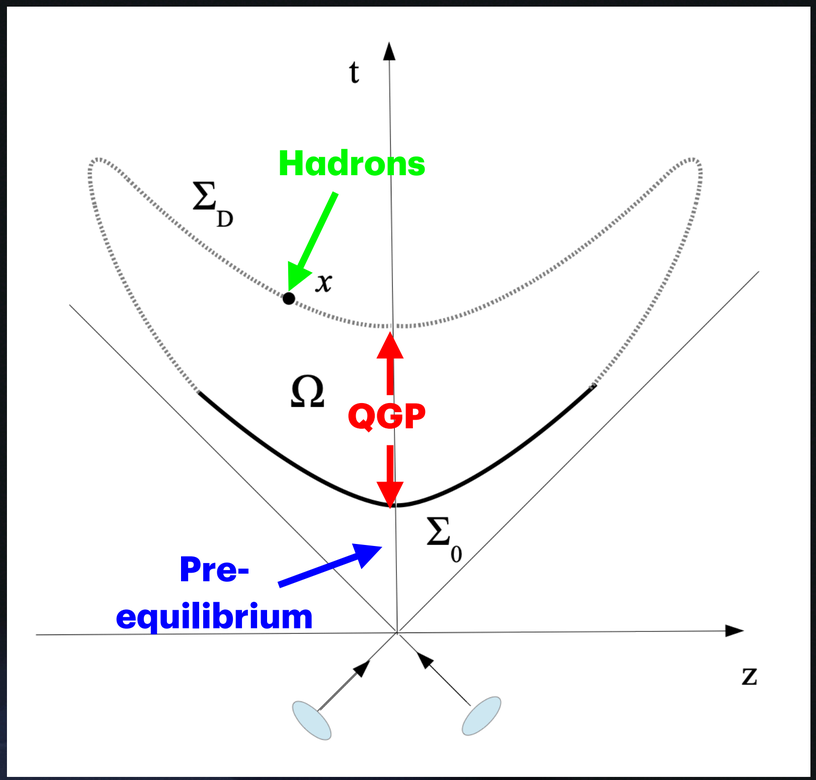
$$\hat{\rho} = \frac{1}{Z}\exp\left[-\int_{\Sigma_0} \mathrm{d}\Sigma_\mu(y)\,\hat{T}^{\mu\nu}(y)\beta_\nu(y)\right]$$

Gauss theorem

$$\hat{\rho} = \frac{1}{Z}\exp\left[-\int_{\Sigma_D} \mathrm{d}\Sigma_\mu(y)\,\hat{T}^{\mu\nu}(y)\beta_\nu(y) + \int_{\Omega} \mathrm{d}^4y\,\hat{T}^{\mu\nu}(y)\partial_\mu\beta_\nu(y)\right]$$

Local equilibrium at decoupling

Dissipation in the plasma



Can the non-equilibrium response always be described locally?

Linear response expansion

We don't use Gauss theorem

Non-equilibrium: local equilibrium + dissipation

$$\Delta\widehat{E} = -\int_{\Sigma_0} \mathrm{d}\Sigma_\mu(y)\,\hat{T}^{\mu\nu}(y)\,\Delta\beta_\nu(y,x)$$

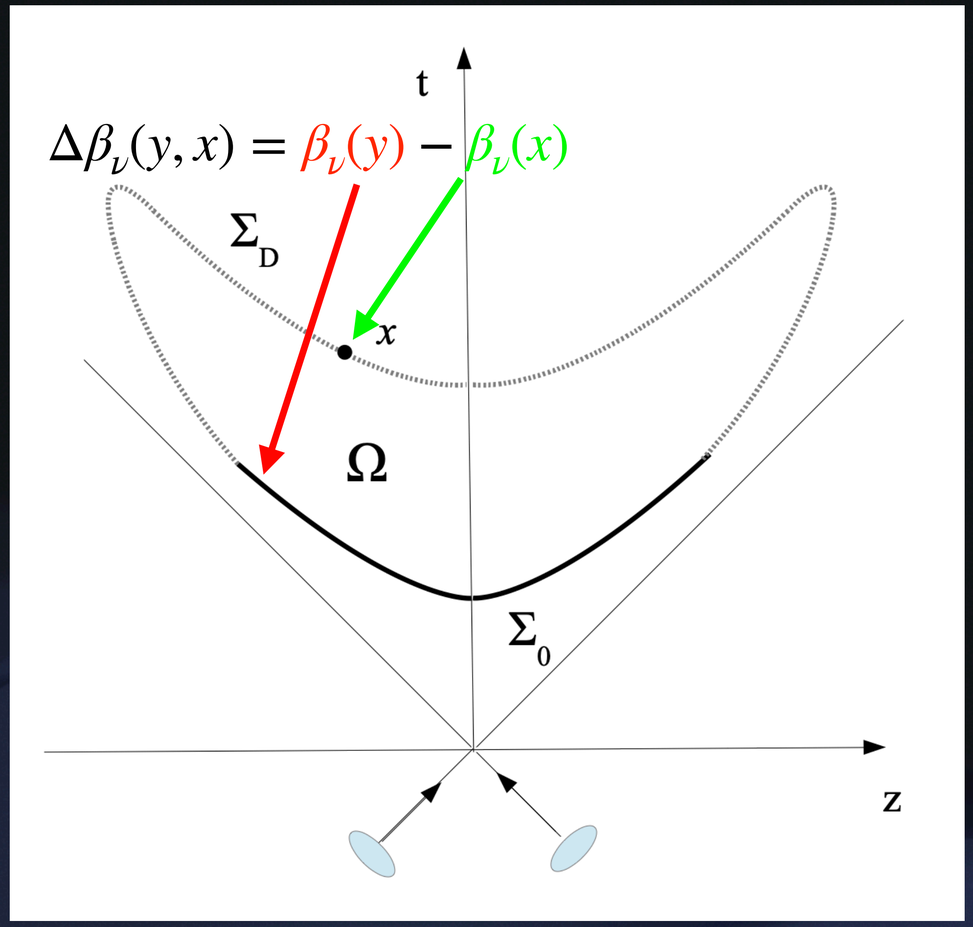
$$\hat{\rho} = \frac{1}{Z}\exp\left[-\beta(x)\cdot\widehat{P} + \Delta\widehat{E}\right]$$

