SPIN HYDRODYNAMICS – recent results

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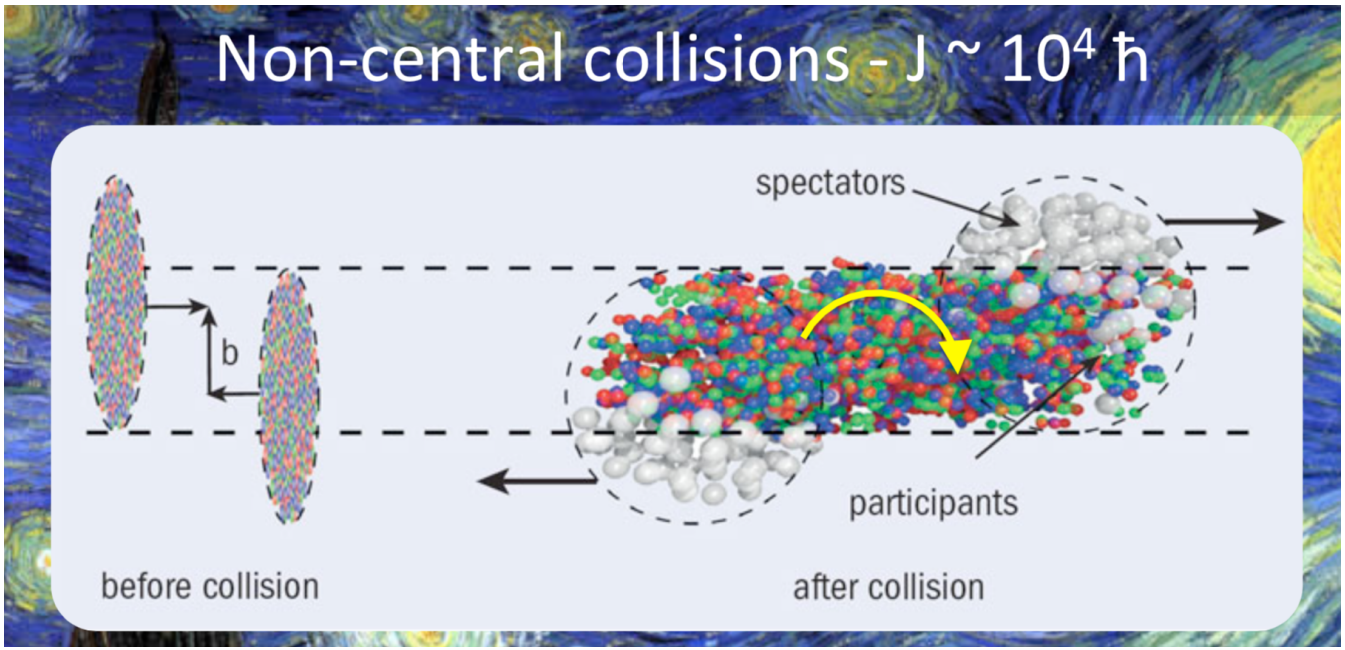
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1: PHYSICS MOTIVATION

Non-central collisions of heavy ions

Non-central heavy-ion collisions create fireballs with large global angular momenta, some part of the angular momentum can be transferred from the orbital to the spin part

$$\boldsymbol{J}_{\mathrm{init}} = \boldsymbol{L}_{\mathrm{init}} = \boldsymbol{L}_{\mathrm{final}} + \boldsymbol{S}_{\mathrm{final}}$$



(Michael Lisa, talk „Strangeness in Quark Matter 2016")

Warning: large angular momentum does not mean large angle of rotation!

$\Delta t = 1$ fm/c = 3×10^{-24} s, $\quad \Delta\phi = \Delta t\, \omega_{\max} = 27 \times 10^{-24} \times 10^{21} = 2.7 \times 10^{-2}$

Orbital and spin part of total angular momentum

Total angular momentum tensor $J^{\mu,\lambda\nu}$ has orbital $L^{\mu,\lambda\nu}$ and spin $S^{\mu,\lambda\nu}$ parts
orbital part is expressed by the energy-momentum tensor (stress-energy tensor), the spin part is called the spin tensor

$$J^{\mu,\lambda\nu} = x^\lambda T^{\mu\nu} - x^\nu T^{\mu\lambda} + S^{\mu,\lambda\nu} \equiv L^{\mu,\lambda\nu} + S^{\mu,\lambda\nu}$$

$$\partial_\mu J^{\mu,\lambda\nu} = T^{\lambda\nu} - T^{\nu\lambda} + \partial_\mu S^{\mu,\lambda\nu} = 0, \qquad \partial_\mu S^{\mu,\lambda\nu} = T^{\nu\lambda} - T^{\lambda\nu}$$

Antisymmetry in the last two indices:

$$J^{\mu,\lambda\nu} = -J^{\mu,\nu\lambda}, \quad L^{\mu,\lambda\nu} = -L^{\mu,\nu\lambda}, \quad S^{\mu,\lambda\nu} = -S^{\mu,\nu\lambda},$$

Warning: different forms of these tensors are used in the literature **(pseudogauge freedom)**
a topic broadly discussed during this conference and later in this talk

At present, the most successful phenomenological description of the data has been achieved by Francesco Becattini and collaborators, based on the Zubarev formalism for non-equilibrium systems

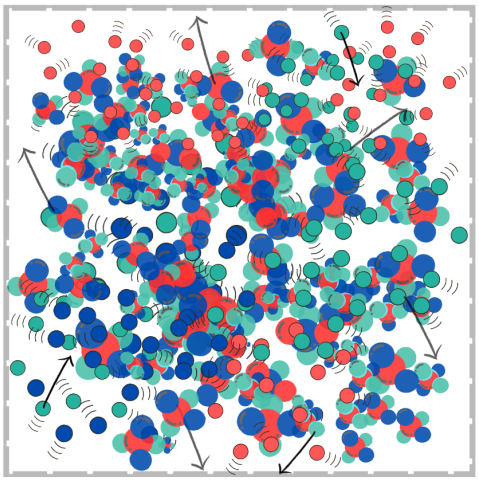
F. Becattini, L. Tinti, V. Chandra, I. Del Zanna, E. Grossi, M. Buzzegoli, G. Inghirami, A. Palermo, I. Karpenko, S. Singh, ...

