Quantum Metric in Relativistic Matter: From Chiral Kinetic Theory to Color Superconductivity

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with Naoki Yamamoto, arXiv: 2509.15731
with Noriyuki Sogabe arXiv:2609.07835

An Essay on “Quantum Geometry”

PHYSICAL REVIEW LETTERS **131,** 240001 (2023)



Essay: Where Can Quantum Geometry Lead Us?

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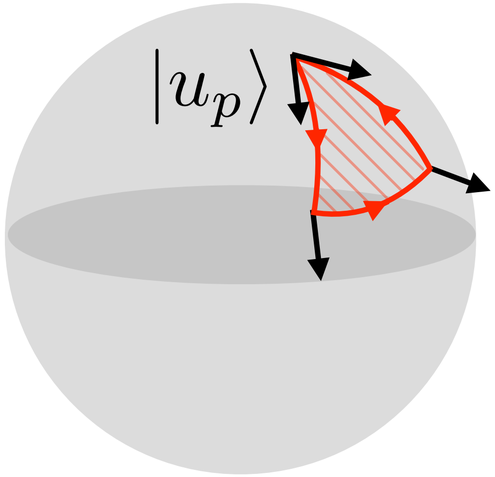
(Received 17 November 2023; published 11 December 2023)

Quantum geometry defines the phase and amplitude distances between quantum states. The phase distance nd thus relates to topological phenomena. The significa e amplitude distance characterized by the quantum metric, h years. Various quantum transport and interaction phenome ed by quantum geometry. For example, quantum geometry allows counterintuitive flow of supercurrent in a flat band where single electrons are immobile. In this Essay, I will discuss my view of the important open problems and future applications of this research topic and will try to ins t its best, quantum geometry can open a new chapter in sformative as room-temperature superconductivity. How e effect of quantum geometry are needed. We also have t ost advanced numerical methods.

Chiral Kinetic Theory
KM, Yamamoto (2025)

Color Superconductivity
KM, Sogabe (2026)

Quantum Geometry = Geometry of Hilbert space

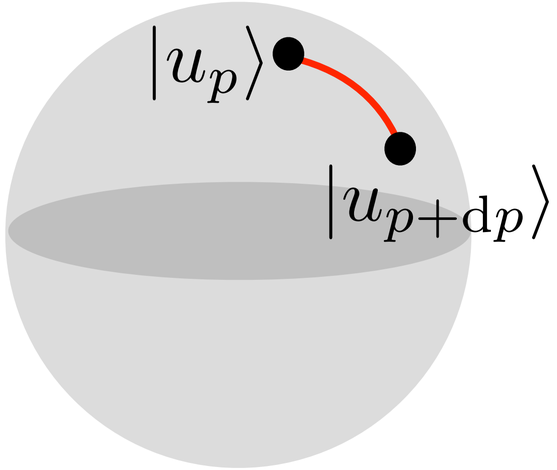


"angle" between two states

$$\alpha = \int_S \boldsymbol{b} \cdot \mathrm{d}\boldsymbol{S}$$

Berry curvature

Berry (1984)



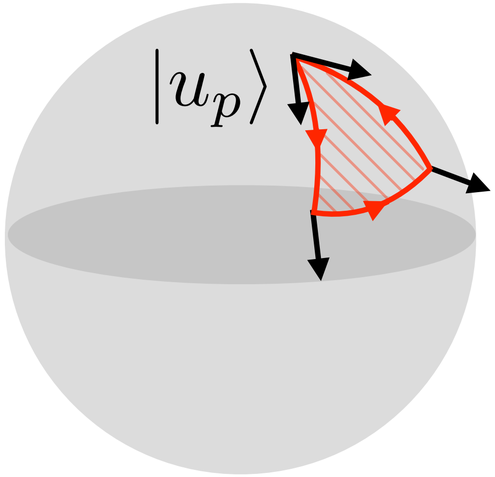
"distance" between nearby states

$$\mathrm{d}s^2 = 1 - |\langle u_p | u_{p+\mathrm{d}p}\rangle|^2 \approx g_{ij}\mathrm{d}p_i\mathrm{d}p_j$$

quantum metric

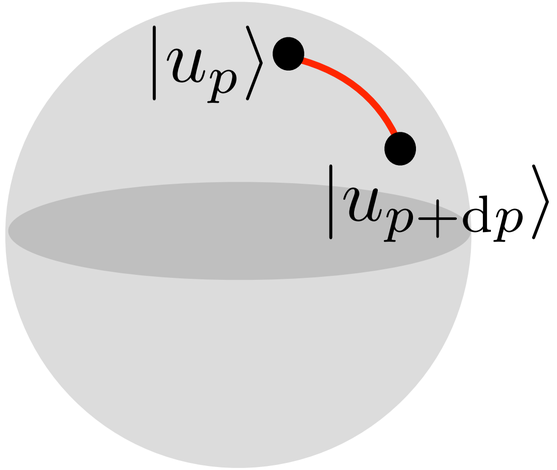
Provost, Vallée (1980)