Wigner function at decoupling

$$W(x,k) \simeq W_{\mathrm{GE}}(x,k) + \Delta_{\mathrm{non-eq}} W(x,k)$$

Cooper-Frye formula

Non-equilbrium correction (dissipative and LTE)

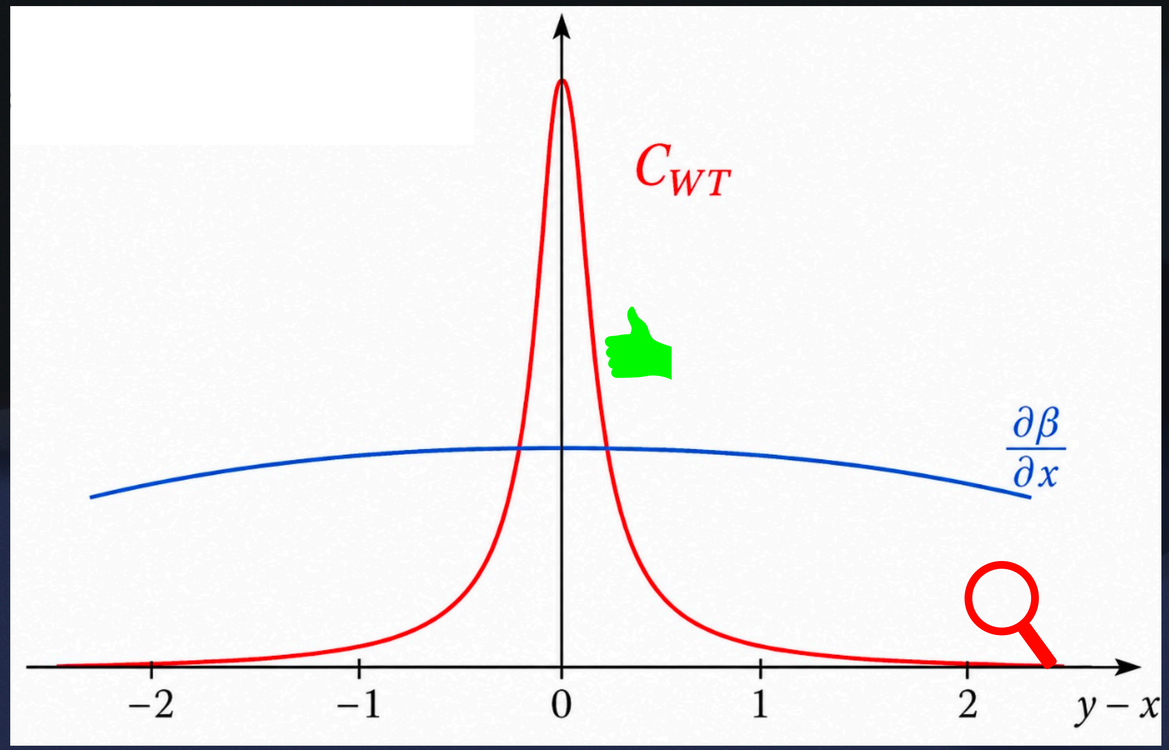


Kubo formulae

Dissipative corrections in linear response theory

$$\Delta_{\rm diss} W(x,k) = -\int \mathrm{d}^4 y\, \partial_\mu \beta_\nu(y)\, C^{\mu\nu}_{WT}(x-y,k)$$

$$C^{\mu\nu}_{WT}(x-y,k) = \int_0^1 \mathrm{d}\lambda \left\langle \widehat{W}(x,k), \mathrm{e}^{-\lambda\beta(x)\cdot\widehat{P}}\, \widehat{T}^{\mu\nu}(y)\, \mathrm{e}^{\lambda\beta(x)\cdot\widehat{P}} \right\rangle_{c,\rm GE}$$