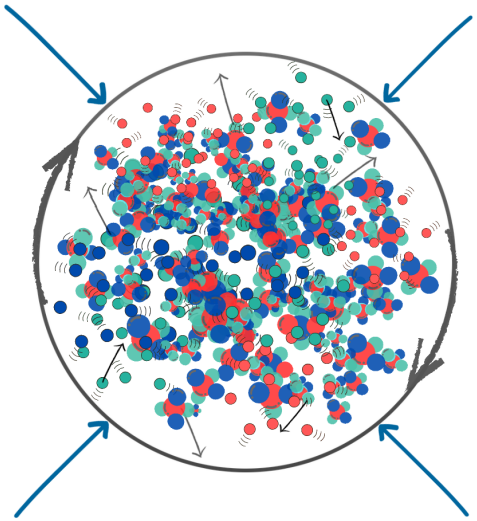
the central quantity (the starting point) in this case is a rank-2 antisymmetric tensor, called the thermal vorticity

$$\omega_{\mu\nu} = \varpi_{\mu\nu} = -1/2\left(\partial_\mu \beta_\nu - \partial_\nu \beta_\mu\right) \qquad \beta^\mu = u^\mu/T, \quad \beta = 1/T$$

u^μ – hydrodynamic flow vector, T – temperature

The formalism presented in this lecture starts from different principles, eventually thermal vorticity appears on the way

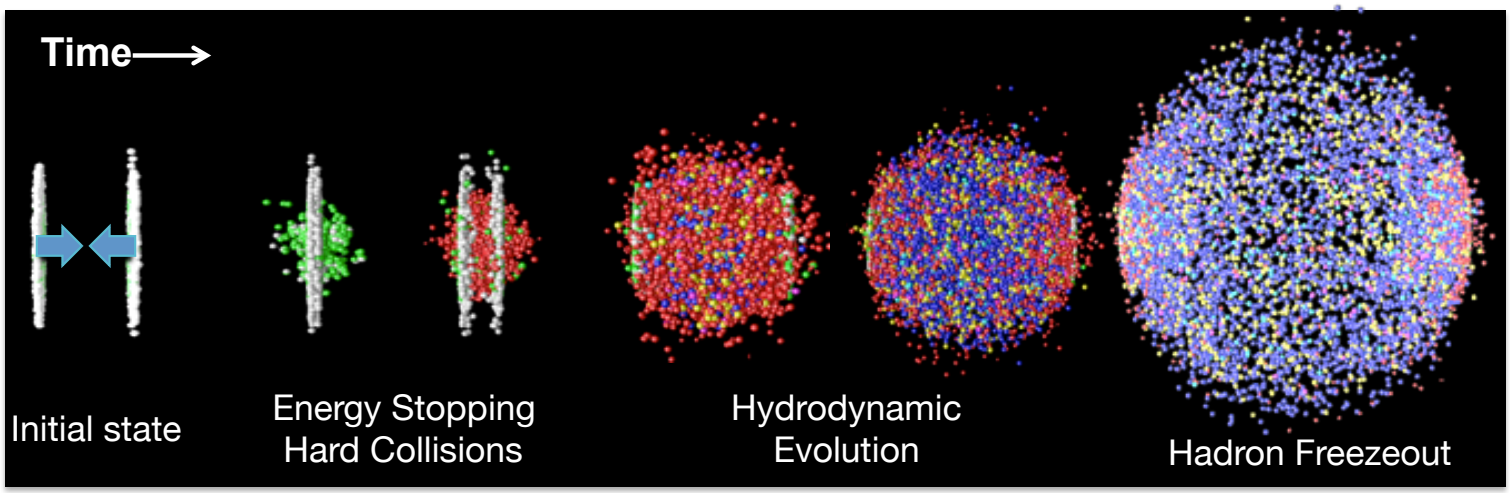




2: STANDARD RELATIVISTIC HYDRODYNAMICS

Standard model (scheme) of heavy-ion collisions

RELATIVISTIC HYDRODYNAMICS FORMS THE BASIC INGREDIENT OF THE STANDARD MODEL OF HEAVY-ION COLLISIONS



T. K. Nayak, Lepton-Photon 2011 Conference

data on spin polarization suggest that spin should be included in the hydrodynamic framework

Perfect fluid hydrodynamics

PERFECT-FLUID HYDRODYNAMICS = local equilibrium + conservation laws

one usually includes energy, linear momentum, baryon number, ...

T (temperature), u^μ (three independent components of flow), μ (baryon chemical potential)
ε (energy density), P (pressure), n (baryon density), σ (entropy density), $\xi = \mu/T$

$$T^{\mu\nu} = [\varepsilon(T,\mu) + P(T,\mu)]\, u^\mu u^\nu - P(T,\mu) g^{\mu\nu} \tag{1}$$

$$\partial_\mu T^{\mu\nu} = 0, \qquad \partial_\mu N^\mu = \partial_\mu (n u^\mu) = 0 \qquad \text{(4+1 eqs.)} \tag{2}$$

five equations for five unknown functions

dissipation does not appear

$$\partial_\mu S^\mu = \partial_\mu (\sigma u^\mu) = 0 \qquad \text{(1 eq.)} \tag{3}$$

entropy conservation follows from the energy-momentum conservation and the form of the energy-momentum tensor
Euler's equation (says that four-acceleration is caused by the pressure gradient)

$$u^\lambda \partial_\lambda u^\mu \equiv a^\mu = \frac{(g^{\mu\nu} - u^\mu u^\nu)}{\varepsilon + P} \partial_\nu P \equiv \frac{1}{\varepsilon + P} \Delta^{\mu\nu} \partial_\nu P \qquad \text{(3 eqs.)} \tag{4}$$

3: CLASSICAL APPROACH TO SPIN

Classical treatment of spin – internal angular momentum

A particle can be characterised by the internal angular momentum tensor $s^{\alpha\beta}$
M. Matthison, Acta Phys. Polon. 6 (1937) 163

Neue Mechanik materieller Systeme

Nowa mechanika systemów materialnych

Von MYRON MATHISSON, Warschau

(Eingegangen am 8. September 1937)



T. Sauer, A. Trautman, *M. Mathisson: What little we know of his life*
Acta Phys. Polon. B39 (2008) 7

Mathisson with Pauli
Copenhagen 1937

$$s^{\alpha\beta} = \frac{1}{m}\epsilon^{\alpha\beta\gamma\delta} p_\gamma s_\delta, \quad s\cdot p = 0, \quad s^\alpha = \frac{1}{2m}\epsilon^{\alpha\beta\gamma\delta} p_\beta s_{\gamma\delta} \tag{5}$$

