# Out-of-equilibrium CME and CSE from lattice QCD

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# Anomalous transport

- Axial anomaly + $\begin{matrix}\text{EM fields}\\ \text{Vorticity}\end{matrix}$ $\rightarrow$ **Anomalous transport phenomena**
- EM fields:
  - Chiral Magnetic Effect (CME)
  - Chiral Separation Effect (CSE)
  - Chiral Magnetic Wave (CMW)
  - ...
- Rotation $\rightarrow$ EGV Wed. 11:55
  - Chiral Vortical Effect (CVE)
  - Axial Vortical Effect (AVE)
  - ...
- Experimental detection in heavy-ion collisions and condensed matter systems, review Kharzeev, Liao, Tribedy '24
- Objective: use Lattice QCD to understand CME/CSE properties in QCD

# Reminder

- We consider homogeneous magnetic field $\vec{B} = B\hat{e}_3$

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- Several groups study them using lattice simulations:

  Yamamoto '11  Polikarpov et al. '11  Buividovich, Smith, Von Smekal '21

  Astrakhantsev et al. '19  Buividovich '25 ...

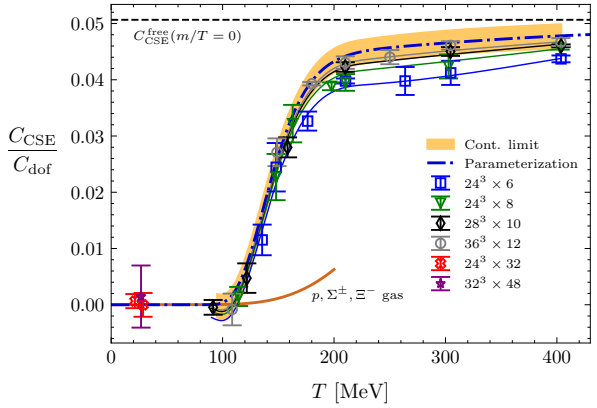
- We study 2-pt functions $\leftrightarrow$ derivatives wrt chemical potentials at finite $B$

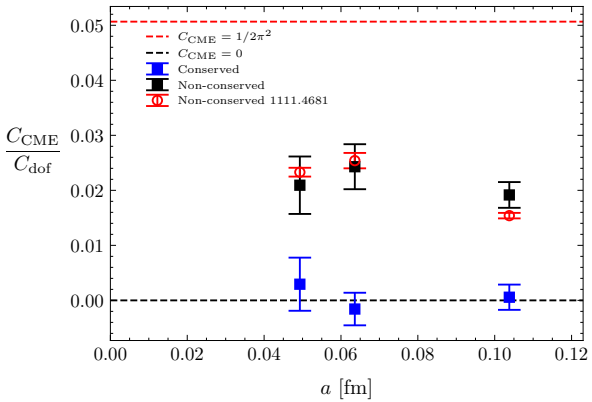
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# Our previous works

## CSE conductivity in physical QCD

Brandt, Endrődi, EGV, Markó '23




## Absence of CME in equilibrium QCD

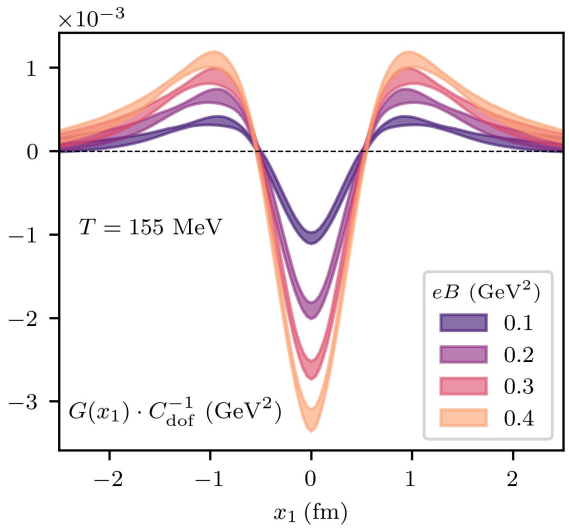
Brandt, Endrődi, EGV, Markó '24

## Local CME with inhomogeneous $B$

Brandt, Endrődi, EGV, Markó, Valois '24

## Initial condition for HICs

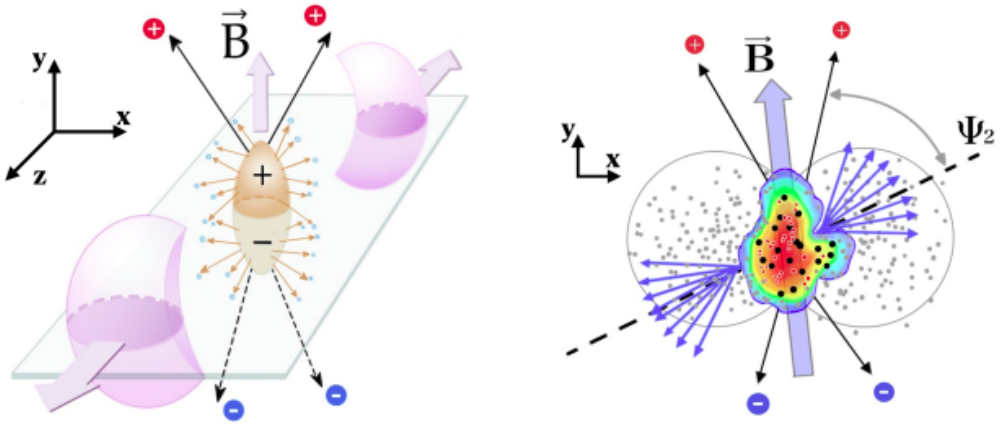
with Constantin, Dash, Elfner, in prep.