Usual assumptions:

- Thermodynamic fields are slowly varying
- Correlator $C^{\mu\nu}_{WT}$ is microscopic

→ Gradient taken out from the integral

Localized Kubo formulae

$$\Delta_{\rm diss} W(x,k) \propto \partial_\mu \beta_\nu(x)\, \upsilon^{\mu\nu}(k,\beta)$$

A. Hosoya, M. Sakagami, M. Takao *Annals Phys.* 154 (1984) 229
M. Buzzegoli, *JHEP* 07 (2025) 255, *JHEP* 07 (2025) 255

Non-equilibrium corrections to the Wigner function

The interacting microscopic kernel

Microscopic Kernel of the correlator

$k_{\pm} = k \pm q/2$

$$\widetilde{C}^{\mu\nu}_{WT}(q,k) \sim \Theta^{\mu\nu}(k,q,\beta) \equiv \langle\, \widehat{\phi}^{\,\dagger}(k_+)\, \widehat{\Phi}\,(k_-),\, \hat{T}^{\mu\nu}(0)\rangle_{c,\mathrm{GE}}$$

Note: all the fields are interacting

Symmetries of equilibrium state

$$\Theta^{\mu\nu}(k,q,\beta) = \sum_i \Theta_i(S)\mathcal{T}^{\mu\nu}(k,q,\beta)$$

Thermo-Gravitational form factors

$$\Theta_i(S)$$

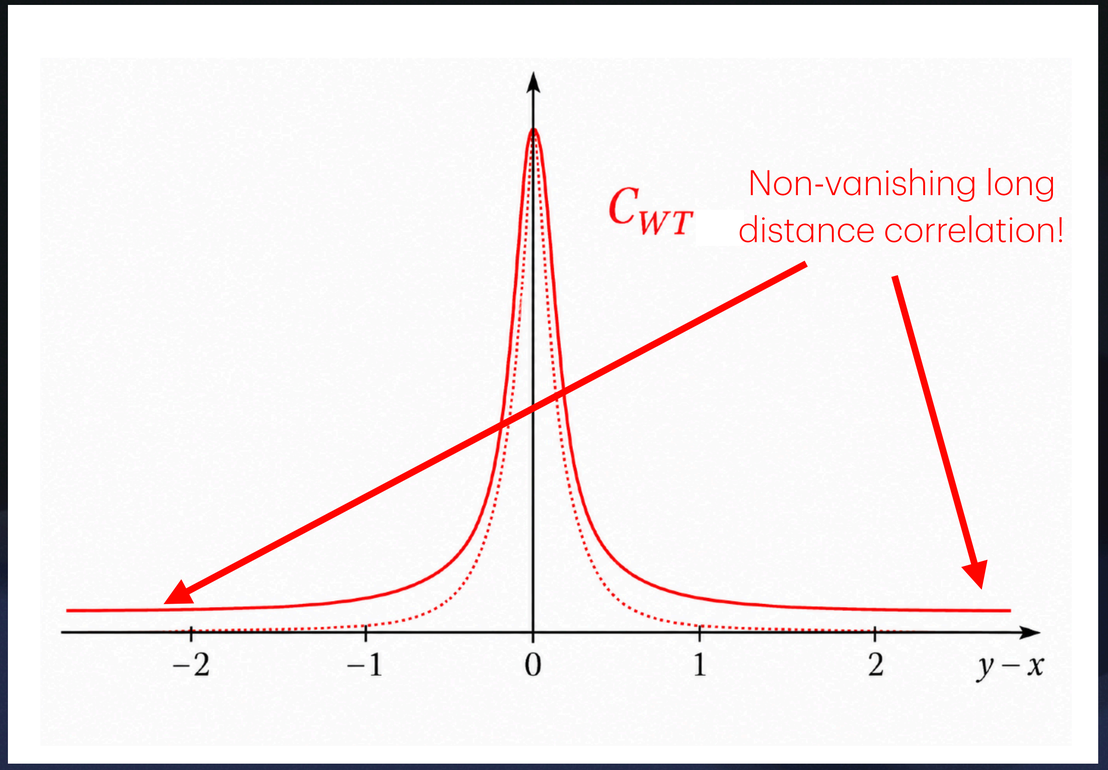
$S = k^2, q^2, \beta^2 \cdots$

Information about the microscopic interactions

Correlators and conservation

Components of the correlators are constrained by global conservation

$$\int \mathrm{d}^3\mathbf{y}\,\hat{T}^{0\nu}(t,\mathbf{y}) = \hat{P}^{\nu} \longrightarrow \int \mathrm{d}^3 y\, C^{0\nu}_{WT}(x-y,k) = -\frac{\partial W_{\mathrm{GE}}(x,k)}{\partial \beta_\nu}$$