A straightforward generalization of the phase-space distribution function $f(x,\boldsymbol{p})$ is a spin dependent distribution $f(x,\boldsymbol{p},s)$
WF, A. Kumar, R. Ryblewski, Prog. Part. Nucl. Phys. 108 (2019) 103709

$$\int dS \ldots = \frac{m}{\pi\mathfrak{s}} \int d^4s\, \delta(s\cdot s + \mathfrak{s}^2)\, \delta(p\cdot s) \ldots \qquad \mathfrak{s}^2 = \frac{1}{2}\left(1+\frac{1}{2}\right) = \frac{3}{4} \tag{6}$$

$$\int dS = \frac{m}{\pi\mathfrak{s}} \int d^4s\, \delta(s\cdot s + \mathfrak{s}^2)\, \delta(p\cdot s) = 2 \tag{7}$$

from now on, we consider spin 1/2 only, extension to spin 1 → see talk by Valeriya Mykhaylova

Local equilibrium function with spin, macroscopic currents

WF, A. Kumar, R. Ryblewski, Prog. Part. Nucl. Phys. 108 (2019) 103709

spin conserving equilibrium distribution functions for particles and antiparticles

$$f^{\pm}_{\rm eq}(x,p,s) = \exp\left(\pm\xi(x) - p\cdot\beta(x) + \tfrac{1}{2}\omega_{\alpha\beta}(x)s^{\alpha\beta}\right) \tag{8}$$

macroscopic currents

baryon current

$$N^{\lambda}_{\rm eq} = \int dP \int dS\; p^{\lambda}\, 1 \left[f^{+}_{\rm eq}(x,p,s) - f^{-}_{\rm eq}(x,p,s)\right] \tag{9}$$

energy-momentum tensor

$$T^{\lambda\mu}_{\rm eq} = \int dP \int dS\; p^{\lambda} p^{\mu} \left[f^{+}_{\rm eq}(x,p,s) + f^{-}_{\rm eq}(x,p,s)\right] \tag{10}$$

spin tensor

$$S^{\lambda,\mu\nu}_{\rm eq} = \int dP \int dS\; p^{\lambda}\, s^{\mu\nu} \left[f^{+}_{\rm eq}(x,p,s) + f^{-}_{\rm eq}(x,p,s)\right] \tag{11}$$

construction of the equilibrium function implies the conservation laws

$$\partial_\mu N^\mu(x) = 0, \quad \partial_\mu T^{\mu\nu}(x) = 0, \quad \partial_\lambda S^{\lambda,\mu\nu}(x) = 0 \tag{12}$$

these are 11 equations for 11 Lagrange multipliers

$$\partial_\mu N^\mu[\xi(x), \beta_\alpha(x), \omega_{\alpha\beta}(x)] = 0 \qquad \text{(1 eq.)} \tag{13}$$

$$\partial_\mu T^{\mu\nu}[\xi(x), \beta_\alpha(x), \omega_{\alpha\beta}(x)] = 0 \qquad \text{(4 eqs.)} \tag{14}$$

$$\partial_\lambda S^{\lambda,\mu\nu}[\xi(x), \beta_\alpha(x), \omega_{\alpha\beta}(x)] = 0 \qquad \text{(6 eqs.)} \tag{15}$$

PERFECT-SPIN HYDRODYNAMICS

WF, B. Friman, A. Jaiswal, E. Speranza, Phys. Rev. C97 (2018) 041901

CAN SPIN-ORBIT INTERACTION BE NEGLECTED?

1. **ATOMIC PHYSICS CONTEXT:**

   $\boldsymbol{L}$ means rotation, for a charged fluid this generates $\boldsymbol{B}$, magnetic fields couples to magnetic moments $\boldsymbol{\mu}$ that are connected to spin $\boldsymbol{S}$, we effectively get $\boldsymbol{L}\cdot\boldsymbol{S}$ coupling,
   this effect (conserving entropy) can be easily included in our framework
   S. Bhadury, WF, A. Jaiswal, A. Kumar, R. Ryblewski, PRL129 (2022) 192301

2. **DIRAC EQUATION LESSON:**

   only $\boldsymbol{J}=\boldsymbol{L}+\boldsymbol{S}$ is conserved, property reflected in asymmetric energy-momentum tensor $T^{\mu\nu}_{\rm can}\neq T^{\nu\mu}_{\rm can}$
   and non-zero divergence $\partial_\lambda S^{\lambda,\mu\nu}_{\rm can}=T^{\nu\mu}_{\rm can}-T^{\mu\nu}_{\rm can}$
   tensors defined with classical spin do not correspond to the canonical forms
   WF, A. Kumar, R. Ryblewski, Prog. Part. Nucl. Phys. 108 (2019) 103709

3. **MICROSCOPIC SPIN-ORBIT INTERACTION:**

   collisions/decays of particles with spin, transfer between $\boldsymbol{L}$ and $\boldsymbol{S}$, for example $\rho^0\to\pi^+\pi^-$
   such processes produce entropy, can be included if we extend the perfect-fluid picture, discussed later

perfect spin hydrodynamics = spin conservation

dissipative spin hydrodynamics = transfer between $\boldsymbol{S}$ and $\boldsymbol{L}$ possible

Joseph Kapusta: strange quark spin or helicity is unchanged from the time they are created to the time they hadronize
J. Kapusta, E. Rrapaj, S. Rudaz, PRC101 (2020) 024907

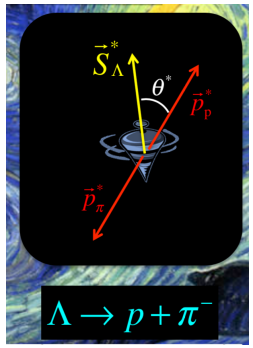
4: QUANTUM (DENSITY MATRIX) APPROACH TO SPIN

Quantum spin description: spin & spinor density matrix I

standard scalar functions $f(x,p)$ are generalized to **2×2 Hermitian matrices** in spin space for each value of the space-time position x and four-momentum p, the sign ± distinguishes particles from antiparticles, σ - Pauli matrices

$\zeta^{\pm}_{*} = 0$ no polarization, $\zeta^{\pm}_{*} = 1$ pure state, $0 < |\zeta^{\pm}_{*}| < 1$ mixed state, **asterisk denotes the particle rest frame (PRF)**

$$f^{\pm}_{rs}(x,p) = f^{\pm}_{0}(x,p)\left[\delta_{rs} + \boldsymbol{\zeta}^{\pm}_{*}(x,\boldsymbol{p})\cdot\sigma_{rs}\right], \quad 0 \leq |\boldsymbol{\zeta}^{\pm}_{*}| \leq 1 \tag{16}$$