Quantum Geometry = Geometry of Hilbert space



e.g. anomalous Hall effect

$$j_i = \epsilon_{ijk} E_j \int_{\boldsymbol{p}} b_k f$$



e.g. nonlinear anomalous Hall effect

Gao, Yang, Niu (2014)

$$j_i = \hbar^2 E_j E_k \int_{\boldsymbol{p}} (\partial_i^p G_{jk} - \partial_j^p G_{ik}) f$$

$$G_{ij} \sim g_{ij}/\Delta\epsilon$$

Berry Curvature Corrected Kinetic Theory

Xiao, Shi, Niu (2005)

$$(1+\hbar \boldsymbol{B}\cdot\boldsymbol{b})\partial_t f + [\boldsymbol{v}+\hbar \boldsymbol{E}\times\boldsymbol{b}+\hbar(\boldsymbol{v}\cdot\boldsymbol{b})\,\boldsymbol{B}]\cdot\boldsymbol{\nabla} f$$

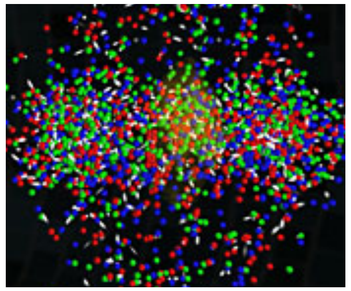
$$+[\boldsymbol{E}+\boldsymbol{v}\times\boldsymbol{B}+\hbar(\boldsymbol{E}\cdot\boldsymbol{B})\boldsymbol{b}]\cdot\boldsymbol{\nabla}_{\boldsymbol{p}} f = (1+\hbar\boldsymbol{B}\cdot\boldsymbol{b})c\,[f]$$

universal form up to $\boldsymbol{b}$

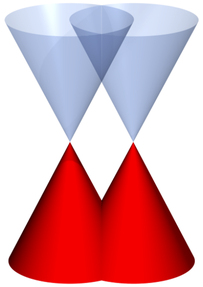
e.g. chiral kinetic theory

Son, Yamamoto (2012) Stephanov, Yin (2012)
Chen, Pu, Wang, Wang (2012)

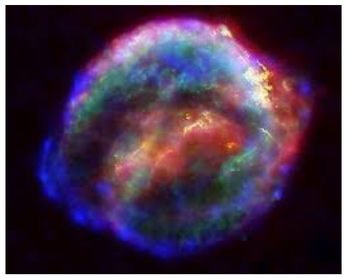
$$\boldsymbol{b} = \frac{\hat{\boldsymbol{p}}}{2|\boldsymbol{p}|^2}$$



QGP



Weyl SM



supernova

Modified Phase Space

- Boltzmann kinetic theory

$$(n, \boldsymbol{j}) = \int_{\boldsymbol{p}} (1, \dot{\boldsymbol{x}}) f \quad \Rightarrow \quad \partial_t n + \boldsymbol{\nabla} \cdot \boldsymbol{j} = 0$$

Modified Phase Space

- Boltzmann kinetic theory

$$(n, \boldsymbol{j}) = \int_{\boldsymbol{p}} (1, \dot{\boldsymbol{x}}) f \quad \Rightarrow \quad \partial_t n + \boldsymbol{\nabla} \cdot \boldsymbol{j} = 0$$

- BC corrected kinetic theory (without anomaly)

$$(n, \boldsymbol{j}) = \int_{\boldsymbol{p}} (1, \dot{\boldsymbol{x}}) \sqrt{\omega} f \quad \Rightarrow \quad \partial_t n + \boldsymbol{\nabla} \cdot \boldsymbol{j} = 0$$

Modified Phase Space

・Boltzmann kinetic theory

$$(n, \boldsymbol{j}) = \int_{\boldsymbol{p}} (1, \dot{\boldsymbol{x}}) f \quad \Rightarrow \quad \partial_t n + \boldsymbol{\nabla} \cdot \boldsymbol{j} = 0$$

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obtained from Poisson bracket

Xiao, Shi, Niu (2005)
Duval, Horvath, Horvathy,
Martina, Stichel (2005)

$$\sqrt{\omega} := \det\{\xi^a, \xi^b\}^{-1} = 1 + \hbar \boldsymbol{b} \cdot \boldsymbol{B} \qquad \xi^a = (p_i, x_i)$$

Foundation Is Missing

	Berry curvature	Quantum metric
transport phenomena	AHE/CME/CVE etc.	nonlinear AHE Gao, Yang, Niu (2014)
definition of currents	$(n, \boldsymbol{j}) = \int_{\boldsymbol{p}} (1, \dot{\boldsymbol{x}}) \sqrt{\omega} f$ $\sqrt{\omega} = 1 + \hbar \boldsymbol{b} \cdot \boldsymbol{B}$	?
conservation (up to anomaly)	$\partial_t n + \boldsymbol{\nabla} \cdot \boldsymbol{j} = 0$	?