# Out-of-equilibrium

- So far we concentrated on a *global thermal equilibrium* system.
- In experiments out-of-equilibrium effects are very relevant

Kharzeev, Liao,Tribedy '24




- Goal :

  Understand out-of-equilibrium CME and CSE using lattice QCD

# Real-time correlators

- Euclidean correlator (accessible from lattice)

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- Imaginary part of $G_R \to$ Spectral function $\sim$ dissipation

$$\rho_{AB}(\omega) = 2\,\mathrm{Im}\; G^R_{AB}(\omega)$$

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- Kramers-Kronig (KK) relations

$$G^E_{AB}(\omega_n) = \int_{-\infty}^{\infty} \frac{\mathrm{d}\omega'}{2\pi} \frac{\rho_{AB}(\omega')}{\omega' - i\omega_n}$$

$$G^R_{AB}(\omega) = \lim_{\epsilon\to 0^+} \int_{-\infty}^{\infty} \frac{\mathrm{d}\omega'}{2\pi} \frac{\rho_{AB}(\omega')}{\omega' - \omega - i\epsilon}$$

# Example: vector-vector correlator

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$$G^E_{\mu\nu}(\tau) = \left\langle J_\mu(-i\tau) J_\nu(0) \right\rangle$$

- Related to equilibrium susceptibility:

$$\chi = \frac{T}{V} \left. \frac{\partial^2 \log \mathcal{Z}}{\partial \mu^2} \right|_{\mu=0} = \frac{1}{T^3} \widetilde{G}^E_{44}(i\omega_n = 0)$$

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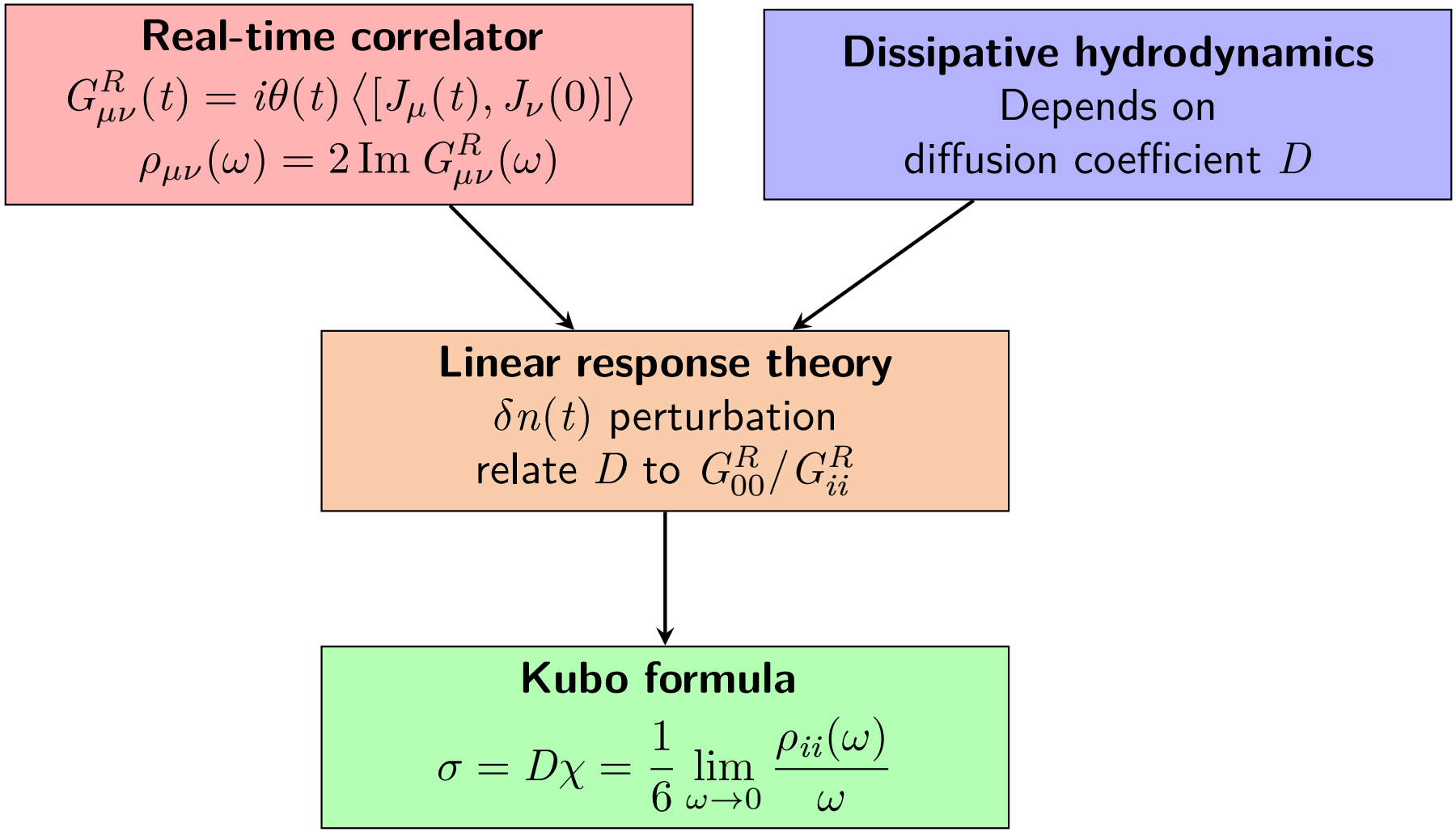
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$$G^R_{\mu\nu}(t) = i\theta(t) \left\langle [J_\mu(t), J_\nu(0)] \right\rangle$$