$\boldsymbol{\zeta}^{\pm}_{*}(x,\boldsymbol{p})$ can be interpreted as a spatial part of the polarization four-vector $\zeta^{\pm\mu}_{*}(x,\boldsymbol{p})$ with a vanishing zeroth component, in the LAB frame – $\zeta^{\pm\mu}_{*}$ boosted with the velocity defined by the particle velocity $\boldsymbol{v}_p$

$$\zeta^{\pm\mu}_{*} = \left(0, \boldsymbol{\zeta}^{\pm}_{*}\right), \quad \zeta^{\mu}_{\pm} = \Lambda^{\mu}{}_{\nu}(\boldsymbol{v}_p)\zeta^{\nu}_{\pm *}, \quad \zeta_{\pm}\cdot p = 0 \tag{17}$$

WIGNER FUNCTION, gradients neglected in the kinetic equation $\left(\not{k} - m + \frac{i\hbar}{2}\not{\partial}\right) W(x,k) = 0$ there will be included at the hydrodynamics level

S. R. de Groot, W. A. van Leeuwen, Ch. G. van Weert, **Relativistic kinetic theory**

$$W^{+}(x,k) = \frac{1}{2}\sum_{r,s=1}^{2}\int \mathrm{d}P\,\delta^{(4)}(k-p)u_r(p)f^{+}_{rs}(x,p)\bar{u}_s(p) \tag{18}$$

$$W^{-}(x,k) = -\frac{1}{2}\sum_{r,s=1}^{2}\int \mathrm{d}P\,\delta^{(4)}(k+p)v_s(p)f^{-}_{rs}(x,p)\bar{v}_r(p) \tag{19}$$

$u_s(p)$ and $v_r(p)$ – Dirac bispinors

Quantum spin description: spin & spinor density matrix II

transition to **4×4 spinor density matrices** $X^{\pm}(x,p)$

$$f^{+}_{rs}(x,p) \quad = \quad \bar{u}_r(p)X^{+}(x,p)u_s(p), \qquad f^{-}_{rs}(x,p) = -\bar{v}_s(p)X^{-}(x,p)v_r(p) \tag{20}$$

WIGNER FUNCTION becomes

$$W^{\pm}(x,k) = \frac{1}{4m}\int \mathrm{d}P\,\delta^{(4)}(k \mp p)(\not{p} \pm m)X^{\pm}(x,p)(\not{p} \pm m) \tag{21}$$

spinor density matrix derived in the Landau–Lifshitz (LL) course for free Dirac particles with momentum p
B. Berestetskii, E. M. Lifshitz, and L. P. Pitaevskii, **Quantum Electrodynamics**
Course of Theoretical Physics, Vol. 4

$$\rho^{\pm}(p) = \pm\frac{1}{4m}\,(\not{p} \pm m)\,\rho'^{\pm}(p)\,(\not{p} \pm m) \tag{22}$$

here $\rho'^{\pm}(p)$ is the auxiliary spin density matrix given by the formula

$$\rho'^{\pm}(p) = 1 + \gamma_5 \not{\zeta}^{\pm}(p) \tag{23}$$

Quantum spin description: local equilibrium spinor density matrix

F. Becattini, V. Chandra, L. Del Zanna, E. Grossi, Annals Phys. 338 (2013) 32
$\Sigma^{\mu\nu} = (i/4)[\gamma^\mu, \gamma^\nu]$ is the Dirac spin operator

$$X^{\pm}(x,p) = \exp\left[\pm\xi(x) - \beta_\mu(x)p^\mu \pm \frac{1}{2}\varpi_{\mu\nu}(x)\Sigma^{\mu\nu}\right] \tag{24}$$

WF, B. Friman, A. Jaiswal, E. Speranza, Phys. Rev. C97 (2018) 041901

$$\varpi_{\mu\nu}(x) \longrightarrow \omega_{\mu\nu}(x) = \frac{\Omega_{\mu\nu}(x)}{T(x)} \tag{25}$$

ratio of the tensor spin chemical potential and temperature, altogether we have 11 Lagrange multipliers that control the conservation of the baryon number (1), energy (1), linear momentum (3), angular momentum (3), and Lorentz boost vectors (3)

$\omega_{\mu\nu}$ can be represented by electric- and magneticlike three-vectors (spin polarization tensor) in analogy to $F_{\mu\nu}$

$$\omega_{\mu\nu} = \begin{bmatrix} 0 & e^1 & e^2 & e^3 \\ -e^1 & 0 & -b^3 & b^2 \\ -e^2 & b^3 & 0 & -b^1 \\ -e^3 & -b^2 & b^1 & 0 \end{bmatrix} \tag{26}$$

this form of $X^{\pm}$ leads to the expected 2 by 2 form, Eq. (16), only for $|\boldsymbol{e}_*| << 1$ and $|\boldsymbol{b}_*| << 1$
WF, B. Friman, A. Jaiswal, E. Speranza, R. Ryblewski, Phys. Rev. D97 (2018) 116017

Quantum spin description: local equilibrium - revised formula

S. Bhadury, Z. Drogosz, WF, S. K. Kar, V. Mykhaylova, **arXiv:2505.02657**, with a spacelike four-vector a^μ ($a^2 < 0$)

$$X^{\pm} = \exp\left(\pm\xi - \beta_\mu p^\mu + \gamma_5 \not{a}\right) \tag{27}$$

STEP 1 – transition to 2 by 2 spin density matrices uses identities

$$\bar{u}_r(p)\gamma_5 \not{\zeta} u_s(p) = 2m\zeta_* \cdot \sigma_{rs}, \qquad \bar{v}_s(p)\gamma_5 \not{\zeta} v_r(p) = -2m\zeta_* \cdot \sigma_{rs} \tag{28}$$

which leads to identification and automatically correct normalization

$$\zeta^\mu = \frac{a^\mu}{\sqrt{-a^2}} \tanh\sqrt{-a^2}, \qquad -\zeta^2 = \zeta_*^2 = \tanh^2\sqrt{\boldsymbol{a}_*^2} \leq 1 \tag{29}$$

STEP 2 – to get the agreement with the previous definition and classical description for $|\boldsymbol{e}_*| << 1$ and $|\boldsymbol{b}_*| << 1$ we choose

$$a_\mu(x,p) = -\frac{1}{2m}\tilde{\omega}_{\mu\nu}(x)p^\nu \tag{30}$$

the particle rest frame (PRF) where $p^\mu = (m,0,0,0)$ we have $\boldsymbol{a}_* = -\frac{\boldsymbol{b}_*}{2}$

similarity to MHD - polarization effects determined by the magnetic(like) component of the spin polarization tensor

