Global equilibrium at $\mathcal{O}(\hbar^2)$: pseudo-gauge ambiguity, Maxwell relations, and thermodynamic consistency

Asaad Daher, David Wagner, Masoud Shokri, Dirk H. Rischke

thanks to

Victor Ambrus, Francesco Becattini, Ashutosh Dash, Xin-Li Sheng

based on

[arXiv:2609.31819[nucl-th]]

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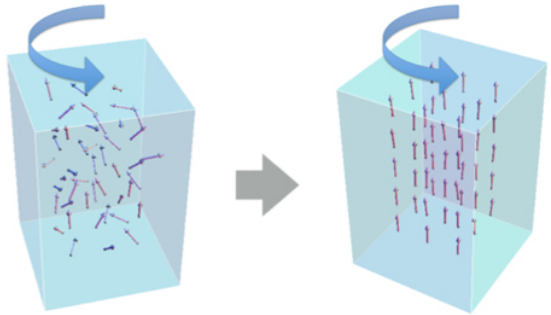
CRC-TR 211 Strong-interaction matter under extreme conditions

Rotation leads to polarization

Condensed matter: **Barnett effect**

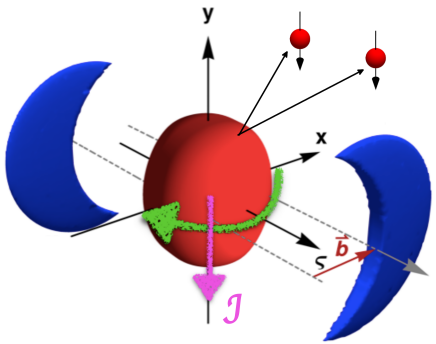
$\Longrightarrow$ spins align under rotation
(due to spin-orbit coupling)

$\Longrightarrow$ non-vanishing magnetization



Non-central heavy-ion collisions:
strong orbital angular momentum

Barnett effect in heavy-ion collisions?



$\Longrightarrow$ hadrons are polarized by vorticity of matter

Z.-T. Liang, X.-N. Wang, PRL 94 (2005) 102301

PRL **94,** 102301 (2005) PHYSICAL REVIEW LETTERS week ending 18 MARCH 2005

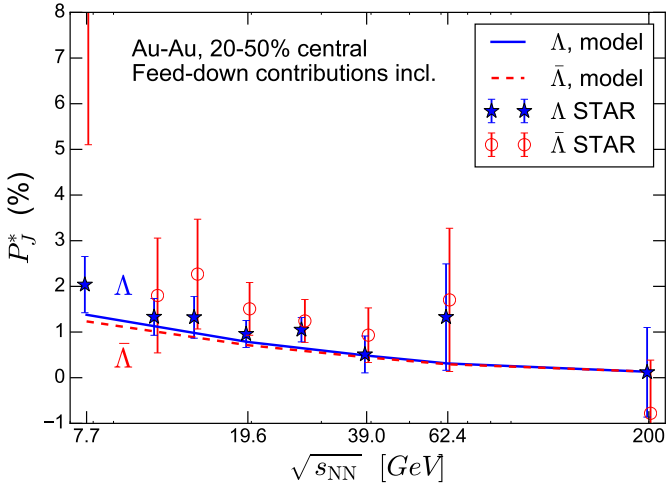
Globally Polarized Quark-Gluon Plasma in Noncentral $A + A$ Collisions

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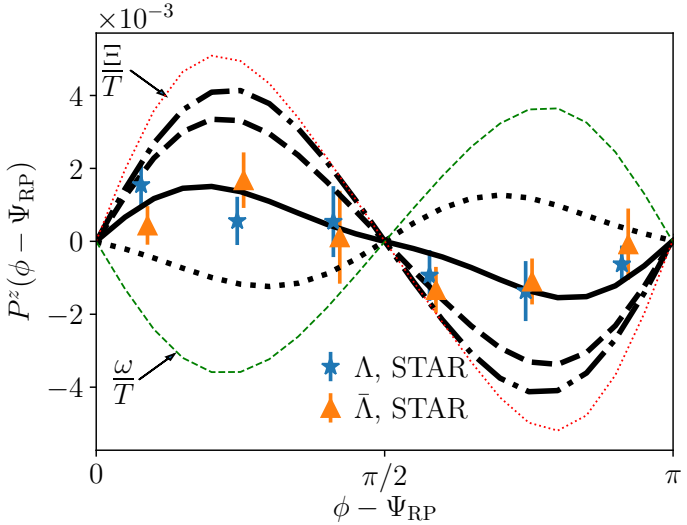




Polarization and hydrodynamics



I. Karpenko, F. Becattini, NPA 967 (2017) 764



F. Becattini, M. Buzzegoli, G. Inghirami, I. Karpenko, A. Palermo, PRL 127 (2021) 272302
(see also S.Y.F. Liu, Y. Yin, JHEP 07 (2021) 188)

Assuming local equilibrium on freeze-out hypersurface $\Sigma_{\rm f.o.}$, a formula inspired by hydrodynamics describes global polarization and local longitudinal polarization (when neglecting temperature gradients on $\Sigma_{\rm f.o.}$) quite well

$$\Pi^\mu \sim \epsilon^{\mu\nu\rho\sigma} k_\nu \int_{\Sigma_{\rm f.o.}} {\rm d}\Sigma \cdot k \left(\varpi_{\rho\sigma} + 2\hat{t}_\rho \xi_{\sigma\lambda} \frac{k^\lambda}{k_0} \right) f_{0\mathbf{k}}$$

where $\varpi_{\rho\sigma} \equiv -\frac{1}{2}\partial_{[\rho}\beta_{\sigma]}$ thermal vorticity
$\xi_{\sigma\lambda} \equiv \frac{1}{2}\partial_{(\sigma}\beta_{\lambda)}$ thermal shear tensor
$\beta^\mu \equiv u^\mu / T$ thermal 4-velocity, $f_{0\mathbf{k}}$ local-equilibrium distribution function