- Kubo formula for electric conductivity:

$$\sigma = \frac{1}{6} \lim_{\omega \to 0} \frac{\rho_{ii}(\omega)}{\omega}$$

# How to derive Kubo formulas?




- Well-known derivation
- Used to calculate $\sigma$ in lattice QCD

# Non-dissipative transport

- Real part of $G_R \rightarrow$ relevant for non-dissipative transport
- **CME**

$$G^R_{\mathrm{CME}}(t) = i\theta(t) \left\langle [J_{05}(t), J_3(0)] \right\rangle_B$$

- **CSE**

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- Out-of-equilibrium non-dissipative coefficients:

$$K_{\mathrm{CME/CSE}} = \frac{1}{eBT} \operatorname{Re} \widetilde{G}^R_{\mathrm{CME/CSE}}(\omega = 0)$$

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- We consider the following "analogy":
  - CSE $\leftrightarrow$ density-density $G_{00}$
  - CME $\leftrightarrow$ current-current $G_{ii}$

# CSE

# 1-loop density-density

- Free fermions $\rightarrow$ 1-loop exact $\rho$
- **Density-density correlator**: e.g. Aarts, Nikolaev '21

$$\rho_{00}(\omega) = 2\pi\chi\left(\frac{m}{T}\right)\omega\delta(\omega)$$

- Using KK relations $\rightarrow$ $G^E_{44}(\tau)$ is $\tau$-independent

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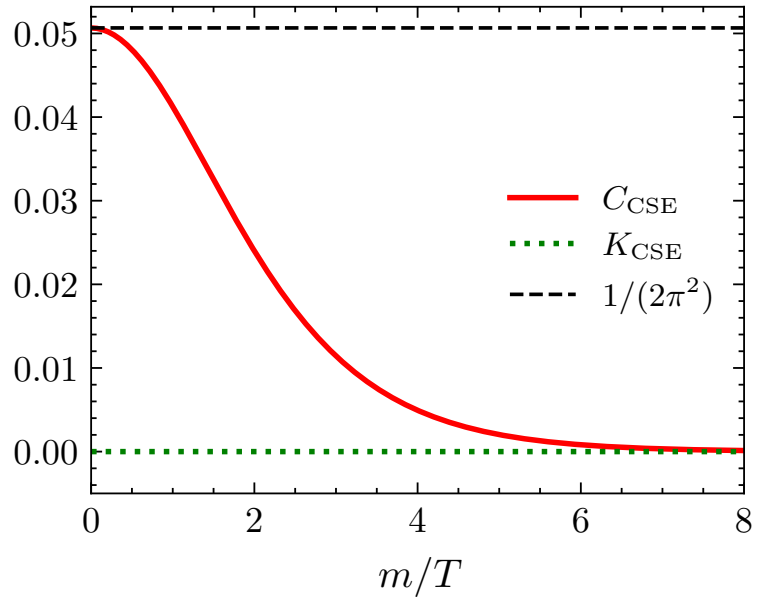
$$\rho_{00}(\omega) = 2\pi\chi\left(\frac{m}{T}\right)\omega\delta(\omega)$$

- Using KK relations $\rightarrow$ $G^E_{44}(\tau)$ is $\tau$-independent
- Interactions: delta protected by charge conservation Arnold, Moore, Yaffe '00
- Delta is present also in QCD $\rightarrow$ $G^E_{44}(\tau)$ is $\tau$-independent also in QCD
- Coefficient $\chi$ is the thermodynamic susceptibility

# 1-loop CSE

- Free fermions Brandt, Endrődi, EGV, Markó, Valois, PoS Lattice '24
- **CSE**:

$$\frac{1}{eB}\,\rho_{\mathrm{CSE}}(\omega) = 2\pi\, C_{\mathrm{CSE}}\left(\frac{m}{T}\right)\omega\delta(\omega)$$




- Using KK relations

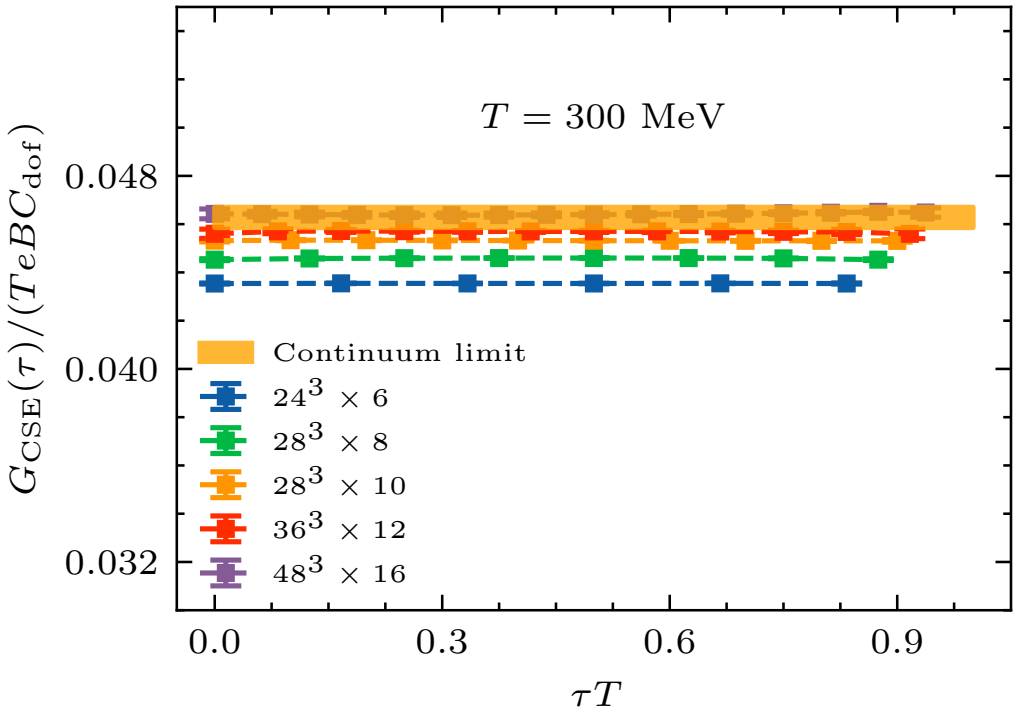
$$C_{\mathrm{CSE}} = \frac{1}{eBT}\,\widetilde{G}^{E}_{\mathrm{CSE}}(i\omega_n = 0) \neq 0$$