Quantum spin description: macroscopic currents

S. R. de Groot, W. A. van Leeuwen, Ch. G. van Weert, **Relativistic kinetic theory**

$$\text{baryon current:} \qquad N^{\lambda}(x) = \sum_{r=1}^{2} \int \mathrm{d}P\, p^{\lambda}\, 1 \left[f_{rr}^{+}(x,p) - f_{rr}^{-}(x,p) \right] \tag{31}$$

$$\text{energy-momentum tensor:} \qquad T^{\lambda\mu}(x) = \sum_{r=1}^{2} \int \mathrm{d}P\, p^{\lambda} p^{\mu} \left[f_{rr}^{+}(x,p) + f_{rr}^{-}(x,p) \right] \tag{32}$$

$$\text{spin tensor:} \qquad S^{\lambda,\mu\nu}(x) = \frac{1}{2} \sum_{r,s=1}^{2} \int \mathrm{d}P\, p^{\lambda} \left[\sigma_{sr}^{+\mu\nu}(p) f_{rs}^{+}(x,p) + \sigma_{sr}^{-\mu\nu}(p) f_{rs}^{-}(x,p) \right] \tag{33}$$

where $\sigma_{sr}^{+\mu\nu}(p) = 1/(2m)\, \bar{u}_s(p) \sigma^{\mu\nu} u_r(p)$ and $\sigma_{sr}^{-\mu\nu}(p) = 1/(2m)\, \bar{v}_r(p) \sigma^{\mu\nu} v_s(p)$, with $\sigma^{\mu\nu} = (i/2)[\gamma^{\mu}, \gamma^{\nu}]$

these forms of currents are commonly known as the GLW versions (GLW pseudogauge)
for free Dirac equation (relativistic gas) these tensors are conserved

$$\partial_{\mu} N^{\mu}(x) = 0, \quad \partial_{\mu} T^{\mu\nu}(x) = 0, \quad \partial_{\lambda} S^{\lambda,\mu\nu}(x) = 0 \tag{34}$$

up to second order in ω averaging over classical spin equivalent to performing traces over spinor indices

$$s^{\mu} \quad \longleftrightarrow \quad \frac{1}{2} \gamma_5 \gamma^{\mu} \tag{35}$$

Zbigniew Drogosz, Phys. Lett. B873 (2026) 140205

5: PSEUDOGAUGE FREEDOM

vs. PSEUDOGAUGE DEPENDENCE

Have we committed blasphemy? ENERGY-MOMENTUM TENSOR SHOULD BE USED IN THE BELINFANTE-ROSENFELD FORM, SINCE THIS FORM APPEARS IN EINSTEIN'S EQUATIONS

Pseudo-gauge transformation (QCD language in the context of the proton spin puzzle: adding boundary terms)

$$T'^{\mu\nu} = T^{\mu\nu} + \frac{1}{2}\partial_\lambda \left(\Phi^{\lambda,\mu\nu} + \Phi^{\nu,\mu\lambda} + \Phi^{\mu,\nu\lambda}\right) \tag{36}$$

$$S'^{\lambda,\mu\nu} = S^{\lambda,\mu\nu} - \Phi^{\lambda,\mu\nu} + \partial_\rho Z^{\mu\nu,\lambda\rho} \tag{37}$$

Canonical forms (directly obtained from Noether's Theorem): asymmetric energy-momentum tensor, spin tensor directly expressed by axial current (couples to weak interactions)

Belinfante-Rosenberg version, $\Phi^{\lambda,\mu\nu} = S^{\lambda,\mu\nu}, Z^{\mu\nu,\lambda\rho} = 0$, (couples to classical gravity); spin tensor appears in modified theories of gravity, couples to torsion

de Groot, van Leuveen, van Weert (GLW) forms: symmetric energy-momentum tensor and conserved spin tensor

Hilgevoord and Wouthuysen (HW) choice: symmetric energy-momentum tensor and conserved spin tensor

there is ongoing discussion if the physics is or is not pseudogauge dependent F. Hehl, Rept. Math. Phys. 9 (1976) 55

Sidney Coleman's old answer: ...we have an infinite family of possible definitions of the local current...some textbooks try to avoid this point, or nervously rub one foot across the other leg and natter about the best definition or the optimum definition...and the right answer is, of course, there's nothing to natter about, there's nothing to be disturbed about...it is something to be pleased about. If we have many objects that satisfy desirable general criteria, then that's better than having just one... the more freedom you have, the better. It's like being passed a plate of cookies and someone starts arguing about which is the best cookie. They're all edible!

GLW → canonical → Belinfante-Rosenfeld

WF, A. Kumar, R. Ryblewski, Prog. Part. Nucl. Phys. 108 (2019) 103709

GLW → canonical: superpotential defined as $\Phi^{\lambda,\mu\nu}_{\rm can} \equiv S^{\mu,\lambda\nu}_{\rm GLW} - S^{\nu,\lambda\mu}_{\rm GLW}$, then we have

$$S^{\lambda,\mu\nu}_{\rm can} = S^{\lambda,\mu\nu}_{\rm GLW} - \Phi^{\lambda,\mu\nu}_{\rm can}$$

and

$$T^{\mu\nu}_{\rm can} = T^{\mu\nu}_{\rm GLW} + \frac{1}{2}\partial_\lambda\left(\Phi^{\lambda,\mu\nu}_{\rm can} + \Phi^{\mu,\nu\lambda}_{\rm can} + \Phi^{\nu,\mu\lambda}_{\rm can}\right)$$

canonical → Belinfante-Rosenfeld: superpotential defined as $\Phi_{\rm Bel} = S^{\lambda,\mu\nu}_{\rm can}$

$$S^{\lambda,\mu\nu}_{\rm Bel} = 0, \quad T^{\mu\nu}_{\rm Bel} = T^{\mu\nu}_{\rm can} + \frac{1}{2}\partial_\lambda\left(S^{\lambda,\mu\nu}_{\rm can} - S^{\mu,\lambda\nu}_{\rm can} - S^{\nu,\lambda\mu}_{\rm can}\right),$$

canonical spin tensor is the "right" observable – connected with spin operator

S. Dey, WF, A. Jaiswal, R. Ryblewski, Phys. Lett. B843 (2023) 137994

6: THERMODYNAMICS WITH SPIN

WF, M. Hontarenko, PRL134 (2025) 082302
Z.. Drogosz, WF, M. Hontarenko, PRD 110 (2024) 096018
Boltzmann's definition of the entropy (*H*-function)

$$S^\mu = -\int dP\, dS\, p^\mu \left[f^+\left(\ln f^+ - 1\right) + f^-\left(\ln f^- - 1\right)\right] \qquad \text{classical spin} \tag{38}$$

$$S^\mu = -\frac{1}{2}\int \mathrm{d}P\, p^\mu \left\{\mathrm{tr}_4\left[X^+\left(\ln X^+ - 1\right)\right] + \mathrm{tr}_4\left[X^-\left(\ln X^- - 1\right)\right]\right\} \qquad \text{quantum spin} \tag{39}$$