One-particle formalism complements Wigner function formalism

Dynamical System with Quantum Metric

w/o mag. field

KM, Yamamoto (2025)

$$L = \underbrace{\boldsymbol{p}\cdot\dot{\boldsymbol{x}} - \hbar\boldsymbol{a}\cdot\dot{\boldsymbol{p}}}_{\pi_a(\xi)\dot{\xi}^a} \underbrace{- \epsilon_{\boldsymbol{p}} - \Phi}_{-H} - \frac{\hbar^2}{2}\dot{p}_i\dot{p}_j G_{ij} \qquad \xi^a = (p_i, x_i)$$

canonical eq.

$$\dot{\xi}^a = \{H, \xi^a\} = -\{\xi^a, \xi^b\}\frac{\partial H}{\partial \xi^b} \quad \rightarrow \quad \sqrt{\omega} := \det\{\xi^a, \xi^b\}^{-1}$$

matching with $\delta \int L\,\mathrm{d}t = 0$

Son, Yamamoto (2012)

Dynamical System with Quantum Metric

w/o mag. field

KM, Yamamoto (2025)

$$L = \boldsymbol{p}\cdot\dot{\boldsymbol{x}} - \hbar\boldsymbol{a}\cdot\dot{\boldsymbol{p}} - \epsilon_{\boldsymbol{p}} - \Phi - \frac{\hbar^2}{2}\dot{p}_i\dot{p}_j G_{ij} \qquad \xi^a = (p_i, x_i)$$

$\pi_a(\xi)\dot{\xi}^a$ $\quad -H$ $\quad$ different form $\sim \dot{\xi}^a\dot{\xi}^b$

canonical eq.

$$\dot{\xi}^a = \{H, \xi^a\} = -\{\xi^a, \xi^b\}\frac{\partial H}{\partial \xi^b} \quad \rightarrow \quad \sqrt{\omega} := \det\{\xi^a, \xi^b\}^{-1}$$

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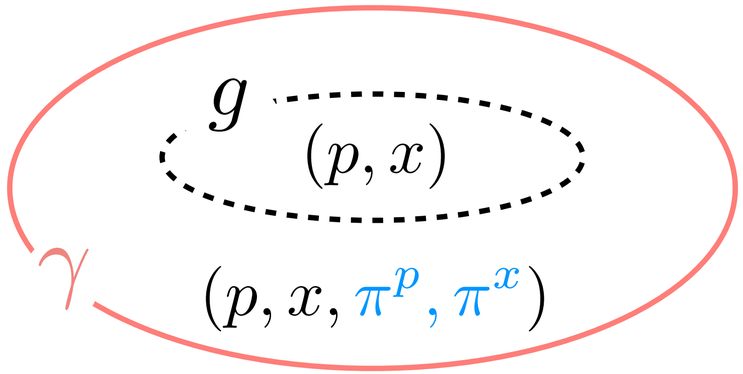
Dynamical Degrees of Freedom

Lagrangian

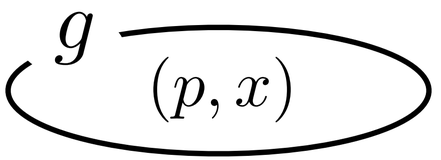
$$L(p, x, \dot{p}, \dot{x})$$

Hamiltonian

$$H(p, x, \pi^p, \pi^x)$$

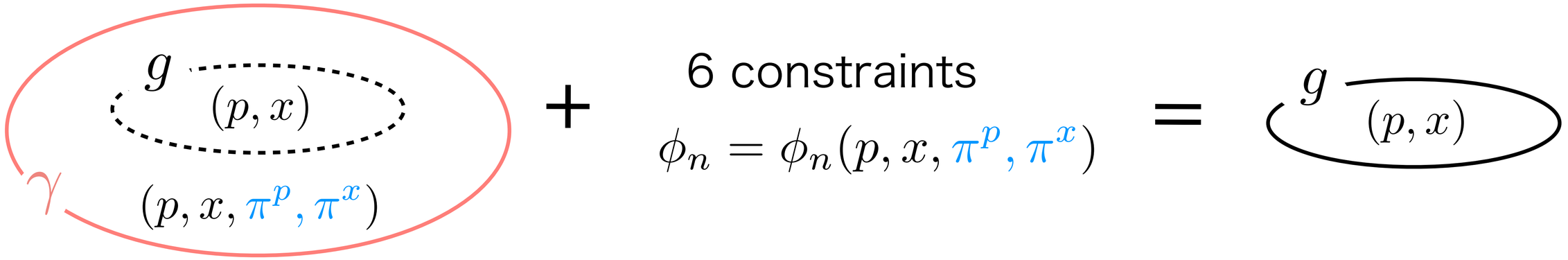


$$\neq$$



Dynamical Degrees of Freedom

Lagrangian $L(p, x, \dot{p}, \dot{x})$ → Hamiltonian $H(p, x, \pi^p, \pi^x)$



Dirac bracket $\{X, Y\}_{\mathrm{D}} = \{X, Y\} - \{X, \phi_n\}\{\phi_n, \phi_m\}^{-1}\{\phi_m, Y\}$