$$K_{\mathrm{CSE}} = \frac{1}{eBT}\,\mathrm{Re}\,\widetilde{G}^{R}_{\mathrm{CSE}}(\omega = 0) = 0$$

- $G^{E}_{\mathrm{CSE}}(\tau)$ is $\tau$-independent
- Charge conservation suggests this also holds in QCD

# Out-of-equilibrium CSE in QCD

- $G^E_{\rm CSE}(\tau)$ is $\tau$-independent
- $\rho_{\rm CSE}(\omega)$ only has the delta term




- In QCD:

$$C_{\rm CSE} \to$$ Brandt, Endrődi, EGV, Markó '23

$$K_{\rm CSE} = 0$$

# CME

# 1-loop current-current

- **Electric conductivity**: e.g. Aarts, Nikolaev '21

$$\rho_{ii}(\omega) = 2\pi\,\chi\left(\frac{m}{T}\right)\omega\delta(\omega) + \Theta(\omega^2-4m^2)\frac{m^2}{4\pi^2}\sqrt{\frac{\omega^2-4m^2}{\omega^2}}(\omega^2+2m^2)\left[1-2n_F\left(\frac{\omega}{2}\right)\right]$$

- $\sigma \to \infty \Rightarrow$ resummation required Arnold, Moore, Yaffe '00

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- Delta smears in the presence of interactions

$$\omega\delta(\omega) \to A\frac{\gamma\omega}{\omega^2+\gamma^2}$$

- Dissipative conductivity $\sigma \propto A/\gamma$ finite in QCD

# CME

- What about CME? We need real part of $G^R_{\mathrm{CME}}$
- Anomaly argument for free massless fermions

$$\partial_0 J_{05}(t) = \frac{1}{2\pi^2}\vec{E}\cdot\vec{B}$$

- Taking derivative wrt $A_3(t) = E_3 t$ and Fourier transform

$$\mathrm{Re}\,\widetilde{G}^R_{\mathrm{CME}}(\omega = 0) = \frac{1}{2\pi^2}B$$

- Coefficient fixed by the anomaly (free massless fermions)

$$K_{\mathrm{CME}} = \frac{1}{2\pi^2}$$

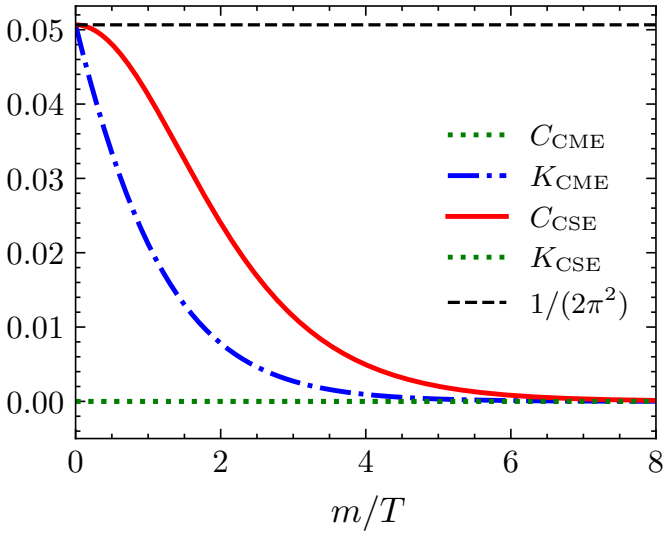
# 1-loop CME

- **CME**:

$$\frac{1}{eB}\rho_{\rm CME}(\omega) = \alpha_{\rm CME}(m)\omega\delta(\omega)$$

$$+\Theta(\omega^2-4m^2)\frac{1}{\pi}\frac{2m^2\tanh\left(\frac{|\omega|}{4T}\right)}{\omega\sqrt{\omega^2-4m^2}}$$

$$+\text{Pauli-Villars}$$



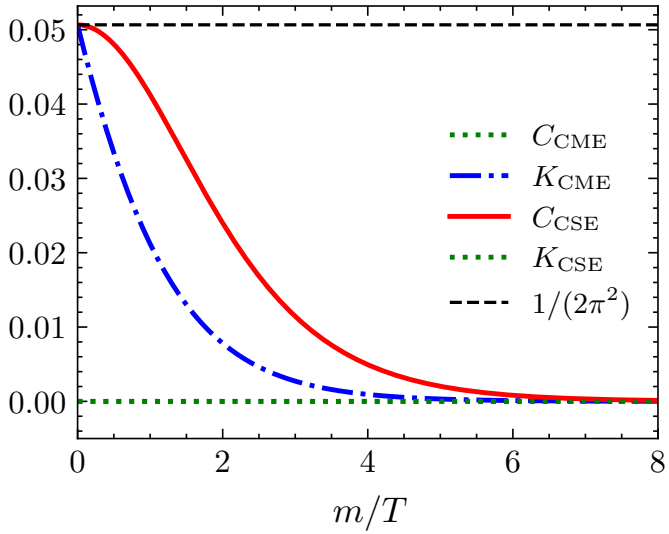
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- Using KK relations