Together with other kinetic-theory expressions, one obtains **tensor forms of thermodynamic relations**
valid for any value of the spin polarization tensor ω

$$S^\mu_{\rm eq} = T^{\mu\alpha}_{\rm eq}\beta_\alpha - \frac{1}{2}\omega_{\alpha\beta}S^{\mu,\alpha\beta}_{\rm eq} - \xi N^\mu_{\rm eq} + \mathcal{N}^\mu, \qquad \mathcal{N}^\mu \neq P\beta^\mu \tag{40}$$

$$dS^\mu_{\rm eq} = -\xi dN^\mu_{\rm eq} + \beta_\lambda dT^{\lambda\mu}_{\rm eq} - \tfrac{1}{2}\omega_{\alpha\beta}dS^{\mu,\alpha\beta}_{\rm eq} \qquad \text{first law of thermodynamics} \tag{41}$$

$$d\mathcal{N}^\mu = N^\mu_{\rm eq}d\xi - T^{\lambda\mu}_{\rm eq}d\beta_\lambda + \tfrac{1}{2}S^{\mu,\alpha\beta}_{\rm eq}d\omega_{\alpha\beta} \qquad \text{Gibbs-Duhem relations} \tag{42}$$

entropy conservation as a consequence of other conservation laws, very close similarity to MHD
M. Gedalin and I. Oiberman, Phys. Rev. E51 (1994) 4901

Stability and causality

DIVERGENCE-TYPE THEORIES
R. P. Geroch and L. Lindblom, Phys. Rev. D 41, 1855 (1990) & Annals of Physics 207, 394 (1991)
L. Gavassino, M. Antonelli, and B. Haskell, Phys. Rev. D 106, 056010 (2022)
N. Abboud, L. Gavassino, R. Singh, and E. Speranza, Phys. Rev. D 112, 094043 (2025)

multi-index notation for conserved tensors: $N^{\lambda A} \equiv (N^\lambda, T^{\lambda\mu}, S^{\lambda,\mu\nu})$, $\partial_\lambda N^{\lambda A} = 0$, $\zeta_A = (\zeta, \zeta_\mu, \zeta_{\mu\nu}) = (\xi, -\beta_\mu, \omega_{\mu\nu})$

thermodynamic relations: $\mathrm{d}S^\lambda = -\zeta_A \,\mathrm{d}N^{\lambda A}$, $\mathrm{d}\mathcal{N}^\lambda = N^{\lambda A}\,\mathrm{d}\zeta_A$

$$\mathcal{N}^\lambda = \frac{\partial \chi}{\partial \zeta_\lambda} = -\frac{\partial \chi}{\partial \beta_\lambda}. \tag{43}$$

Treating the fields ζ_A as independent variables, we have $\mathrm{d}N^{\lambda A} = M^{\lambda AB}\mathrm{d}\zeta_B$, and rewrite the conservation laws in the form

$$M^{\lambda AB}\partial_\lambda \zeta_B = 0, \tag{44}$$

where

$$M^{\lambda AB} = \left(\frac{\partial N^{\lambda A}}{\partial \zeta_B}\right) = -\left(\frac{\partial^3 \chi}{\partial\zeta_B \partial\zeta_A \partial\beta_\lambda}\right). \tag{45}$$

This object plays a central role in assessing the causality and stability of the hydrodynamic theory based on the conservation laws. These conditions are satisfied if the four-vector

$$M^\lambda \equiv M^{\lambda AB} Z_A Z_B \tag{46}$$

defined for any set of nonvanishing real elements $Z_A = (Z, Z_\mu, Z_{\mu\nu})$ is future-oriented and timelike

SPIN HYDRO BASED ON THE NEW WIGNER FUNCTION IS NON-LINEARLY CAUSAL AND STABLE

7: GOING OFF EQUILIBRIUM

W. Israel, J.M.Stewart, Annals Phys. 118 (1979) 341 & Phys.Lett. A58 (1976) 213
here we use the IS method to construct the Navier-Stokes theory

replacement of the equilibrium currents by the general ones (equilibrium + non-equilibrium corrections)

$$S^{\mu} = T^{\mu\alpha}\beta_{\alpha} - \frac{1}{2}\omega_{\alpha\beta}S^{\mu,\alpha\beta} - \xi N^{\mu} + \mathcal{N}^{\mu}_{\rm eq} \tag{47}$$

Conservations laws, now for total angular momentum $J = L + S$

$$\partial_{\mu}N^{\mu} = 0, \qquad \partial_{\mu}T^{\mu\nu} = 0, \qquad \partial_{\mu}S^{\mu,\alpha\beta} = T^{\beta\alpha} - T^{\alpha\beta} \tag{48}$$

entropy production

$$\partial_{\mu}S^{\mu} \quad = \quad -\delta N^{\mu}\partial_{\mu}\xi + \delta T^{\mu\lambda}_{s}\partial_{\mu}\beta_{\lambda} + \delta T^{\mu\lambda}_{a}\left(\partial_{\mu}\beta_{\lambda} - \omega_{\lambda\mu}\right) - \frac{1}{2}\delta S^{\mu,\alpha\beta}\partial_{\mu}\omega_{\alpha\beta} \geq 0 \tag{49}$$

K. Hattori, M. Hongo, X.-G. Huang, M. Matsuo, and H. Taya, Phys. Lett. B795, 100 (2019)
F. Becattini, A. Daher, Phys. Lett. B850, 138533 (2024)