In Fourier space

$$\widetilde{C}^{0\nu}_{WT}(q^0,\mathbf{q}=0) = -\frac{\delta(q^0)}{(2\pi)^3}\frac{\partial W_{\mathrm{GE}}(x,k)}{\partial \beta_\nu}$$

Global conservation leads to long-lived correlations

Decomposition of $\Theta^{\mu\nu}$

$$w_\vartheta^\mu = \frac{\cos\vartheta}{\sqrt{k^2}}k^\mu + \frac{\sin\vartheta}{\sqrt{\beta^2}}\beta^\mu$$

$$\Theta^{\mu\nu}(q) \sim \int_{-\pi/2}^{\pi/2} \mathrm{d}\vartheta\, \delta(q\cdot w_\vartheta)\Gamma_\vartheta^{\mu\nu}(q) + Z^{\mu\nu}(q) + \Xi^{\mu\nu}(q) + \text{others}\,\delta$$

Ballistic | Protected | Unprotected | Long tails

Contributes to the global conservation

$$Z^{0\nu}(q^0, \mathbf{q}=0) \propto \delta(q^0)$$

Doesn't contribute to the global conservation

$$\Xi^{0\nu}(q^0, \mathbf{q}=0) = 0$$

Unprotected
Local corrections → Viscous-like coefficients

Ballistic + Protected
Non-local corrections →

$$\int \mathrm{d}\vartheta \Gamma_1(\vartheta) w_\vartheta^\nu + 2\pi Z^\nu = -\frac{(2\pi)^5}{2}\frac{\partial W_{\mathrm{GE}}}{\partial \beta_\nu}$$

Ballistic thermal form factor

Residue of the non-analytic part

Localized vs non-localized correlators

$$C^{\mu\nu}_{WT} = C^{\mu\nu}_{WT}\Big|_{\Gamma} + C^{\mu\nu}_{WT}\Big|_{Z} + C^{\mu\nu}_{WT}\Big|_{\Xi}$$

Localized correlator produces a local transport coefficient

$$\int_{t_0}^{t} \mathrm{d}y^0 \int \mathrm{d}^3\mathbf{y} C^{\mu\nu}\Big|_{\Xi} \equiv \upsilon^{\mu\nu}(k,\beta) \sim \eta,\ \zeta_{\mathrm{bulk}}$$

Non-localized correlators diverge at large times:

$$\int_{t_0}^{t} \mathrm{d}y^0 \int \mathrm{d}^3\mathbf{y} C^{\mu\nu}_{WT}\Big|_{\Gamma},\quad \int_{t_0}^{t} \mathrm{d}y^0 \int \mathrm{d}^3\mathbf{y} C^{\mu\nu}_{WT}\Big|_{Z} = (t-t_0) K^{\mu\nu}_{\Gamma,Z}(k,\beta)$$

Conserved-mode contributions require a nonlocal treatment.

Non-equilibrium corrections
Local transport vs Memory effects

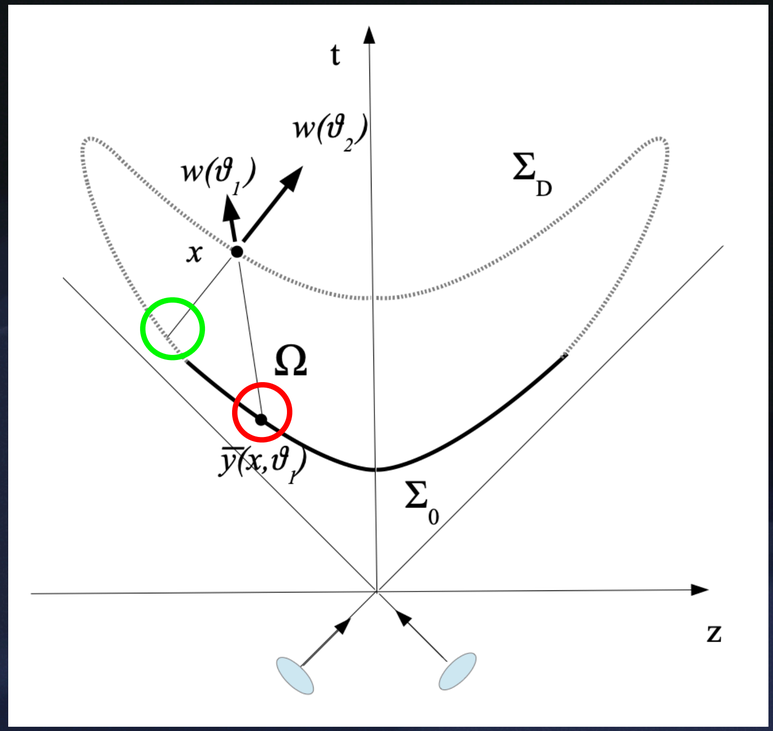
Memory induced correction

$$\Delta_{\text{ball}} W(x,k) \simeq \int \mathrm{d}\vartheta \Gamma_1(\vartheta) w^\mu_\vartheta w^\nu_\vartheta \frac{\hat{n}_\mu(\bar{y})}{k\cdot\hat{n}(\bar{y})} \left[\beta_\nu(\bar{y}) - \beta_\nu(x)\right] + \mathcal{O}(\partial\beta(\bar{y}))$$