$$C_{\rm CME} = \frac{1}{eBT}\widetilde{G}^E_{\rm CME}(i\omega_n = 0) = 0$$

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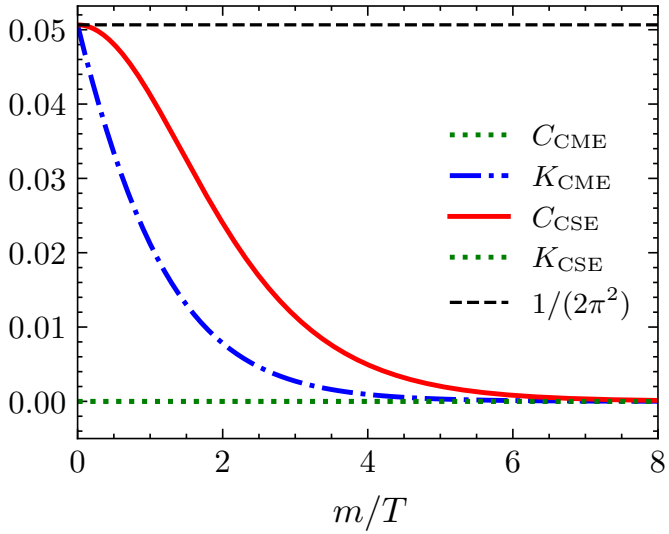
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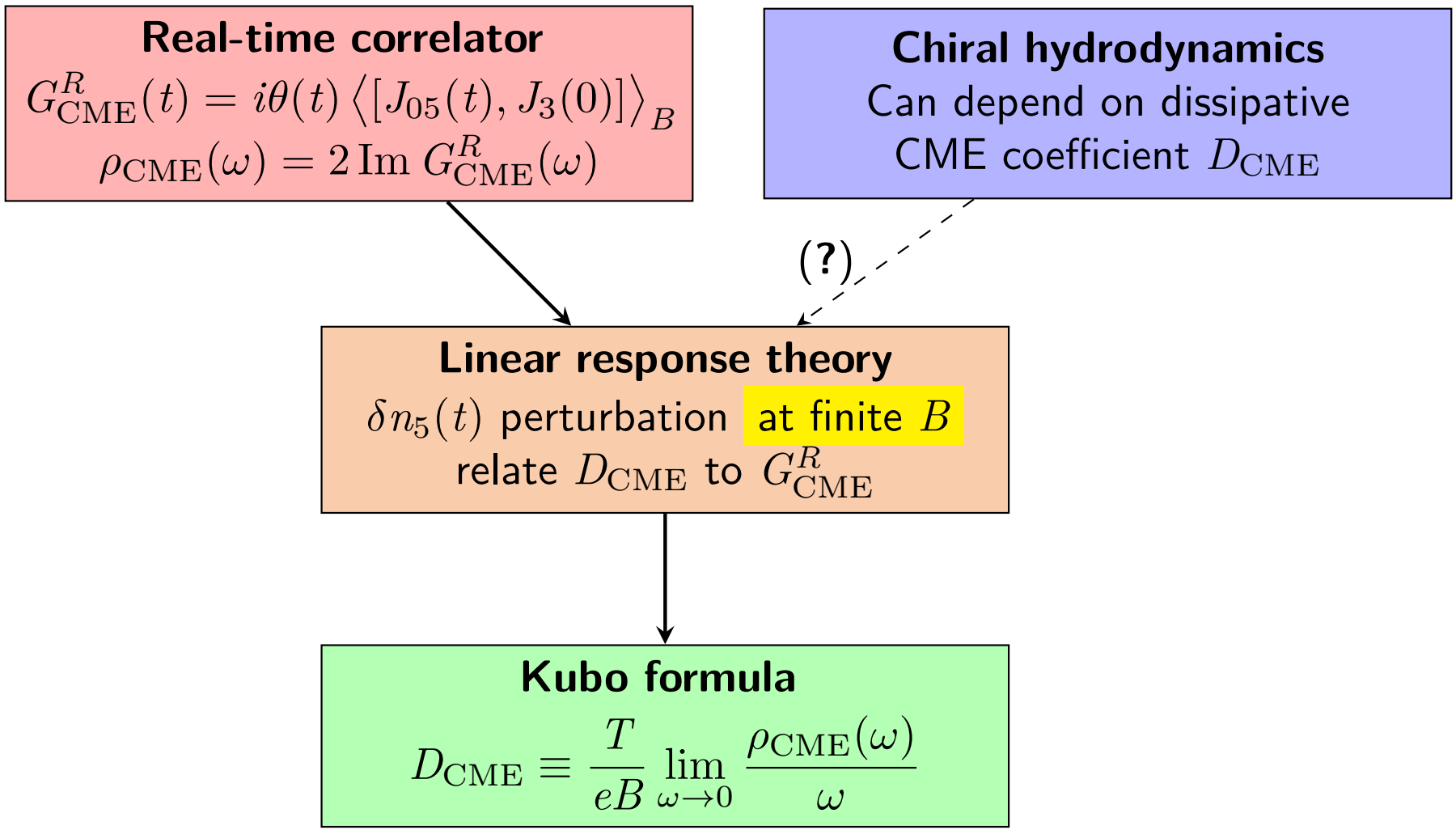
- Consistent with anomaly argument for $m = 0$
- For massive fermions not fixed by the anomaly

# Dissipative CME

- Could there be a dissipative part of the CME? $\rightarrow$ $D_{\mathrm{CME}}$ (analogy with $\sigma$)

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- Missing precise connection with chiral hydrodynamics!