$\delta T^{\mu\lambda}_{s}$ and $\delta T^{\mu\lambda}_{a}$, are symmetric and antisymmetric dissipative corrections to the energy-momentum tensor

second-order theory also developed
R. Biswas, A. Daher, A. Das, WF, R. Ryblewski, Phys. Rev. D108, 014024 (2023)

dissipative spin hydrodynamics = transfer between ***S*** and ***L*** possible

the second law of thermodynamics imposes constraints of non-equilibrium currents, they should be proportional to appropriate "gradients" multiplied by the kinetic coefficients, an example of the contribution is
K. Hattori, M. Hongo, X.-G. Huang, M. Matsuo, and H. Taya, Phys. Lett. B795, 100 (2019)

$$\delta T_a^{\nu\mu} = d^\mu u^\nu - d^\nu u^\mu + \cdots, \qquad d^\mu = \lambda\beta^{-1}(\beta a^\mu + \beta^2\nabla^\mu T - 2\omega^{\mu\nu}u_\nu), \qquad \lambda > 0 \tag{50}$$

$$\partial_\lambda S^{\lambda,\mu\nu} = \delta T_a^{\nu\mu} - \delta T_a^{\mu\nu} \tag{51}$$

"kinetic force/gradient"

$$\omega_{\lambda\mu} + \frac{1}{2}\left(\partial_\lambda\beta_\mu - \partial_\mu\beta_\lambda\right) \tag{52}$$

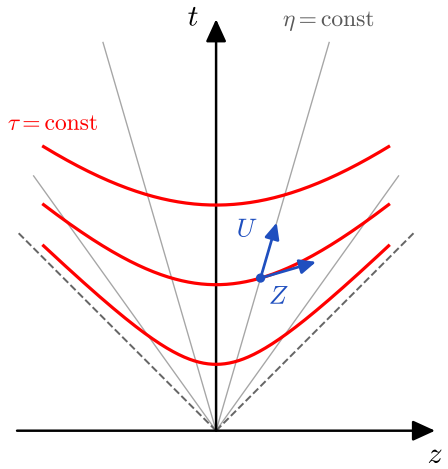
brings spin polarization tensor towards thermal vorticity

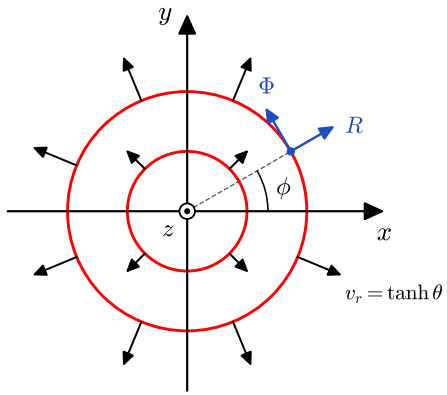
$$\omega_{\lambda\mu} \to \varpi_{\lambda\mu} = -\frac{1}{2}\left(\partial_\lambda\beta_\mu - \partial_\mu\beta_\lambda\right) \tag{53}$$

can we reach global equilibrium?

Boost-invariant and cylindrically symmetric expansion 1

Z. Drogosz, WF, J. Witkowski, arXiv:2611.06945
boost-invariant and cylindrically symmetric vector basis





spin polarization tensor: $\omega_{\mu\nu} = k_\mu U_\nu - k_\nu U_\mu + t_{\mu\nu} = k_\mu U_\nu - k_\nu U_\mu + \epsilon_{\mu\nu\alpha\beta} U^\alpha \omega^\beta$ (54)

$$k^\mu = C_{kr} R^\mu + C_{k\phi} \Phi^\mu + C_{kz} Z^\mu, \quad \omega^\mu = C_{\omega r} R^\mu + C_{\omega\phi} \Phi^\mu + C_{\omega z} Z^\mu \tag{55}$$

$$\boldsymbol{C}_k = \left(C_{kr}, C_{k\phi}, C_{kz}\right), \qquad \boldsymbol{C}_\omega = \left(C_{\omega r}, C_{\omega\phi}, C_{\omega z}\right) \tag{56}$$

thermal vorticity: $\varpi_{\mu\nu} = k^{\rm eq}_\mu U_\nu - k^{\rm eq}_\nu U_\mu = C^{\rm eq}_{kr}\left(R_\mu U_\nu - R_\nu U_\mu\right), \quad C^{\rm eq}_{kr} = \frac{\beta}{2}\left(D\theta - \beta\nabla_R T\right) = \frac{\beta}{2}\left(a_r - \beta\nabla_R T\right)$ (57)

Boost-invariant and cylindrically symmetric expansion 2

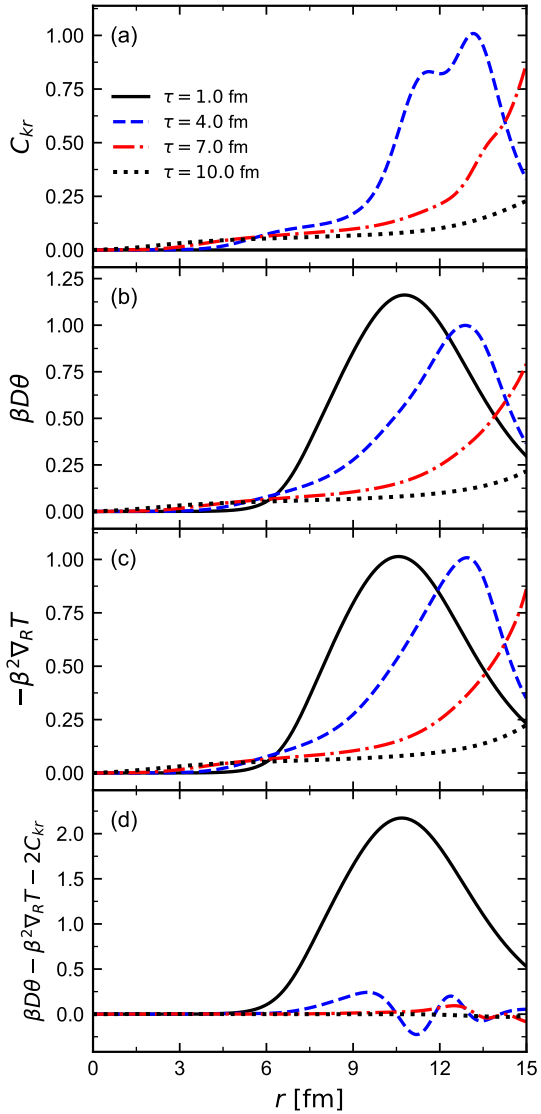
Z. Drogosz, WF, J. Witkowski, arXiv:2611.06945
Israel-Stewart approach used R. Biswas, A. Daher, A. Das, WF, R. Ryblewski