$$\Delta_Z W(x,k) \simeq Z^\nu(k,\beta)\left[\beta_\nu(\bar{y}_\perp) - \beta_\nu(x)\right]$$

Memory on the initial state

- Depends on fields on the initial hypersurface
- Depends on two dynamical coefficients
- $\neq 0$ only when $(x-y)^\mu = w^\mu_\vartheta \tau$ intersects Σ_0



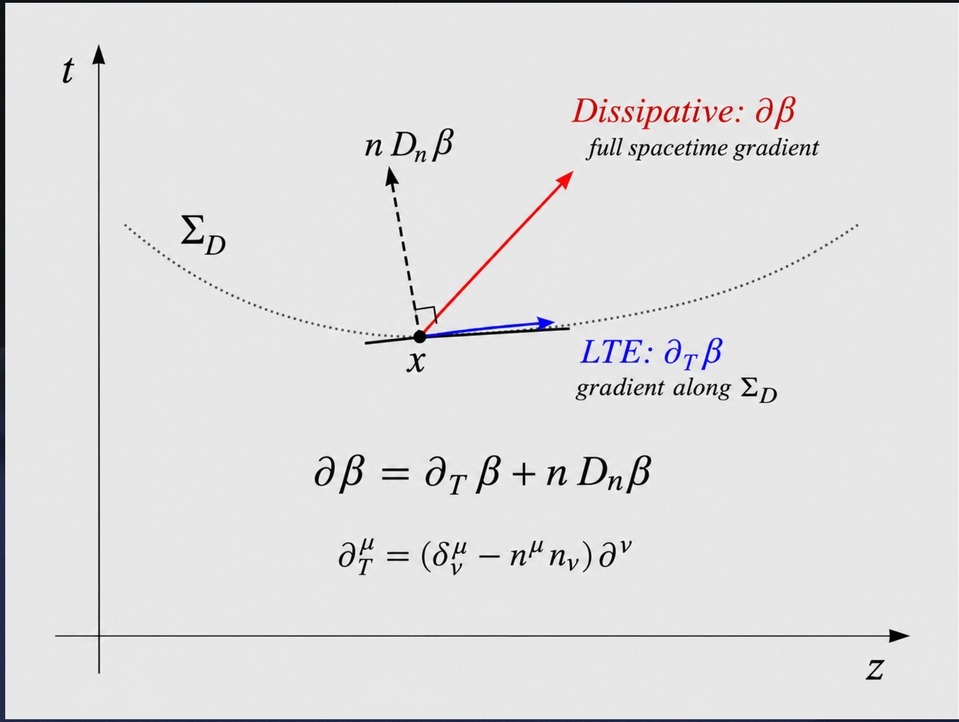
Local corrections: LTE and dissipation

$$\Pi^{\mu}_{\nu} = \delta^{\mu}_{\nu} - \hat{n}^{\mu}\hat{n}_{\nu}$$
$$\partial^{T}_{\rho} = \Pi^{\lambda}_{\rho}\partial_{\lambda}$$

Using Gauss theorem and extracting the gradient

$$\Delta W_{\Xi}(x,k) \simeq -\mathcal{S}^{\rho\nu}\partial^{T}_{\rho}\beta_{\nu}(x) + \mathscr{V}^{\mu\nu}\partial_{\mu}\beta_{\nu}(y)$$

$$\mathcal{S}^{\rho\nu} = \int_{\Sigma_D} \mathrm{d}\Sigma_{\mu}(y)(x-y)^{\rho} C^{\mu\nu}_{\Xi}(x-y,k) \qquad \mathscr{V}^{\mu\nu} = \int_{\Omega} \mathrm{d}^4 y\, C^{\mu\nu}_{\Xi}(x-y,k)$$



Local equilibrium and dissipative corrections terms from the same microscopic correlator

Ballistic term and free field limit

Using free fields in the Wigner function

$$Z^{\mu\nu}(q) = \Xi^{\mu\nu}(q) = 0 \implies \Theta^{\mu\nu}_{\text{free}} \propto \delta(k \cdot q)\, k^\mu k^\nu$$