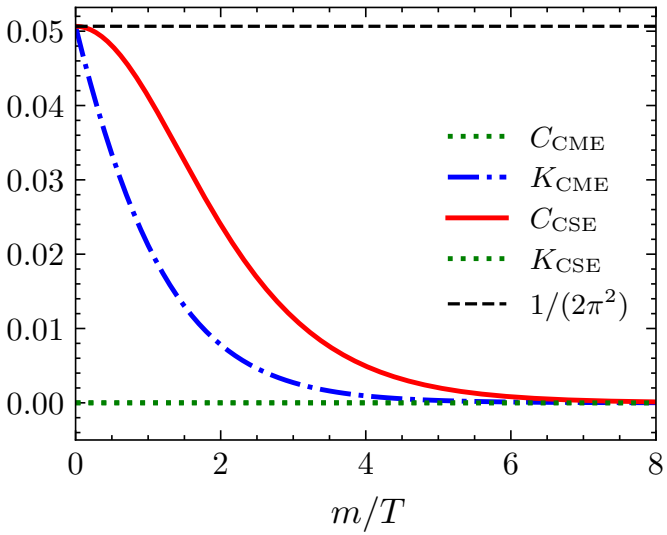
# Dissipative part

- **CME**:

$$\frac{1}{eB}\rho_{\mathrm{CME}}(\omega) = \alpha_{\mathrm{CME}}(m)\omega\delta(\omega)$$

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$$+\text{Pauli-Villars}$$




- $D_{\mathrm{CME}} \to \infty$ for free fermions
- Remember: for $\sigma$ the $\omega\delta(\omega)$ in the spectral function smears in QCD
- What happens if $\omega\delta(\omega)$ smears in QCD and $D_{\mathrm{CME}}$ finite?

# CME in QCD

- If $\omega\delta(\omega)$ smears, then KK relations enforce

$$\mathrm{Re}\,\widetilde{G}^{R}_{\mathrm{CME}}(\omega=0)=\widetilde{G}^{E}_{\mathrm{CME}}(\omega_n=0)$$

- But we know the Euclidean correlator at $\omega_n=0$ vanishes (equilibrium result)

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- $\omega\delta(\omega)$ smears $\Rightarrow$ non-dissipative CME $K_{\mathrm{CME}}$ vanishes!

$$K_{\mathrm{CME}}=0$$

- Need to understand the fate of the delta in QCD

# CME in QCD

- Notice we can also have delta + transport peak:

$$\rho_{\mathrm{CME}}(\omega) \sim \omega \left[ A\delta(\omega) + B\frac{\gamma}{\omega^2+\gamma^2} \right] \Rightarrow K_{\mathrm{CME}} \neq 0, D_{\mathrm{CME}} \neq 0$$

- There are four possibilities:

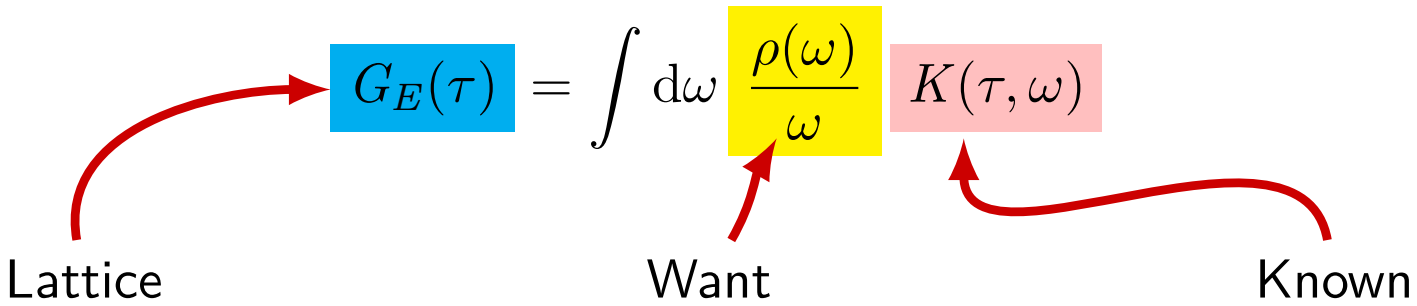
| | $\delta(\omega)$ | Transport peak | $K_{\mathrm{CSE}}$ | $D_{\mathrm{CSE}}$ |
|---|---|---|---|---|
| a) | Yes | Yes | Finite | Finite |
| b) | Yes | No | Finite | 0 |
| c) | No | Yes | 0 | Finite |
| d) | No | No | 0 | 0 |

- b) and c) are the most likely scenarios (delta smears or not)

- Perturbative tools not yet available $\rightarrow$ can we study this on the lattice?