Dirac (1950) Bergmann (1949)

$$\sqrt{\omega} = \det\{\xi^a, \xi^b\}_{\mathrm{D}}^{-1} = 1 + \hbar^2 G_{ij}\partial_i E_j$$

KM, Yamamoto (2025)

QM-Corrected Kinetic Theory

KM, Yamamoto (2025)

- 4+3-dim kinetic equation (w/o mag. field and collisions)

$$\left[\partial_t + (\boldsymbol{\nabla}_{\boldsymbol{p}}\epsilon'_{\boldsymbol{p}} + \hbar\boldsymbol{E}\times\boldsymbol{b}' + \hbar^2\boldsymbol{V})\cdot\boldsymbol{\nabla} + \boldsymbol{E}\cdot\boldsymbol{\nabla}_{\boldsymbol{p}}\right] f = 0$$

$$\epsilon'_{\boldsymbol{p}} = \epsilon_{\boldsymbol{p}} - \frac{\hbar^2}{2}G_{ij}E_iE_j \qquad b'_i = \epsilon_{ijk}\partial^p_j(a_k + \hbar G_{kl}E_l) \qquad V_i = -G_{ij}\partial^p_k\epsilon_{\boldsymbol{p}}\partial_k E_j$$

- current consistent with conservation law and QFT

Yang, Gao, Liang, Wang (2020)
KM (2023) Yang, Gao, Pu (2025)

$$(n, \boldsymbol{j}) = \int_{\boldsymbol{p}} (1, \dot{\boldsymbol{x}})\sqrt{\omega}f \qquad G_{ij} = \frac{1}{4|\boldsymbol{p}|^3}(\delta_{ij} - \hat{p}_i\hat{p}_j)$$

$$n^{(EE)} = -\frac{\hbar^2 C}{24\pi^2}|\boldsymbol{E}|^2 \qquad \boldsymbol{j}^{(EE)} = \frac{\hbar^2 C}{12\pi^2}\left[-\boldsymbol{E}(\boldsymbol{E}\cdot\boldsymbol{u}) + |\boldsymbol{E}|^2\boldsymbol{u}\right]$$

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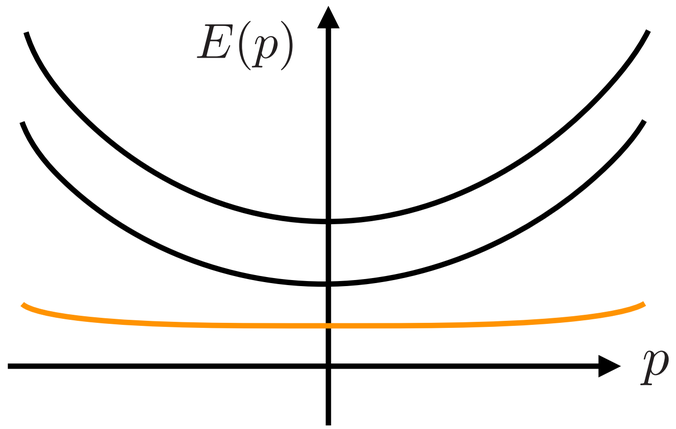
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Flat-band Superconductor



“dispersionless” energy dispersion

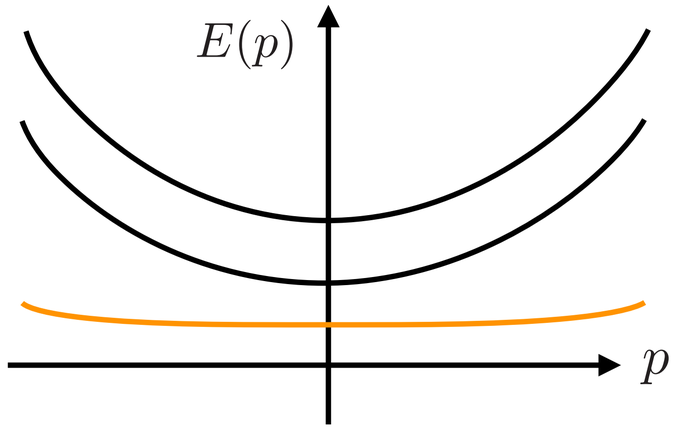
e.g. twisted bilayer graphene

Cao et al. (2018)

- Huge density of states → high-temperature superconductor?