$a_{[\mu} b_{\nu]} \equiv a_\mu b_\nu - a_\nu b_\mu$
$a_{(\mu} b_{\nu)} \equiv a_\mu b_\nu + a_\nu b_\mu$

$\Longrightarrow$ Second-order dissipative spin hydrodynamics

N. Weickgenannt, D. Wagner, E. Speranza, DHR, PRD 106 (2022) 096014, PRD 106 (2022) L091901

D. Wagner, PRD 111 (2025) 016008

Conservation equations

$$\partial_\mu J^\mu = 0$$
$$\partial_\mu T^{\mu\nu} = 0$$
$$\hbar\,\partial_\mu S^{\mu,\nu\lambda} = T^{[\lambda\nu]}$$

Relaxation equations for dissipative quantities

$$\tau_\Pi \dot{\Pi} + \Pi = -\zeta\,\theta + \dots$$
$$\tau_n \Delta^{\mu\nu}\dot{n}_\nu + n^\mu = \varkappa\,\nabla^\mu\alpha + \dots$$
$$\tau_\pi \Delta^{\mu\nu}_{\alpha\beta}\dot{\pi}^{\alpha\beta} + \pi^{\mu\nu} = 2\eta\,\sigma^{\mu\nu} + \dots$$
$$\tau_t \Delta^{\mu\nu}_{\alpha\beta}\dot{t}^{\alpha\beta} + t^{\mu\nu} = \frac{\mathfrak{d}}{T}\sigma^{\mu\nu} + \dots$$

$\dot{A} \equiv u\cdot\partial A$, $\Delta^{\mu\nu} \equiv g^{\mu\nu} - u^\mu u^\nu$, $\Delta^{\mu\nu}_{\alpha\beta} \equiv \frac{1}{2}\Delta^{(\mu}{}_{\alpha}\Delta^{\nu)}{}_{\beta} - \frac{1}{3}\Delta^{\mu\nu}\Delta_{\alpha\beta}$

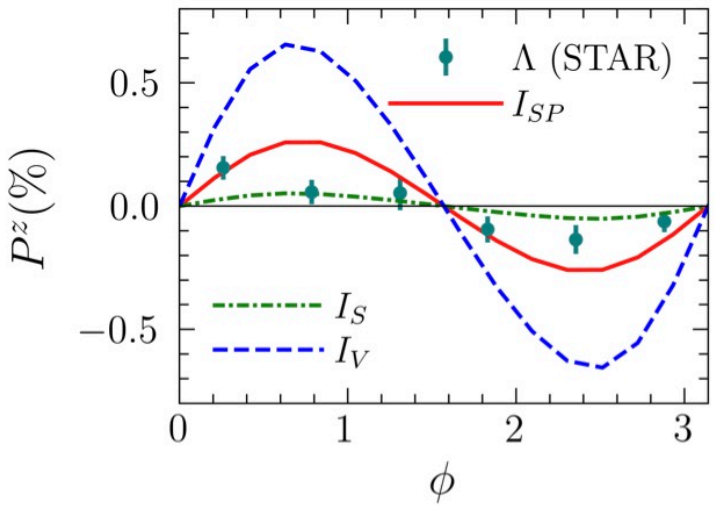
$\Longrightarrow$ dynamical solution of spin hydrodynamics

Sapna, S. K. Singh, D. Wagner,

PRC 112 (2025) 5, 054902

$\Longrightarrow$ spin hydrodynamics can describe local longitudinal polarization!

$\Longrightarrow$ D. Wagner's talk



Solving spin hydrodynamics

Equations of motion (EoM's) of spin hydrodynamics are solved perturbatively in $\hbar$

$\Longrightarrow$ Decompose $T^{\mu\nu} = \frac{1}{2} T^{(\mu\nu)} + \frac{1}{2} T^{[\mu\nu]}$

where: $\frac{1}{2} T^{(\mu\nu)} = \varepsilon\, u^\mu u^\nu - (P + \Pi)\Delta^{\mu\nu} + \pi^{\mu\nu} \sim \mathcal{O}(1)$

while: $\hbar\, S^{\mu,\nu\lambda} \sim \mathcal{O}(\hbar^2)$, $T^{[\mu\nu]} = -\frac{\hbar}{2} \int \mathrm{d}\Gamma\, \Sigma^{\mu\nu}_{\mathbf{k}\mathfrak{s}}\, \mathfrak{C}[f] \sim \mathcal{O}(\hbar^2)$

$\Sigma^{\mu\nu}_{\mathbf{k}\mathfrak{s}} \equiv -\frac{1}{m} \epsilon^{\mu\nu\alpha\beta} k_\alpha \mathfrak{s}_\beta$ dipole moment tensor, $\mathfrak{C}[f]$ collision term in Boltzmann equation

EoM's to $\mathcal{O}(\hbar)$: $\partial_\mu J^\mu = 0$, $\partial_\mu T^{(\mu\nu)} = 0$, plus relaxation equations for Π, n^μ, $\pi^{\mu\nu}$

Note: spin-polarization effects enter equation of state (EoS) at order $\mathcal{O}(\hbar^2)$!

$\Longrightarrow$ To order $\mathcal{O}(\hbar)$, EoM's are identical to EoM's of ordinary dissipative fluid dynamics, spin dynamics does not affect bulk evolution!

Spin dynamics at order $\mathcal{O}(\hbar^2)$: $\hbar\, \partial_\mu S^{\mu,\nu\lambda} = T^{[\lambda\nu]}$, $\tau_{\mathfrak{t}} \Delta^{\mu\nu}_{\alpha\beta} \dot{\mathfrak{t}}^{\alpha\beta} + \mathfrak{t}^{\mu\nu} = \frac{\mathfrak{d}}{T} \sigma^{\mu\nu} + \ldots$

$\Longrightarrow$ solve spin dynamics on background given by ordinary dissipative fluid dynamics!