# Spectral reconstruction

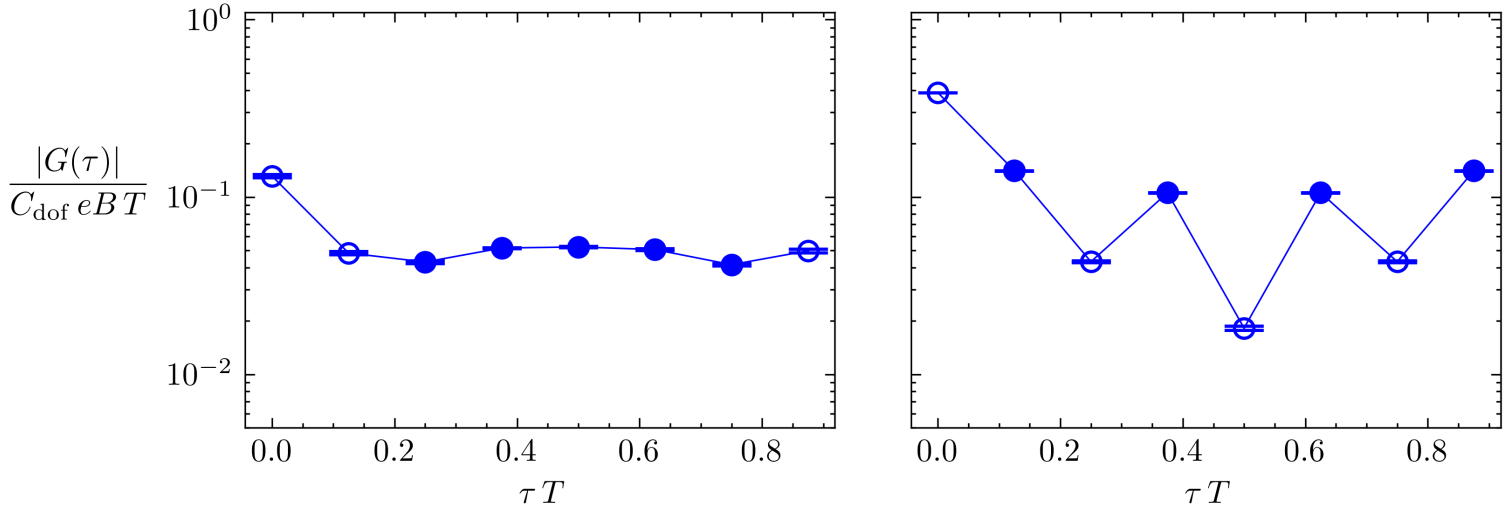
- $G_R, \rho$ not directly accessible from lattice QCD
- Spectral representation of Euclidean correlators




- On the lattice: $N_t \sim \mathcal{O}(10)$ ill-posed inverse problem
- Many methods on the market $\rightarrow$ applied to get other transport coefficients
- Examples: Maximum entropy method, Backus-Gilbert, machine learning, ...

# Staggered fermions

- Staggered fermions: optimal for finite $T$ QCD
- Technical issue: staggered correlators in $\tau$ can oscillate due to taste mixing

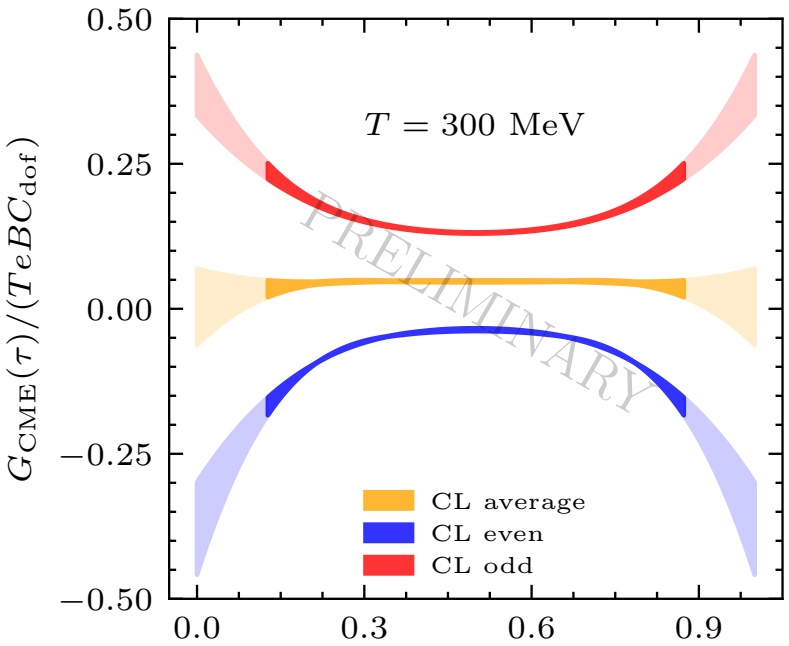


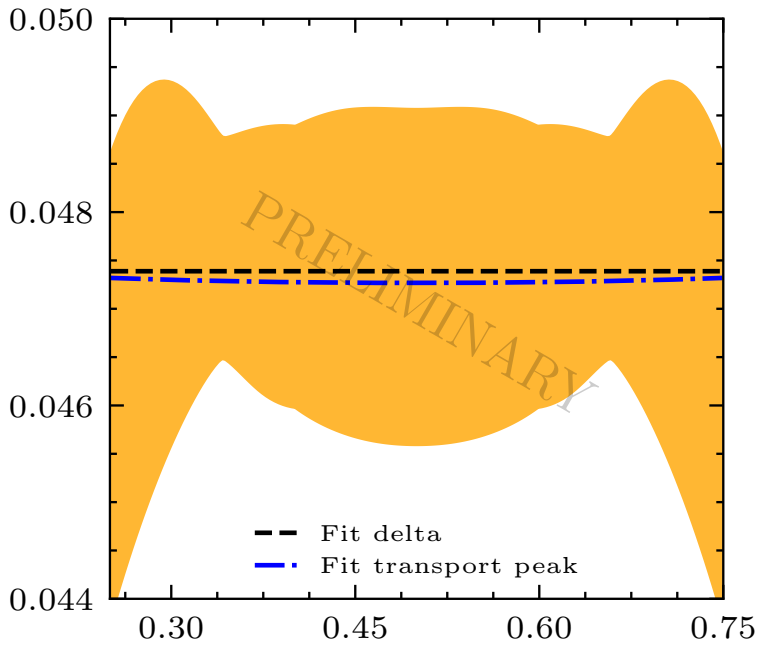

Wilson Staggered

- At finite $a \to$ half of the points available for reconstruction
- Continuum limit should be taken first