Kopnin, Heikkilä, Volovik (2011)

Flat-band Superconductor



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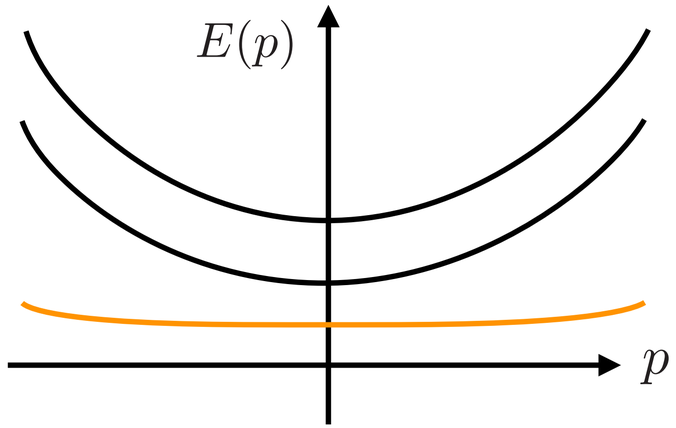
  Kopnin, Heikkilä, Volovik (2011)

- Quenched group velocity → no supercurrent? no Meissner effect?

$$J_{\mathrm{conv}}$$

$$v = \partial E / \partial p$$

Flat-band Superconductor



"dispersionless" energy dispersion

e.g. twisted bilayer graphene

Cao et al. (2018)

- Huge density of states → high-temperature superconductor?

  Kopnin, Heikkilä, Volovik (2011)

- Quenched group velocity → ~~no supercurrent? no Meissner effect?~~

  Peotta, Törmä (2015)

$$J = J_{\mathrm{conv}} + J_{\mathrm{geom}}$$

J_{conv}: $v = \partial E / \partial p$

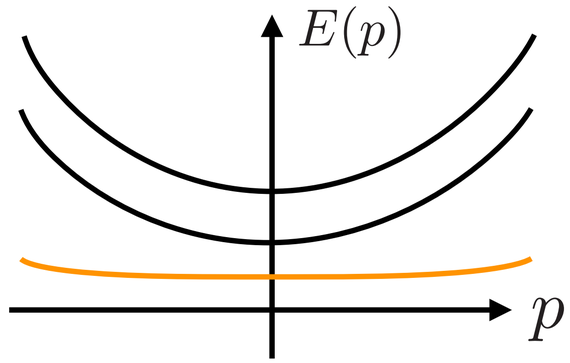
J_{geom}: quantum metric $g_{ij} = \mathrm{tr}\left[\mathcal{P}\partial^p_{(i}\mathcal{P}\partial^p_{j)}\mathcal{P}\right]$

$\mathcal{P} = |u_{\boldsymbol{p}}\rangle\langle u_{\boldsymbol{p}}|$

Flat-Band in QCD?

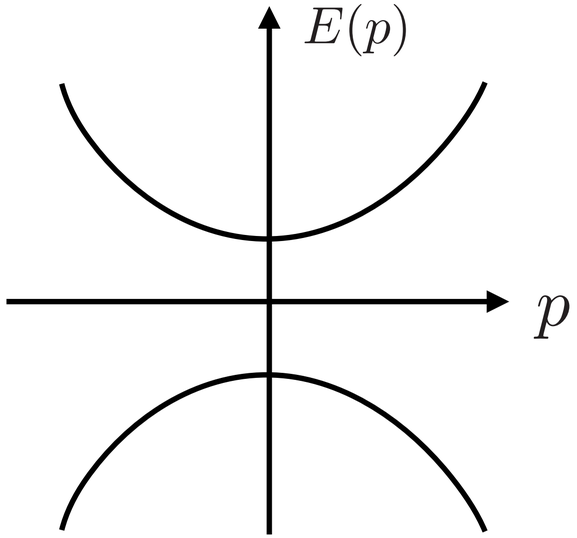
cond-mat

TB graphene



QCD

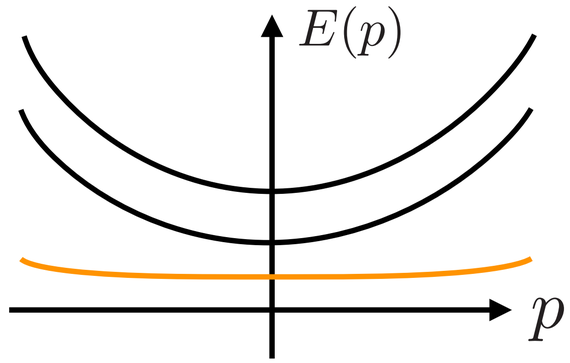
typical case



Flat-Band in QCD?