Ballistic correction = leading order free-streaming

$$W(x,k) = W(y_0, \mathbf{x} - \mathbf{v}_k(x^0 - y^0)) \longrightarrow \Delta W(x,k) \sim k^\mu(\beta_\nu(\bar{y}) - \beta_\nu(x))$$

Free-streaming memory survives without dissipation

Non equilibrium correction to the spectrum

The spectra of the hadrons emitted at decoupling is given by

$$\frac{\mathrm{d}N}{\mathrm{d}^3p} = \frac{\mathrm{d}N}{\mathrm{d}^3p}\bigg|_{\mathrm{CF}} + \frac{\mathrm{d}N}{\mathrm{d}^3p}\bigg|_{\mathrm{local}} + \frac{\mathrm{d}N}{\mathrm{d}^3p}\bigg|_{\mathrm{mem}}$$

Non-equilibrium correction to the Cooper-Frye formula

$$\frac{\mathrm{d}N}{\mathrm{d}^3k}\bigg|_{\mathrm{loc}} \sim -\int_{\Sigma_{\mathrm{D}}} \mathrm{d}\Sigma(x)\cdot k\left[-\mathcal{S}^{\mu\nu}\partial^T_\mu\beta_\nu(x) + \mathcal{V}^{\mu\nu}\partial_\mu\beta_\nu(x)\right]$$

$$\frac{\mathrm{d}N}{\mathrm{d}^3k}\bigg|_{\mathrm{mem}} \sim \int_{\Sigma_{\mathrm{D}}} \mathrm{d}\Sigma(x)\cdot k\left(\int \mathrm{d}\vartheta\Gamma(\vartheta)w^\nu_\vartheta + Z^\nu(k,\beta)\right)(\beta_\nu(\bar{y}) - \beta_\nu(x))$$

Conclusion

Ab *initio* linear response computation

Non-equilibrium Wigner corrections expressed through interacting quantum correlators.

Beyond local gradients

Long-lived correlations can generate memory of the initial state.

Conservation constrains the memory

Sum rules relate the combined long-wavelength response to equilibrium susceptibilities.

感谢您的关注

Thank you for your attention

Backup Slides

Volume term and dissipation

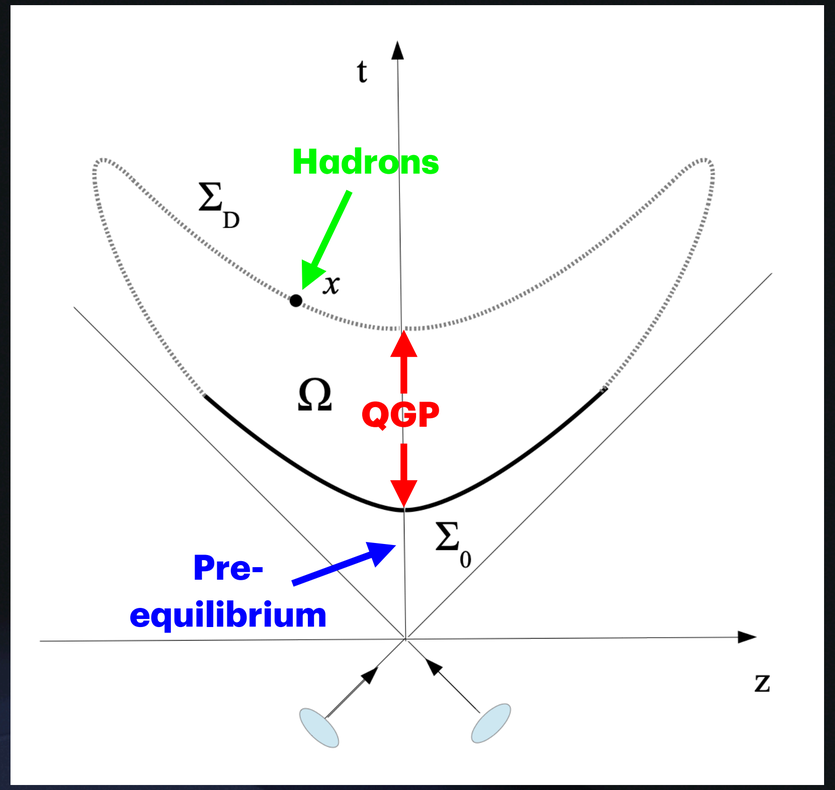
Separation in local equilibrium and dissipative part

F.Becattini, M.Buzzegoli, E.Grossi, *Particles* 2 (2019) 2, 197-207

$$\hat{\rho} = \frac{1}{Z}\exp\left[-\int_{\Sigma_D} d\Sigma_\mu(y)\,\hat{T}^{\mu\nu}(y) + \int_\Omega d^4y\,\hat{T}^{\mu\nu}(y)\partial_\mu\beta_\nu(y)\right]$$

Local equilibrium at decoupling

Dissipation in the plasma



Dissipation produces entropy

$$\partial_\mu s^\mu \sim [\mathrm{Tr}(\hat{\rho}\,\hat{T}^{\mu\nu}(x)) - \mathrm{Tr}(\hat{\rho}_{\mathrm{LTE}}\,\hat{T}^{\mu\nu}(x))]\partial_\mu\beta_\nu(x)$$

This does not guarantee that the red part is completely dissipative!