However: EoM's of spin hydrodynamics are not solved self-consistently to order $\mathcal{O}(\hbar^2)$

$\Longrightarrow$ Need to account for $T^{[\mu\nu]} \sim \mathcal{O}(\hbar^2)$ in $\partial_\mu T^{\mu\nu} = 0$

$\Longrightarrow$ Need to account for $\mathcal{O}(\hbar^2)$ corrections in EoS $\Longrightarrow$ this talk!

Conservation laws

Charge	$\partial_\mu \hat{J}^\mu = 0$
Energy-momentum	$\partial_\mu \hat{T}^{\mu\nu} = 0$
Angular momentum	$\partial_\mu \hat{J}^{\mu,\nu\lambda} = 0$

with: $\hat{J}^{\mu,\nu\lambda} = x^\nu \hat{T}^{\mu\lambda} - x^\lambda \hat{T}^{\mu\nu} + \hbar \hat{S}^{\mu,\nu\lambda}$

$\Longrightarrow \quad \hbar \partial_\mu \hat{S}^{\mu,\nu\lambda} = \hat{T}^{[\lambda\nu]}$

Conserved charges

$$\hat{\mathbb{Q}} = \int_\Sigma \mathrm{d}\Sigma_\mu \hat{J}^\mu$$

$$\hat{\mathbb{P}}^\nu = \int_\Sigma \mathrm{d}\Sigma_\mu \hat{T}^{\mu\nu}$$

$$\hat{\mathbb{J}}^{\nu\lambda} = \int_\Sigma \mathrm{d}\Sigma_\mu \hat{J}^{\mu,\nu\lambda}$$

with Σ Cauchy hypersurface

remain invariant under:

Pseudo-gauge transformations

$$\hat{J}'^\mu = \hat{J}^\mu + \hbar \partial_\lambda \hat{B}^{\lambda\mu}$$

$$\hat{T}'^{\mu\nu} = \hat{T}^{\mu\nu} + \frac{\hbar}{2} \partial_\lambda \left(\hat{\Phi}^{\lambda,\mu\nu} - \hat{\Phi}^{\mu,\lambda\nu} - \hat{\Phi}^{\nu,\lambda\mu} \right)$$

$$\hat{S}'^{\mu,\nu\lambda} = \hat{S}^{\mu,\nu\lambda} - \hat{\Phi}^{\mu,\nu\lambda} + \hbar \partial_\rho \hat{Z}^{\nu\lambda,\mu\rho}$$

$\hat{B}^{\lambda\mu} = -\hat{B}^{\mu\lambda}$, $\hat{\Phi}^{\lambda,\mu\nu} = -\hat{\Phi}^{\lambda,\nu\mu}$, $\hat{Z}^{\nu\lambda,\mu\rho} = -\hat{Z}^{\lambda\nu,\mu\rho} = -\hat{Z}^{\nu\lambda,\rho\mu}$

$$\mathcal{Z} = \mathrm{Tr}\exp\left[-\int_\Sigma \mathrm{d}\Sigma_\mu \left(\hat{T}^{\mu\nu}\mathfrak{b}_\nu - \alpha\hat{J}^\mu - \frac{1}{2}\Omega_{\nu\lambda}\hat{J}^{\mu,\nu\lambda}\right)\right]$$

F. Becattini,
PRL 108 (2012) 244502

$\mathfrak{b}_\nu$, α, $\Omega_{\nu\lambda}$ Lagrange multipliers to control mean values of energy-momentum, charge, and angular momentum

(α reduced chemical potential, $\Omega_{\nu\lambda}$ reduced spin potential)

$\Longrightarrow$ introduce thermal 4-velocity $\beta_\nu = \mathfrak{b}_\nu + \Omega_{\nu\lambda}x^\lambda$

$$\mathcal{Z} = \mathrm{Tr}\exp\left[-\int_\Sigma \mathrm{d}\Sigma_\mu \left(\hat{T}^{\mu\nu}\beta_\nu - \alpha\hat{J}^\mu - \frac{\hbar}{2}\Omega_{\nu\lambda}\hat{S}^{\mu,\nu\lambda}\right)\right]$$

$\Longrightarrow$ Density operator $\hat{\rho} = \frac{1}{\mathcal{Z}}\exp\left[-\int_\Sigma \mathrm{d}\Sigma_\mu \left(\hat{T}^{\mu\nu}\beta_\nu - \alpha\hat{J}^\mu - \frac{\hbar}{2}\Omega_{\nu\lambda}\hat{S}^{\mu,\nu\lambda}\right)\right]$

$\Longrightarrow$ Define thermodynamic-potential 4-current ϕ^μ via $\ln\mathcal{Z} = \int_\Sigma \mathrm{d}\Sigma_\mu\phi^\mu - \text{v.e.v.}$

F. Becattini, D. Rindori, PRD 99 (2019) 125011

v.e.v. $= 0$ for Boltzmann statistics

$$\alpha = \text{const.}\,, \quad \mathfrak{b}_\nu = \text{const.}\,, \quad \Omega_{\nu\lambda} = \text{const.}$$

$\Longrightarrow$ thermal 4-velocity is Killing vector: $\xi_{\nu\lambda} = \frac{1}{2}\partial_{(\nu}\beta_{\lambda)} = 0$

$\Longrightarrow$ thermal vorticity equals reduced spin potential: $\varpi_{\nu\lambda} = -\frac{1}{2}\partial_{[\nu}\beta_{\lambda]} = \Omega_{\nu\lambda} = \text{const.}$

$\Longrightarrow$ define fluid 4-velocity u_ν via $\beta_\nu \equiv \beta u_\nu \quad \Longrightarrow \quad u \cdot u = 1$

$\Longrightarrow$ inverse temperature: $\beta \equiv \sqrt{\beta \cdot \beta} \equiv 1/T$