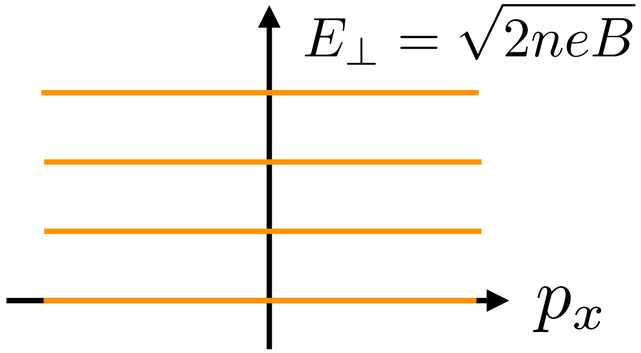
cond-mat

TB graphene



QCD

magnetized quarks



Ferrer, Incera, Manuel (2005)

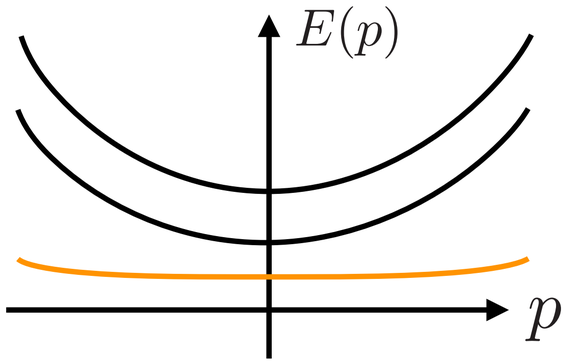
Noronha, Shovkovy (2007)

Fukushima, Warringa (2007)

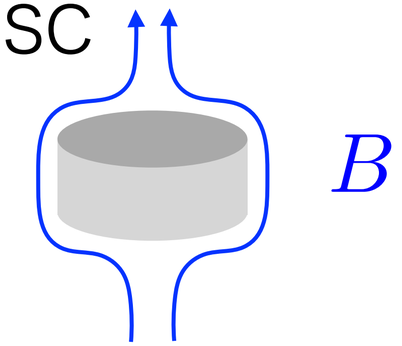
Flat-Band in QCD?

cond-mat

TB graphene

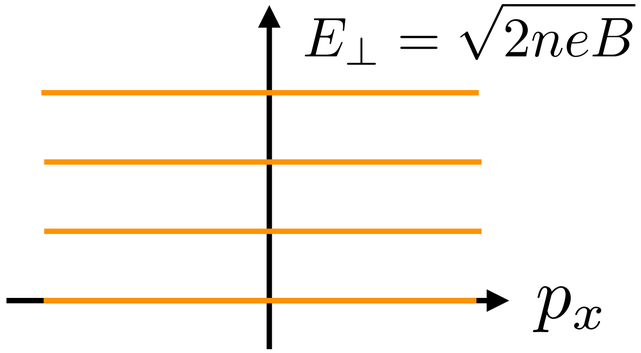


U(1) SC



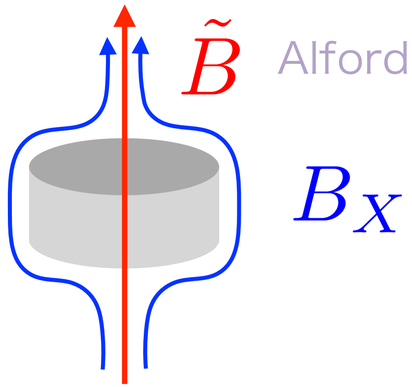
QCD

magnetized quarks



Ferrer, Incera, Manuel (2005)
Noronha, Shovkovy (2007)
Fukushima, Warringa (2007)

2SC



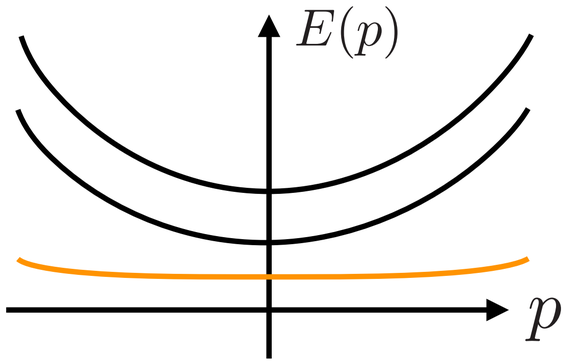
Alford, Berges, Rajagopal (1999)

“Landau-level flat-band” superconductor

Flat-Band in QCD?

cond-mat

TB graphene

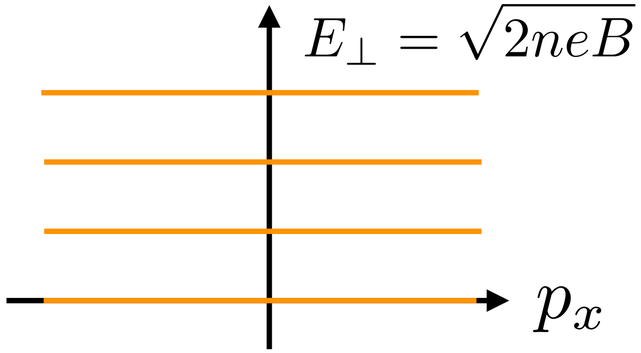


U(1) SC Peotta, Törmä (2015)

finite m_{M} due to g_{ij}

QCD

magnetized quarks



Ferrer, Incera, Manuel (2005)
Noronha, Shovkovy (2007)
Fukushima, Warringa (2007)