# Reconstruction of $D_{\mathrm{CME}}$ in QCD

- Continuum limit is the average of even and odd points
- Difficult to obtain a reliable extrapolation for small and large $\tau$
- Very tough to distinguish a narrow transport peak from a delta
- Needs deeper analysis and theoretical insight



# Summary

Take home messages

- Study of out-of-equilibrium CME and CSE in QCD using lattice simulations
- Spectral function calculated at 1-loop in perturbation theory
- Usual non-dissipative $K_{\mathrm{CME}}$ and dissipative $D_{\mathrm{CME}}$ parts deeply connected
- Crucial to understand this in QCD

# Backup slides

# CME in equilibrium

- Consider system with $\vec{B} = B\,\hat{e}_3$ and $\mu_5$ in global thermal equilibrium
- Global CME vanishes in equilibrium

$$C_{\mathrm{CME}} = 0 \quad \Rightarrow \quad J_3 = C_{\mathrm{CME}}\,\mu_5 B = 0$$

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- Bloch's theorem N. Yamamoto '15
  Global conserved currents cannot flow in equilibrium ground state

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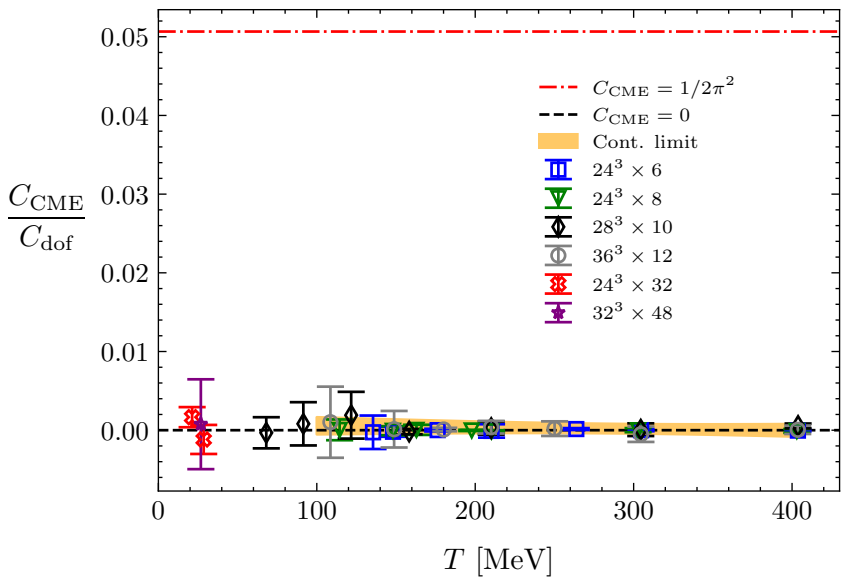
$$C_{\mathrm{CME}} = 0 \quad \Rightarrow \quad J_3 = C_{\mathrm{CME}}\,\mu_5 B = 0$$

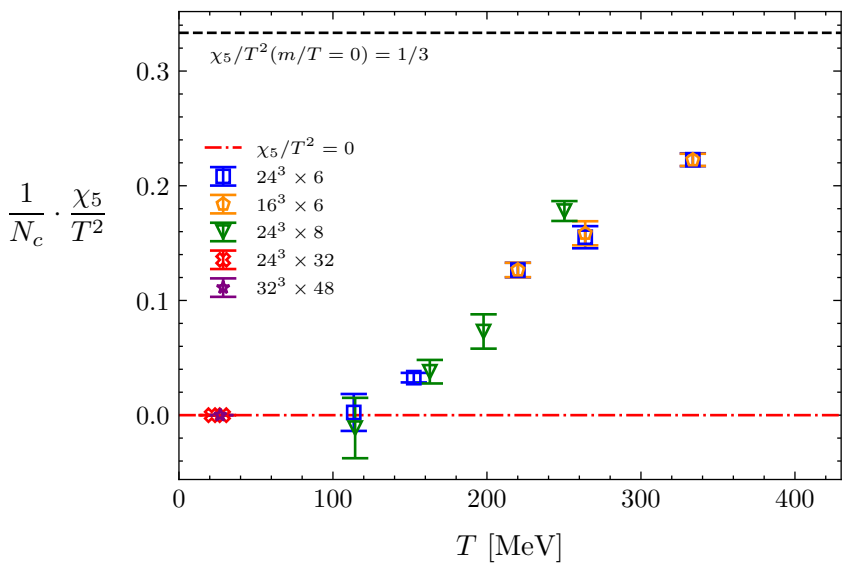
- Bloch's theorem N. Yamamoto '15
  Global conserved currents cannot flow in equilibrium ground state
- QFT perspective: role of regularization is crucial
- Regulators breaking gauge invariance (cutoff) lead to incorrect non-vanishing results

# CME in QCD

- **CME vanishes** in equilibrium, also in QCD
- Chiral density is **finite** at $\mu_5 \neq 0$

$$n_5(\mu_5) = \frac{T}{V}\left.\frac{\mathrm{d}^2 \log \mathcal{Z}}{\mathrm{d}\mu_5^2}\right|_{\mu_5=0} \mu_5 + \mathcal{O}(\mu_5^3) = \chi_5\, \mu_5 + \mathcal{O}(\mu_5^3)$$




Brandt, Endrődi, EGV, Markó '24

# CME midpoint

- At 1-loop:

$$\frac{1}{eBT} G_{\mathrm{CME}}(\tau T = 1/2) = C_{\mathrm{CSE}}$$

- Also fulfilled in QCD at (not so) high temperatures