$\Longrightarrow$

$$\mathcal{Z} = \mathrm{Tr}\exp\left(-\mathfrak{b}_\nu \hat{\mathbb{P}}^\nu + \alpha\hat{\mathbb{Q}} + \frac{1}{2}\,\Omega_{\nu\lambda}\hat{\mathbb{J}}^{\nu\lambda}\right) \qquad \hat{\rho} = \frac{1}{\mathcal{Z}}\,\exp\left(-\mathfrak{b}_\nu \hat{\mathbb{P}}^\nu + \alpha\hat{\mathbb{Q}} + \frac{1}{2}\,\Omega_{\nu\lambda}\hat{\mathbb{J}}^{\nu\lambda}\right)$$

$$\mathbb{Q} = \mathrm{Tr}\left(\hat{\rho}\,\hat{\mathbb{Q}}\right) = \left.\frac{\partial \ln \mathcal{Z}}{\partial\alpha}\right|_{\mathfrak{b}_\rho,\Omega_{\sigma\tau}} = \int_\Sigma \mathrm{d}\Sigma_\mu \left.\frac{\partial\phi^\mu}{\partial\alpha}\right|_{\mathfrak{b}_\rho,\Omega_{\sigma\tau}}$$

$$\mathbb{P}^\nu = \mathrm{Tr}\left(\hat{\rho}\,\hat{\mathbb{P}}^\nu\right) = -\left.\frac{\partial \ln \mathcal{Z}}{\partial\mathfrak{b}_\nu}\right|_{\alpha,\Omega_{\sigma\tau}} = -\int_\Sigma \mathrm{d}\Sigma_\mu \left.\frac{\partial\phi^\mu}{\partial\mathfrak{b}_\nu}\right|_{\alpha,\Omega_{\sigma\tau}}$$

$$\mathbb{J}^{\nu\lambda} = \mathrm{Tr}\left(\hat{\rho}\,\hat{\mathbb{J}}^{\nu\lambda}\right) = \left.\frac{\partial \ln \mathcal{Z}}{\partial\Omega_{\nu\lambda}}\right|_{\alpha,\mathfrak{b}_\rho} = \int_\Sigma \mathrm{d}\Sigma_\mu \left.\frac{\partial\phi^\mu}{\partial\Omega_{\nu\lambda}}\right|_{\alpha,\mathfrak{b}_\rho}$$

$\Longrightarrow$ Define local conserved currents via

$$J^{\mu}(x) \equiv \frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\Sigma_{\mu}} = \mathrm{Tr}\left[\hat{\rho}\,\hat{J}^{\mu}(x)\right] = \left.\frac{\partial\phi^{\mu}(x)}{\partial\alpha}\right|_{\mathfrak{b}_{\rho},\Omega_{\sigma\tau}}$$

$$T^{\mu\nu}(x) \equiv \frac{\mathrm{d}\mathbb{P}^{\nu}}{\mathrm{d}\Sigma_{\mu}} = \mathrm{Tr}\left[\hat{\rho}\,\hat{T}^{\mu\nu}(x)\right] = -\left.\frac{\partial\phi^{\mu}(x)}{\partial\mathfrak{b}_{\nu}}\right|_{\alpha,\Omega_{\sigma\tau}} \qquad (\star)$$

$$J^{\mu,\nu\lambda}(x) \equiv \frac{\mathrm{d}\mathbb{J}^{\nu\lambda}}{\mathrm{d}\Sigma_{\mu}} = \mathrm{Tr}\left[\hat{\rho}\,\hat{J}^{\mu,\nu\lambda}(x)\right] = \left.\frac{\partial\phi^{\mu}(x)}{\partial\Omega_{\nu\lambda}}\right|_{\alpha,\mathfrak{b}_{\rho}}$$

$\Longrightarrow$ Replace $\mathfrak{b}_{\nu}$ by β_{ν} as independent thermodynamic variable

$\Longrightarrow$

$$J^{\mu}(x) = \left.\frac{\partial\phi^{\mu}(x)}{\partial\alpha}\right|_{\beta_{\rho},\Omega_{\sigma\tau}}$$

$$T^{\mu\nu}(x) = -\left.\frac{\partial\phi^{\mu}(x)}{\partial\beta_{\nu}}\right|_{\alpha,\Omega_{\sigma\tau}} \qquad (\star\star)$$

$$\hbar\, S^{\mu,\nu\lambda}(x) = \left.\frac{\partial\phi^{\mu}(x)}{\partial\Omega_{\nu\lambda}}\right|_{\alpha,\beta_{\rho}}$$

Caveat: holds up to contributions from boundary of Σ!

$\Longrightarrow$ M. Shokri's talk

$\Longrightarrow$ Equations $(\star), (\star\star)$ give rise to:

Gibbs-Duhem relation

$$\begin{aligned} \mathrm{d}\phi^\mu &= J^\mu \mathrm{d}\alpha - T^{\mu\nu}\mathrm{d}\mathfrak{b}_\nu + \frac{1}{2} J^{\mu,\nu\lambda}\mathrm{d}\Omega_{\nu\lambda} \\ &= J^\mu \mathrm{d}\alpha - T^{\mu\nu}\mathrm{d}\beta_\nu + \frac{\hbar}{2} S^{\mu,\nu\lambda}\mathrm{d}\Omega_{\nu\lambda} \end{aligned} \qquad (+)$$

Define entropy 4-current S^μ via:

$$S = -\mathrm{Tr}\,(\hat{\rho}\,\ln\hat{\rho}) = \int_\Sigma \mathrm{d}\Sigma_\mu\, S^\mu$$

Euler relation

$\Longrightarrow$

$$\begin{aligned} \phi^\mu &= S^\mu + J^\mu \alpha - T^{\mu\nu}\mathfrak{b}_\nu + \frac{1}{2} J^{\mu,\nu\lambda}\Omega_{\nu\lambda} \\ &= S^\mu + J^\mu \alpha - T^{\mu\nu}\beta_\nu + \frac{\hbar}{2} S^{\mu,\nu\lambda}\Omega_{\nu\lambda} \end{aligned} \qquad (\nabla)$$

As antisymmetric tensor, reduced spin potential can be decomposed as:

$$\Omega^{\mu\nu} = \beta \left(u^{[\mu} \varkappa^{\nu]} + \epsilon^{\mu\nu\alpha\beta} u_\alpha \omega_\beta \right)$$