See also
Y. Y. Li, S. Liu ArXiV: 2611.08482

Localization and Kubo formule

Kubo-Mori equilibrium correlator

$$C^{\mu\nu}_{OT}(x-y) = \int_0^1 \mathrm{d}\lambda \, \langle \, \widehat{O}\,(x), \mathrm{e}^{-\lambda\beta(x)\cdot\widehat{P}}\, \widehat{T}^{\mu\nu}(y)\, \mathrm{e}^{\lambda\beta(x)\cdot\widehat{P}} \, \rangle_{c,\mathrm{GE}}$$

Slowly varying thermodynamic fields + localization of the correlator

$$\Delta_{\mathrm{diss}} O(x) \sim \partial_\mu \beta_\nu(x) \int \mathrm{d}^4 y \, C^{\mu\nu}_{OT}(y-x) \sim \partial_\mu \beta_\nu(x) \int \mathrm{d}^4 y \, G^{\mu\nu R}_{OT}(y-x)$$

$$G^{\mu\nu R}_{OT}(t-t') \propto \theta(t-t') \langle [\, \widehat{O}\,(x), \hat{T}^{\mu\nu}(y)] \rangle_{\mathrm{GE}}$$

Localization of the correlation is not valid when involves conserved quantities

$$C^{\mu\nu}_{OT}(t,\mathbf{z}) = T \int_{-\infty}^{t} \mathrm{d}t' G^{\mu\nu R}_{OT}(t',\mathbf{z}) + C^{\mu\nu}(-\infty,\mathbf{z})$$

Viscous coefficient

Memory

Gradient expansion for the ballistic term

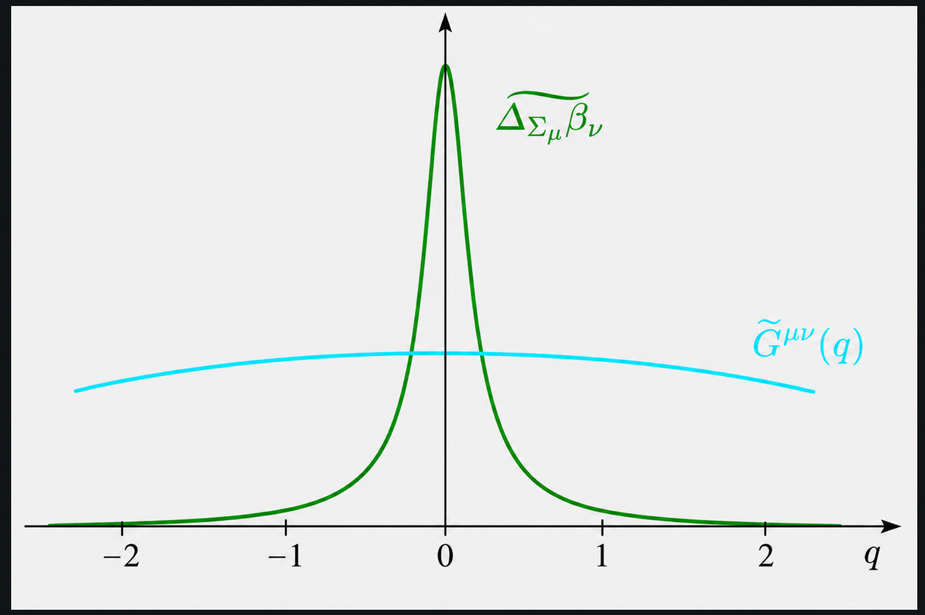
Ballistic term: δ times analytic tensor

$$\delta(q \cdot w_\vartheta) \times \Gamma^{\mu\nu}(q)$$

$$\Delta_{\text{ball}} W(x,k) \sim \int \mathrm{d}^4 q\, \delta(k \cdot w_\vartheta)\, \widetilde{\Delta_{\Sigma_\mu} \beta_\nu}(q)\, \widetilde{G}^{\mu\nu}_{WT}(q) \Big|_\Gamma$$

FT of td field on the initial hypersurface

Microscopic regular kernel



$$\mathfrak{D}(y) = \partial^y - \frac{\hat{n}(y)}{|k \cdot \hat{n}(y)|} k \cdot \partial^y$$