2SC KM, Sogabe (2026)

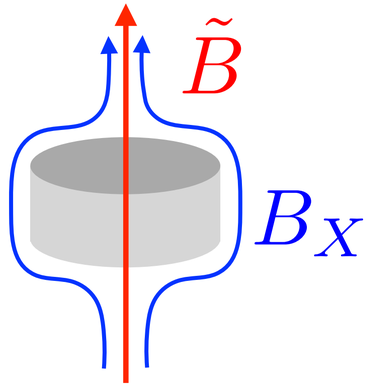
finite $m_{\mathrm{M}}^{\perp}$ due to g_{ij}

Mean-Field Theory of 2SC

Nambu-Gor'kov Propagator for Weyl fermions

$$G^{-1} = -\partial_\tau - (-\mathrm{i}\sigma_i D_i - \mu)\tau_3 + \Delta\tau_+ + \Delta^\dagger\tau_-$$

constant gap



Meissner mass of B_X

$$(m_{\mathrm{M}}^{\perp})^2 \sim \mathrm{Tr}[G\sigma_i^{\perp} G\sigma_i^{\perp}]$$

Transition b/w adjacent LLs provides

$$\langle n|\sigma_i^{\perp}|n\pm 1\rangle = \mathrm{i}(\epsilon_n - \epsilon_{n\pm 1})\langle n|x_i^{\perp}|n\pm 1\rangle$$

source of g_{ij}

cf. Berry connection

$$a_i(\boldsymbol{p}) = -\mathrm{i}\langle u_{\boldsymbol{p}}|\partial_i^p|u_{\boldsymbol{p}}\rangle$$

Strong Magnetic Field Limit

- naive guess

no transverse kinematics
(dimensional reduction)

$$m_{\mathrm{M}}^{\perp} \overset{?}{\to} 0$$

Strong Magnetic Field Limit

・naive guess　no transverse kinematics (dimensional reduction) → $m_{\mathrm{M}}^{\perp} \overset{?}{\to} 0$

・actual result

$$(m_{\mathrm{M}}^{\perp})^2 \longrightarrow \frac{3q_X^2|\Delta|^2}{2\pi^2}\log\frac{8\tilde{q}\tilde{B}}{|\Delta|^2}$$

cf. $(m_{\mathrm{M,CFL}}^{\perp})^2 \sim \mu^2$

Son, Stephanov (1999)
Rischke (2000)

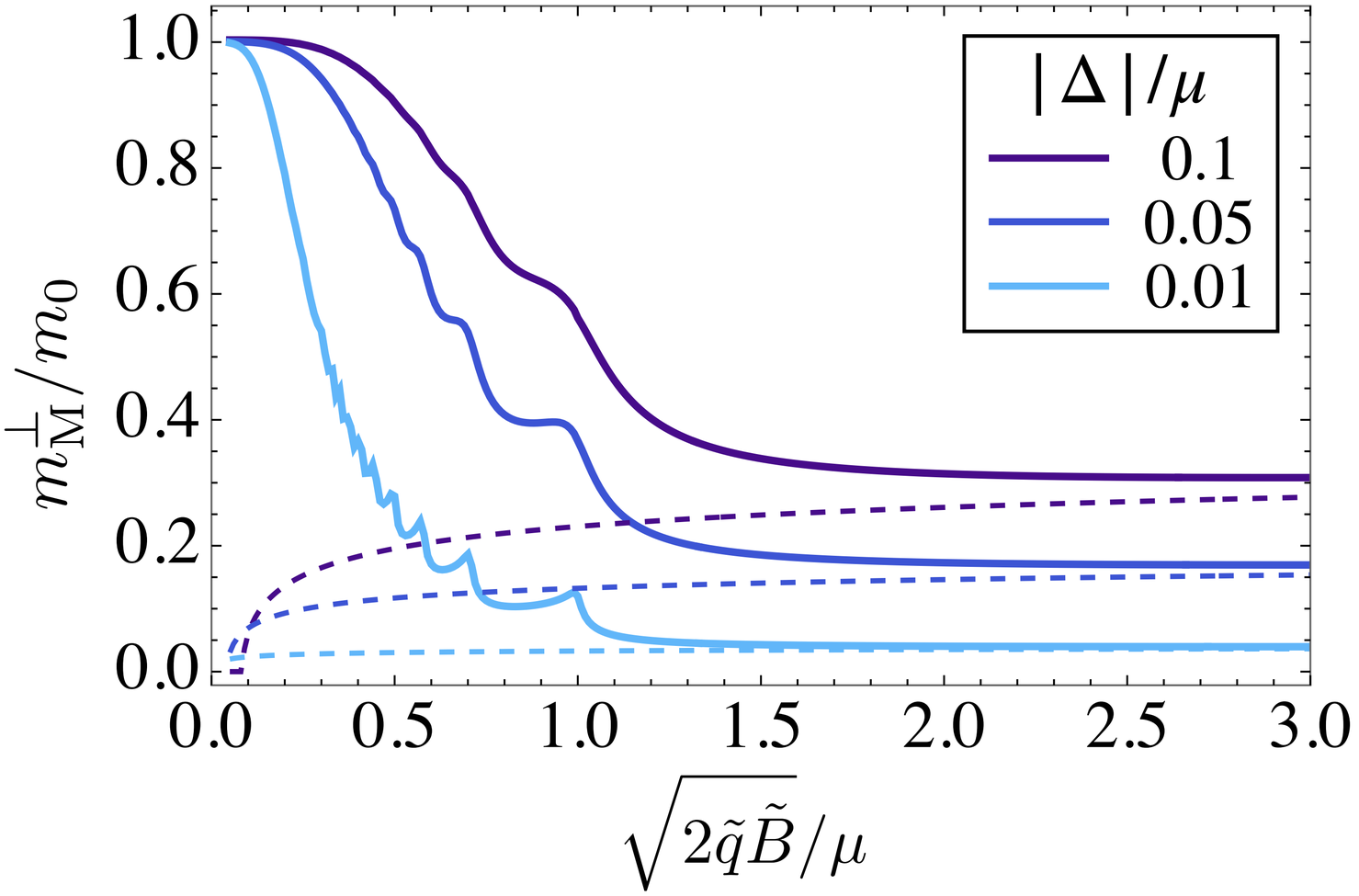
surviving thanks to LLL topology (Chern number)