$\varkappa^\mu = -T \Omega^{\mu\nu} u_\nu$ acceleration-like 4-vector ("electric"), $\varkappa^2 \equiv -\varkappa \cdot \varkappa > 0$

$\omega^\mu = \frac{T}{2} \epsilon^{\mu\nu\alpha\beta} u_\nu \Omega_{\alpha\beta}$ rotation-like 4-vector ("magnetic"), $\omega^2 \equiv -\omega \cdot \omega > 0$

$\Longrightarrow$ Define $L^\mu \equiv \epsilon^{\mu\nu\alpha\beta} u_\nu \varkappa_\alpha \omega_\beta$, $\ell^\mu \equiv L^\mu / \sqrt{-L \cdot L}$

One can show that

$$\phi^\mu = \beta P u^\mu + \beta P_\times \ell^\mu \quad (\square)$$ Daher, Wagner, Shokri, DHR [arXiv:2609.31819[nucl-th]]

In homogeneous equilibrium, $\Omega_{\nu\lambda} = \ell^\mu = 0$, $\beta = \text{const.}$, $P(\alpha, \beta) = \text{const.}$

$\Longrightarrow$ $\ln \mathcal{Z} = \int_\Sigma d\Sigma_\mu \, \phi^\mu = \beta P \int_\Sigma d\Sigma_\mu \, u^\mu \equiv \beta P V$

$\Longrightarrow$ P is thermodynamic pressure!

In accelerating and rotating equilibrium, however,

$\beta = \beta(x)$, $P = P(x)$, $P_\times = P_\times(x)$, $u^\mu = u^\mu(x)$, $\ell^\mu = \ell^\mu(x)$

Longitudinal and transverse Euler relations

Projecting Euler relation (∇) onto u^μ

$\Longrightarrow$ **Longitudinal Euler relation**

$$\varepsilon + P = T\,s + \mu\,n + \frac{\hbar}{2}\,\omega_{\nu\lambda} S^{\nu\lambda}$$

$\varepsilon \equiv u_\mu T^{\mu\nu} u_\nu$ energy density
$s \equiv u_\mu S^\mu$ entropy density
$n \equiv u_\mu J^\mu$ charge density
$S^{\nu\lambda} \equiv u_\mu S^{\mu,\nu\lambda}$ spin tensor
$\mu = T\alpha$ chemical potential
$\omega_{\nu\lambda} = T\Omega_{\nu\lambda}$ spin potential

Projecting Euler relation (∇) onto ℓ^μ

$\Longrightarrow$ **Transverse Euler relation**

$$q_\times + P_\times = T\,s_\times + \mu\,n_\times + \frac{\hbar}{2}\,\omega_{\nu\lambda} S_\times^{\nu\lambda}$$

$q_\times \equiv -\ell_\mu T^{\mu\nu} u_\nu$
$s_\times \equiv -\ell_\mu S^\mu$
$n_\times \equiv -\ell_\mu J^\mu$
$S_\times^{\nu\lambda} \equiv -\ell_\mu S^{\mu,\nu\lambda}$

cf. also W. Florkowski, M. Hontarenko, PRL 134 (2025) 082302

Projecting Gibbs-Duhem relation $(+)$ onto u^μ

Longitudinal Gibbs-Duhem relation

$$\Longrightarrow \quad \mathrm{d}P = s\,\mathrm{d}T + n\,\mathrm{d}\mu + \frac{\hbar}{2} S^{\nu\lambda}\mathrm{d}\omega_{\nu\lambda} + \left(P_\times \ell^\mu - u_\nu T^{\nu\lambda}\Delta_\lambda{}^\mu\right)\mathrm{d}u_\mu$$

New term! cf. A.D. Gallegos, U. Gursoy, A. Yarom, [arXiv:2203.05044 [hep-th]]

$\Longrightarrow$ fluid 4-velocity becomes independent thermodynamic variable!

$$\Longrightarrow \quad s = \left.\frac{\partial P}{\partial T}\right|_{\mu,\omega_{\sigma\tau},u_\rho}, \quad n = \left.\frac{\partial P}{\partial \mu}\right|_{T,\omega_{\sigma\tau},u_\rho}, \quad \hbar S^{\nu\lambda} = \left.\frac{\partial P}{\partial \omega_{\nu\lambda}}\right|_{T,\mu,u_\rho}$$

Note: T, $u_\rho = \text{const.} \iff \beta_\rho = \text{const.} \iff \varpi_{\sigma\tau} = \text{const.}$!

Projecting Gibbs-Duhem relation $(+)$ onto ℓ^μ

Transverse Gibbs-Duhem relation

$$\Longrightarrow \quad \mathrm{d}P_\times = s_\times\,\mathrm{d}T + n_\times\,\mathrm{d}\mu + \frac{\hbar}{2} S_\times^{\nu\lambda}\mathrm{d}\omega_{\nu\lambda} + \left(P\ell^\mu + \ell_\nu T^{\nu\lambda}\Delta_\lambda{}^\mu\right)\mathrm{d}u_\mu$$

With $s = \frac{\partial P}{\partial T}\big|_{\mu,\omega_{\sigma\tau}}$ and the longitudinal Euler relation

$$\Longrightarrow \quad T^2 \left.\frac{\partial (P/T)}{\partial T}\right|_{\mu,\omega_{\sigma\tau},u_\rho} = -P + s = \varepsilon - \mu n - \frac{\hbar}{2}\,\omega_{\nu\lambda} S^{\nu\lambda}$$

$$\Longrightarrow \quad P = T \int_0^T \frac{\mathrm{d}T'}{T'^2} \left(\varepsilon - \mu n - \frac{\hbar}{2}\,\omega_{\nu\lambda} S^{\nu\lambda}\right)_{\mu,\omega_{\sigma\tau},u_\rho}$$