$D = u^\mu \partial_\mu$
$\nabla_R = R^\mu \partial_\mu$

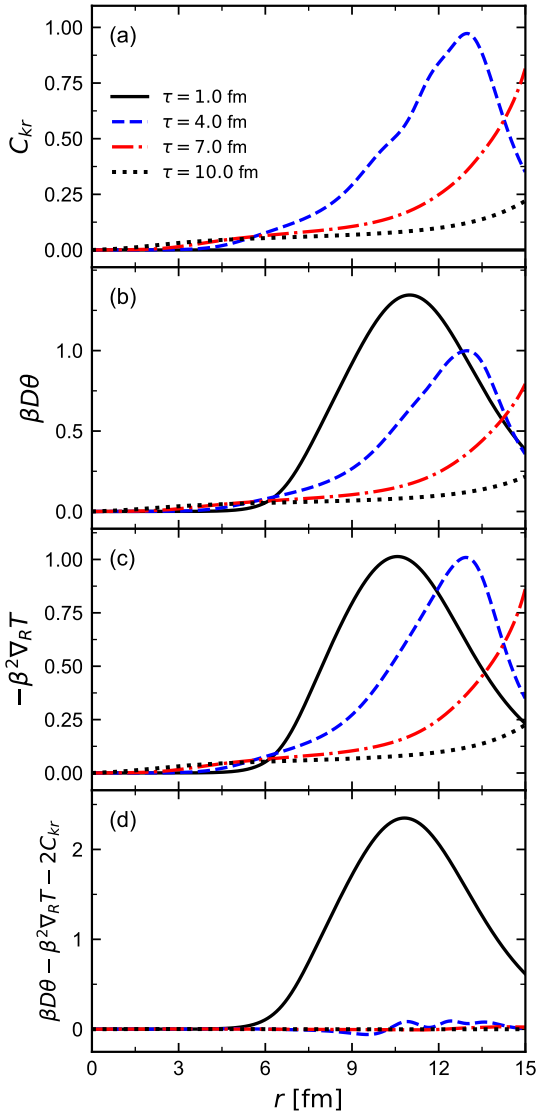
$\beta = 1/T$
$v_r = \tanh(\theta)$

spin kinetic coefficients scaled with entropy density
$\lambda = \lambda_0 \sigma$
with $\lambda_0 = 0.02$

spin polarization tensor approaches thermal vorticity
$\omega \longrightarrow \tilde{\omega}$



Relaxation time $\tau_d = 1.0$ fm



Relaxation time $\tau_d = 0.5$ fm

Conclusions

1. The concept of perfect spin hydrodynamics has been proposed for a relativistic gas of spin 1/2 Dirac fermions. Analogous structures should hold for other systems.
2. With a suitably chosen pseudo-gauge (GLW), a set of consistent hydrodynamic and thermodynamic relations is obtained. Analogy to magneto-hydrodynamics.
3. The correct expansion is twofold – in the magnitude of $\omega = \Omega/T$ and gradients of hydrodynamic variables. Similarly to expansion in the reduced chemical potential $\xi = \mu/T$.
4. Two formulations (based on the classical-spin treatment, or using the spin density matrix) have been given and shown to be exactly consistent up to the second order in ω. Integrations over classical spin is equivalent to taking traces gamma matrices.
5. Direct connection to the Landau-Lifschitz textbook result.
6. Extension to dissipative hydrodynamics has been formulated with the Israel-Stewart method. Dissipation introduces spin orbit interaction.
7. Generalizations to Fermi-Dirac and Bose-Einstein statistics have been made (papers by Sudip Kumar Kar and Valeriya Mykhaylova).
8. We are able to verify nonlinear causality and symmetric hyperbolicity of the equations of motion of perfect spin hydrodynamics constructed with our equilibrium functions, which ensures local well-posedness of the initial value problem and stability of the theory.
9. Hybrid approach: perfect spin hydrodynamics constructed with relativistic gas in mind and kinetic theory, going beyond perfect fluid achieved with general methods.
10. Realistic simulations ahead of us.
    Sapna, S. K. Singh, D. Wagner, Phys. Rev. C112 (2025) 054902
    S. K. Singh, R. Ryblewski, WF, Phys. Rev. C111 (2025) 024907

BACK-UP SLIDES 1: THERMODYNAMIC RELATIONS

inclusion of spin, $\Omega_{\alpha\beta}$ - spin chemical potential

$S^{\alpha\beta}$ - spin density tensor introduced in J. Weyssenhoff, A. Raabe, Acta Phys. Polon. 9 (1947) 7

$$\varepsilon + P = T\sigma + \mu n + \tfrac{1}{2}\Omega_{\alpha\beta}S^{\alpha\beta} \tag{58}$$

$$d\varepsilon = Td\sigma + \mu dn + \tfrac{1}{2}\Omega_{\alpha\beta}dS^{\alpha\beta} \qquad dP = \sigma dT + nd\mu + \tfrac{1}{2}S^{\alpha\beta}d\Omega_{\alpha\beta} \tag{59}$$

multiplication of the above equations by the hydrodynamic flow vector u gives **the tensor (Israel-Stewart) form**
W. Israel, J.M.Stewart, Annals Phys. 118 (1979) 341 & Phys.Lett. A58 (1976) 213

$$S^{\mu}_{\rm eq} = P\beta^{\mu} - \xi N^{\mu}_{\rm eq} + \beta_{\lambda}T^{\lambda\mu}_{\rm eq} - \tfrac{1}{2}\omega_{\alpha\beta}S^{\mu,\alpha\beta}_{\rm eq} \tag{60}$$

$$dS^{\mu}_{\rm eq} = -\xi dN^{\mu}_{\rm eq} + \beta_{\lambda}dT^{\lambda\mu}_{\rm eq} - \tfrac{1}{2}\omega_{\alpha\beta}dS^{\mu,\alpha\beta}_{\rm eq}, \quad d(P\beta^{\mu}) = N^{\mu}_{\rm eq}d\xi - T^{\lambda\mu}_{\rm eq}d\beta_{\lambda} + \tfrac{1}{2}S^{\mu,\alpha\beta}_{\rm eq}d\omega_{\alpha\beta} \tag{61}$$

spin tensor

$$S^{\mu,\alpha\beta}_{\rm eq} = u^{\mu}S^{\alpha\beta}_{\rm eq} \tag{62}$$

analog to the perfect-fluid forms of $N^{\mu}_{\rm eq}$ and $T^{\lambda\mu}_{\rm eq}$, however, in kinetic theory we find

$$S^{\mu,\alpha\beta}_{\rm eq} = u^{\mu}S^{\alpha\beta}_{\rm eq} + \text{problem} \tag{63}$$

problem = term that is not proportional to u^{μ}