$$\frac{\tilde{q}\tilde{B}}{2\pi}\mathrm{tr}(g_n) \geq \frac{\tilde{q}\tilde{B}}{2\pi}\frac{|\boldsymbol{b}_n|}{2} = \frac{|C|}{4\pi}$$

Roy (2014)
Peotta, Törmä (2015)
Ozawa, Mera (2021)

quantum-geometric inequality (saturated for LLL)

Weak to Strong Magnetic Field



- zero magnetic field limit

$$m_0 \sim \mu$$ Rischke (2000)

- dashed lines

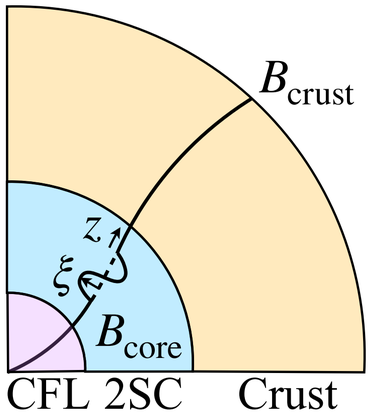
$$(m_{\rm M}^{\perp})^2 \longrightarrow \frac{3q_X^2|\Delta|^2}{2\pi^2}\log\frac{8\tilde{q}\tilde{B}}{|\Delta|^2}$$

- scaling transition

$$m_{\rm M}^{\perp} \propto \mu \longrightarrow m_{\rm M}^{\perp} \propto |\Delta|$$

Magnetar Quasi-Periodic Oscillation

giant flare → magnetic flux oscillation → X-ray observation



Alford, Sedrakian (2010)

Propagation speed depends on Meissner mass

- $f_{\rm 2SC} \sim 1\,{\rm kHz}$
- $Z_{\rm crust} \ll Z_{\rm 2SC} \ll Z_{\rm CFL}$

for $|\Delta|/\mu \sim 10^{-2}$–$10^{-1}$, $R_{\rm 2SC} \sim 1\,{\rm km}$

→ quasi-normal mode in 2SC

$m_{\rm M}^{\perp} \sim |\Delta|$ is a possible solution of kHz QPO unlike $m_{\rm M,CFL}^{\perp} \sim \mu$

Israel et al. (2005)

Summary

- (chiral) kinetic theory

$$(n, \boldsymbol{j}) = \int_{\boldsymbol{p}} (1, \dot{\boldsymbol{x}}) \sqrt{\omega} f$$

$\sqrt{\omega} = 1$ (classical)

$\sqrt{\omega} = 1 + \hbar \boldsymbol{b} \cdot \boldsymbol{B}$ (BC)

$\sqrt{\omega} = 1 + \hbar^2 G_{ij} \partial_i E_j$ (QM) KM, Yamamoto (2025)

- color superconductor (2SC)

KM, Sogabe (2026)

$m_{\mathrm{M}}^{\perp} \propto |\Delta|$ connects QCD to cond-mat and astro-ph

- various possible directions

  — Landau-level quantum geometry in hot magnetized matter

  — quantum geometric inequality as a new constraint on QCD