F. Becattini, A. Daher, X.-L. Sheng, PLB 850 (2024) 138533 (I1)

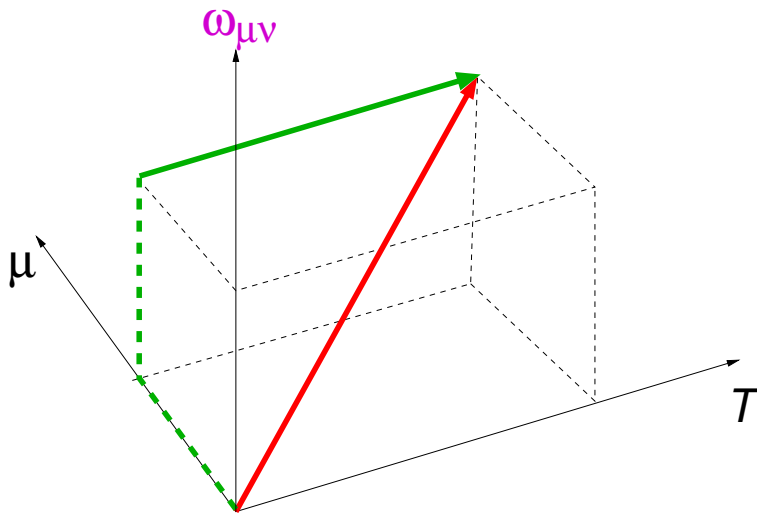
On the other hand, with Eqs. ($\star\star$), ($\square$) and longitudinal Gibbs-Duhem relation

$$\Longrightarrow \quad \varepsilon = -u_\mu u_\nu \left.\frac{\partial \phi^\mu}{\partial \beta_\nu}\right|_{\alpha,\Omega_{\sigma\tau}} = - \left.\frac{\partial (\beta P)}{\partial \beta}\right|_{\alpha,\Omega_{\sigma\tau},u_\rho}$$

$$\Longrightarrow \quad P = \frac{1}{\beta} \int_\beta^\infty \mathrm{d}\beta' \; \varepsilon|_{\alpha,\Omega_{\sigma\tau},u_\rho}$$

V.E. Ambrus, A. Gecić, [arXiv:2509.17640 [hep-th]] (I2)

Visualizing integration in $(T, \mu, \omega_{\mu\nu})$ space



$$(I1)\ P = T \int_0^T \frac{\mathrm{d}T'}{T'^2} \left(\varepsilon - \mu n - \frac{\hbar}{2}\, \omega_{\nu\lambda} S^{\nu\lambda} \right)_{\mu, \omega_{\sigma\tau}, u_\rho}$$

$$(I2)\ P = \frac{1}{\beta} \int_\beta^\infty \mathrm{d}\beta'\ \varepsilon|_{\alpha, \Omega_{\sigma\tau}, u_\rho}$$

P twice cont. differentiable $\Longleftrightarrow$ Maxwell relations $\Longleftrightarrow$ integration path-independent

Maxwell relations

$$\left.\frac{\partial n}{\partial T}\right|_{\omega_{\sigma\tau}} = \left.\frac{\partial s}{\partial \mu}\right|_{\omega_{\sigma\tau}}, \quad \left.\frac{\partial s}{\partial \omega_{\mu\nu}}\right|_{\mu} = \hbar \left.\frac{\partial S^{\mu\nu}}{\partial T}\right|_{\mu}, \quad \left.\frac{\partial n}{\partial \omega_{\mu\nu}}\right|_{T} = \hbar \left.\frac{\partial S^{\mu\nu}}{\partial \mu}\right|_{T}$$

$\Longrightarrow$ Provided: (i) Maxwell relations hold,

(ii) $P(0, \mu, \omega_{\mu\nu}) = 0$ (which holds, e.g., for Boltzmann statistics),

$\Longrightarrow$ both integral identities yield identical results!

Note: not automatically fulfilled in every pseudo-gauge!

Wigner function

$$W_{\alpha\beta}(x,k) = \int \mathrm{d}^4 y\, e^{-ik\cdot y} \left\langle : \hat{\rho}\, \hat{\bar{\psi}}_\beta \left(x + \frac{y}{2}\right) \hat{\psi}_\alpha \left(x - \frac{y}{2}\right) : \right\rangle$$

Dyson-Schwinger equation

$$\left[k^\mu \left(\gamma_\mu + \frac{i\hbar}{2}\,\partial_\mu\right) - m\right] W(x,k) = I_{\text{coll}}$$

In equilibrium, $I_{\text{coll}} = 0$

Mass-shell constraint and equation of motion

$$\Longrightarrow \quad \left(k^2 - m^2 - \frac{\hbar^2}{4}\Box\right) W(x,k) = 0\,, \quad k\cdot\partial W(x,k) = 0 \qquad (\lozenge)$$

Clifford decomposition

$$W = \frac{1}{4}\left(\mathcal{F} + i\gamma_5\mathcal{P} + \gamma^\mu\mathcal{V}_\mu + \gamma_5\gamma^\mu\mathcal{A}_\mu + \frac{i}{4}\,[\gamma^\mu,\gamma^\nu]\,\mathcal{S}_{\mu\nu}\right)$$

$\Longrightarrow$ From Eqs. $(\lozenge)$: only $\mathcal{F}$ and components of $\mathcal{A}_\mu$ transverse to k^μ are independent!

Assign spin 4-vector $\mathfrak{s}^\mu$ to (anti-)particles, extend phase space by spin d.o.f.'s:

$$\Longrightarrow \quad \mathrm{d}\Gamma \equiv \mathrm{d}K\,\mathrm{d}S(K)\,, \quad \mathrm{d}K \equiv \frac{\mathrm{d}^4k}{(2\pi\hbar)^4}\,, \quad \mathrm{d}S(K) \equiv \sqrt{\frac{k^2}{3\pi^2}}\,\mathrm{d}^4\mathfrak{s}\,\delta(k\cdot\mathfrak{s})\delta(\mathfrak{s}\cdot\mathfrak{s}+3)$$