Expansion in q $\Longrightarrow$ Expansion in the gradients

X.L. Sheng, F.Becattini, D.Roselli, *Phys.Rev.C* 114 (2026)3,3

$$\Delta_{\text{ball}} W(x,k) \simeq \sum_{N=0} K^{\mu\nu}_{\alpha_1 \ldots \alpha_N} \mathfrak{D}^{\alpha_1}_y \circ \ldots \circ \mathfrak{D}^{\alpha_N}_y \left[\frac{k_\nu \hat{n}_\mu(y)}{k \cdot \hat{n}(y)} \Delta\beta_\nu(y,x) \right] \Big|_{y=\bar{y}}$$

$N-$th derivative of $\widetilde{G}(q)$ in $q = 0$

Geometry and gradients on the initial hypersurface

The non-analytic term

Long-tails correlator $\Longrightarrow$ higher order memory terms

$$Z^{\mu\nu}_{\text{tail}}(q) = \frac{\theta(-q^2)}{\sqrt{-q^2}} Z^1_{\text{tail}}(q^\mu q^\nu - q^2 g^{\mu\nu}) \qquad \int \mathrm{d}^3 y \, C^{0\nu}_{WT}\Big|_{\text{tail}} \sim (t - t_0)^{-7/2}$$

$X^\mu(q) = (n \cdot q) q^\mu - q^2 n^\mu$

Diffusive pole $\Longrightarrow$ clear dissipative correction

$$Z^{\mu\nu}_{\mathscr{D}}(q) \equiv \frac{Z^1_{\mathscr{D}} \, \mathscr{D} X^\mu(q) X^\nu(q)}{((n \cdot q)^2 - q^2)\left[(n \cdot q)^2 + \mathscr{D}^2((n \cdot q)^2 - q^2)^2\right]} \qquad \lim_{\mathbf{q}\to 0} Z^{0\nu}_{\mathscr{D}} = Z^1_{\mathscr{D}} X^\nu_\perp \equiv V^\nu_{\mathscr{D}}$$

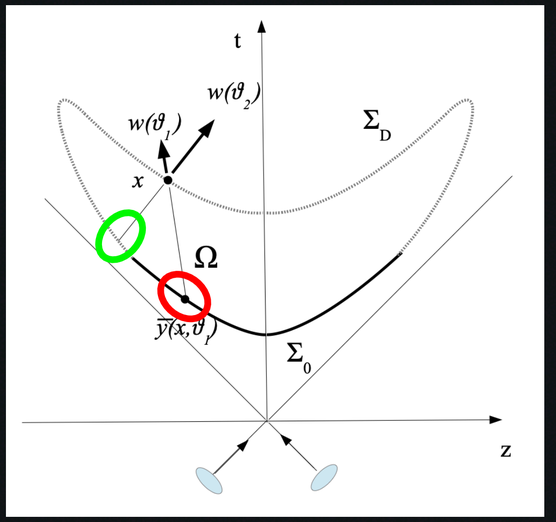
$Z^{\mu\nu}$ contains both memory and pure dissipative terms

Relaxation towards equilibrium

Memory term depends on values of field on Σ_0

$$\beta_\nu(\bar{y}) - \beta_\nu(x)$$

It vanishes if there is no intersection

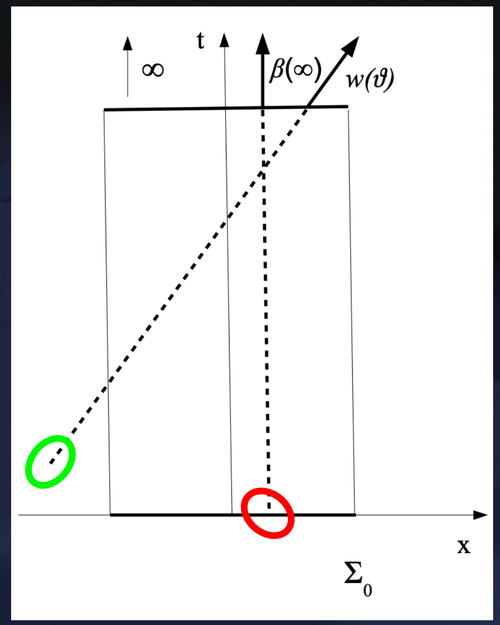


Relaxation to global equilibrium

$$\lim_{t\to\infty} W(x,k) = W_{\mathrm{GE}}(x,k)$$

Only intersection from $k = 0$

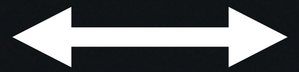
$$\Delta W(x,k) \to 0$$



Diffusion and non-ballistic modes

Ballistic mode

$\delta(q \cdot w) \times [\text{analytic in } q = 0]$



Non-ballistic mode

$Z^{\mu\nu}$ non $-$ analytic in $q = 0$

Example: diffusive mode

$$Z^{00}_{\mathscr{D}}(q) \propto \frac{\mathscr{D}\mathbf{q}^2}{(q^0)^2 + \mathscr{D}^2\mathbf{q}^4} \qquad \ell_{\text{diff}} \simeq \sqrt{\mathscr{D}\Delta t}$$



Non-analyitic memory need not to be ballistic