$\Longrightarrow$ **Distribution function**

$$\mathfrak{f}(x,k,\mathfrak{s}) = \frac{1}{2}\left[\mathcal{F}(x,k) - \mathfrak{s}^\mu \mathcal{A}_\mu(x,k)\right]$$

$$\mathcal{F}(x,k) \equiv \int \mathrm{d}S(K)\,\mathfrak{f}(x,k,\mathfrak{s})$$

$$\mathcal{A}_\mu(x,k) \equiv \int \mathrm{d}S(K)\,\mathfrak{s}_\mu\,\mathfrak{f}(x,k,\mathfrak{s}) \qquad (\clubsuit)$$

For on-shell particles:

$$\mathfrak{f}(x,k,\mathfrak{s}) = 4\pi\hbar\, m\,\theta(k_0)\,\delta\left(k^2 - m^2 - F(k)\right)\left[f^+(x,k,\mathfrak{s}) + f^-(x,k,\mathfrak{s})\right]$$

Assume global equilibrium and Boltzmann statistics:

$$f^\pm_{\mathrm{eq}}(x,k,\mathfrak{s}) \equiv \exp\left(-\beta_\mu k^\mu \pm \alpha + \frac{\hbar}{4}\,\Omega_{\mu\nu}\Sigma^{\mu\nu}_{k\mathfrak{s}}\right)$$

$\Sigma^{\mu\nu}_{k\mathfrak{s}} \equiv -\frac{1}{m}\,\epsilon^{\mu\nu\alpha\beta}\,k_\alpha\,\mathfrak{s}_\beta$
dipole-moment tensor

$\Longrightarrow$ From mass-shell constraint $(\diamondsuit)$: $F_{\mathrm{eq}}(k) \equiv -\frac{\hbar^2}{4}\varpi_\mu{}^\lambda\varpi_{\lambda\nu}k^\mu k^\nu$

$\Longrightarrow$ modified mass-shell condition:

$$m^2 = k^\mu\left(g_{\mu\nu} + \frac{\hbar^2}{4}\varpi_\mu{}^\lambda\varpi_{\lambda\nu}\right)k^\nu$$

Canonical

$$J^\mu_C = \frac{1}{m}\int \mathrm{d}\Gamma \left[k^\mu\left(1-\frac{\hbar^2}{4m^2}\Box\right)+\frac{\hbar}{2}\Sigma^{\mu\lambda}_{k\mathfrak{s}}\partial_\lambda\right]\frac{\partial \mathfrak{f}(x,k,\mathfrak{s})}{\partial\alpha}$$

$$T^{\mu\nu}_C = \frac{1}{m}\int \mathrm{d}\Gamma\, k^\nu \left[k^\mu\left(1-\frac{\hbar^2}{4m^2}\Box\right)+\frac{\hbar}{2}\Sigma^{\mu\lambda}_{k\mathfrak{s}}\partial_\lambda\right]\mathfrak{f}(x,k,\mathfrak{s})$$

$$S^{\lambda,\mu\nu}_C = -\frac{1}{2}\epsilon^{\lambda\mu\nu\alpha}\int \mathrm{d}K\,\mathfrak{s}_\alpha\,\mathfrak{f}(x,k,\mathfrak{s})$$

GLW

$$J^\mu_{GLW} = \frac{1}{m}\int \mathrm{d}\Gamma\, k^\mu\,\frac{\partial \mathfrak{f}(x,k,\mathfrak{s})}{\partial\alpha}$$

$$T^{\mu\nu}_{GLW} = \frac{1}{m}\int \mathrm{d}\Gamma\, k^\mu k^\nu\,\mathfrak{f}(x,k,\mathfrak{s})$$

$$S^{\lambda,\mu\nu}_{GLW} = \frac{1}{2m}\int \mathrm{d}\Gamma\, k^\lambda\left[\Sigma^{\mu\nu}_{k\mathfrak{s}} - \frac{\hbar}{2m}\left(k^\mu\partial^\nu - k^\nu\partial^\mu\right)\right]\mathfrak{f}(x,k,\mathfrak{s})$$

$\Longrightarrow$ Blue term in GLW pseudo-gauge can be eliminated by a further pseudo-gauge transformation! D. Wagner, PRD 111 (2025) 016008

kinetic theory

$\Longrightarrow$

$$J^\mu = \frac{1}{m}\int \mathrm{d}\Gamma\, k^\mu\,\frac{\partial \mathfrak{f}(x,k,\mathfrak{s})}{\partial\alpha}$$

$$T^{\mu\nu} = \frac{1}{m}\int \mathrm{d}\Gamma\, k^\mu k^\nu\,\mathfrak{f}(x,k,\mathfrak{s})$$

$$S^{\lambda,\mu\nu} = \frac{1}{2m}\int \mathrm{d}\Gamma\, k^\lambda\,\Sigma^{\mu\nu}_{k\mathfrak{s}}\,\mathfrak{f}(x,k,\mathfrak{s})$$

cf. W. Israel, Lett. Nuovo Cim. 7 (1973) 860

Note: although in global equilibrium $\Omega_{\mu\nu} \equiv \varpi_{\mu\nu}$, keep them distinct at first

Reason: partial derivatives of pressure performed at $\varpi_{\mu\nu} = \text{const.}$

$\Longrightarrow$ also T', β' integrations performed at $\varpi_{\mu\nu} = \text{const.}$

$\Longrightarrow$ Decompose $\varpi^{\mu\nu} = \beta \left(u^{[\mu} a^{\nu]} + \epsilon^{\mu\nu\alpha\beta} u_\alpha b_\beta \right)$, $a^2 \equiv -a \cdot a$, $b^2 \equiv -b \cdot b$

$\Longrightarrow$ General structure of conserved currents in kinetic-theory pseudogauge:

$$J^\mu = n\, u^\mu + \hbar^2 n^\mu_\times$$

$$T^{\mu\nu} = \varepsilon\, u^\mu u^\nu - P\, \Delta^{\mu\nu} + \hbar^2 q^{(\mu}_\times u^{\nu)} + \hbar^2 \Pi_\Omega \left[\varkappa^\mu \varkappa^\nu + \omega^\mu \omega^\nu + \Delta^{\mu\nu} \left(\varkappa^2 + \omega^2 \right) \right] + \hbar^2 \Pi_\varpi \left[a^\mu a^\nu + b^\mu b^\nu + \Delta^{\mu\nu} \left(a^2 + b^2 \right) \right]$$

$$\hbar\, S^{\lambda,\mu\nu} = \hbar^2 \left[u^\lambda \left(A\, u^{[\mu} \varkappa^{\nu]} + B\, \epsilon^{\mu\nu\alpha\beta} u_\alpha \omega_\beta \right) + C\, \Delta^{\lambda[\mu} \varkappa^{\nu]} + D\, u^{[\mu} \epsilon^{\nu]\lambda\alpha\beta} u_\alpha \omega_\beta \right]$$

$$n = n_0 + \hbar^2 \left[n_{2\varkappa}\, \varkappa^2 + n_{2\omega}\, \omega^2 + n_{2a}\, a^2 + n_{2b}\, b^2 \right]$$

$$\varepsilon = \varepsilon_0 + \hbar^2 \left[\varepsilon_{2\varkappa}\, \varkappa^2 + \varepsilon_{2\omega}\, \omega^2 + \varepsilon_{2a}\, a^2 + \varepsilon_{2b}\, b^2 \right]$$

$$n^\mu_\times = n_{\times L}\, L^\mu + n_{\times K}\, K^\mu, \quad q^\mu_\times = q_{\times L}\, L^\mu + q_{\times K}\, K^\mu, \quad K^\mu \equiv \epsilon^{\mu\nu\alpha\beta} u_\nu a_\alpha b_\beta$$

where n_0, $n_{2\varkappa}$, $n_{2\omega}$, n_{2a}, n_{2b}, ε_0, $\varepsilon_{2\varkappa}$, $\varepsilon_{2\omega}$, ε_{2a}, ε_{2b},

$n_{\times L}$, $n_{\times K}$, $q_{\times L}$, $q_{\times K}$, Π_Ω, Π_ϖ, A, B, C, D are functions of α, β

Consequences:

(i) $T^{\mu\nu}u_\nu = \varepsilon u^\mu + \hbar^2 q^\mu_\times$

$\Longrightarrow$ ε is not time-like eigenvalue of $T^{\mu\nu}$, u^μ is not corresponding eigenvector

$\Longrightarrow$ u^μ is not fluid 4-velocity in Landau frame

(ii) with ε, n, $S^{\mu\nu} \equiv u_\lambda S^{\lambda,\mu\nu}$, compute thermodynamic pressure P from Eqs. (I1), (I2)

$\Longrightarrow$ agreement!

(iii) $n = \beta P \tanh\alpha$ $\Longrightarrow$ ideal gas EoS!

(iv) check validity of $n = \left.\dfrac{\partial P}{\partial \mu}\right|_{T,\omega_{\sigma\tau},u_\rho}$, $\hbar\, S^{\mu\nu} = \left.\dfrac{\partial P}{\partial \omega_{\mu\nu}}\right|_{T,\mu,u_\rho}$

$\Longrightarrow$ fulfilled!

(v) check consistency: $s = \left.\dfrac{\partial P}{\partial T}\right|_{\mu,\omega_{\sigma\tau},\varpi_{\sigma\tau}} \equiv \beta\left(\varepsilon + P - \mu n - \dfrac{\hbar}{2}\,\omega_{\nu\lambda}S^{\nu\lambda}\right)$

$\Longrightarrow$ fulfilled!

(vi) check validity of Maxwell relations

$\Longrightarrow$ fulfilled!

$\Longrightarrow$ kinetic-theory pseudo-gauge is thermodynamically consistent!

GLW pseudo-gauge:

(i) Since $n_{GLW} = n$, but $S^{\nu\lambda}_{GLW} \neq S^{\nu\lambda}$, third Maxwell relation not fulfilled!

(ii) Since $T^{\mu\nu}_{GLW} = T^{\mu\nu} \implies \varepsilon_{GLW} = \varepsilon \implies P_{GLW} = P$ if computed from Eq. (I2), but because $S^{\nu\lambda}_{GLW} \neq S^{\nu\lambda}$, $P_{GLW} \neq P$ if computed from Eq. (I1)

$\implies$ GLW pseudo-gauge is thermodynamically inconsistent!

Canonical pseudo-gauge:

(i) If $\Omega_{\mu\nu}$ and $\varpi_{\mu\nu}$ are strictly kept distinct, canonical pseudo-gauge is thermodynamically inconsistent!

(ii) However: by an educated application of $\Omega_{\mu\nu} = \varpi_{\mu\nu}$ in some selected terms, it can be made thermodynamically consistent!
(for details, see A. Daher, D. Wagner, M.Shokri, DHR [arXiv:2609.31819[nucl-th]])

Conclusions and Outlook

- Derived thermodynamic relations for system in global equilibrium with acceleration and rotation

  $\Longrightarrow$ thermodynamic-potential 4-current $\quad \phi^\mu = \beta\, P\, u^\mu + \beta\, P_\times\, \ell^\mu$

  $\Longrightarrow$ longitudinal and transverse Euler and Gibbs-Duhem relations

  $\Longrightarrow$ fluid 4-velocity becomes independent thermodynamic variable

- Maxwell relations hold $\Longleftrightarrow$ integral identities for P yield identical results
- Explicitly computed conserved currents, thermodynamical quantities for massive spin-1/2 Boltzmann particles in kinetic-theory, GLW, canonical pseudo-gauge
- In global equilibrium, $\mathcal{Z}$ pseudo-gauge invariant,
  while local thermodynamic quantities (n, ε, P, s, $S^{\mu\nu}$) depend on pseudo-gauge
- Mass-shell modification at order $\mathcal{O}(\hbar^2)$: $\quad m^2 = k^\mu \left(g_{\mu\nu} + \frac{\hbar^2}{4} \varpi_\mu{}^\lambda \varpi_{\lambda\nu} \right) k^\nu$
- Thermodynamic consistency:
  - kinetic-theory pseudo-gauge ✓
  - GLW pseudo-gauge –
  - canonical pseudo-gauge (✓)

Outlook: Investigate role of boundary terms for thermodynamic consistency