Linear causality and stability constraints on relativistic second-order magnetohydrodynamics

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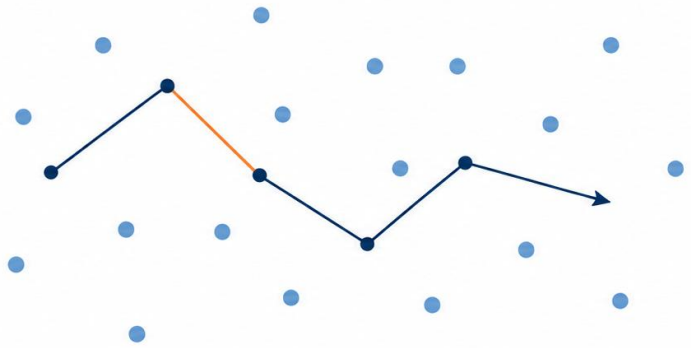
Hydrodynamics of the QGP

A description of collective motion

Conserved densities evolve slowly after microscopic degrees of freedom relax.

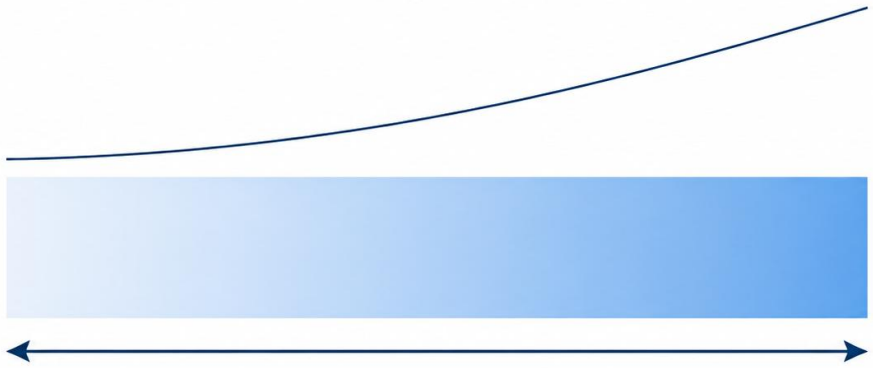
Mean free path

λ_{mfp}



Macroscopic variation scale

L

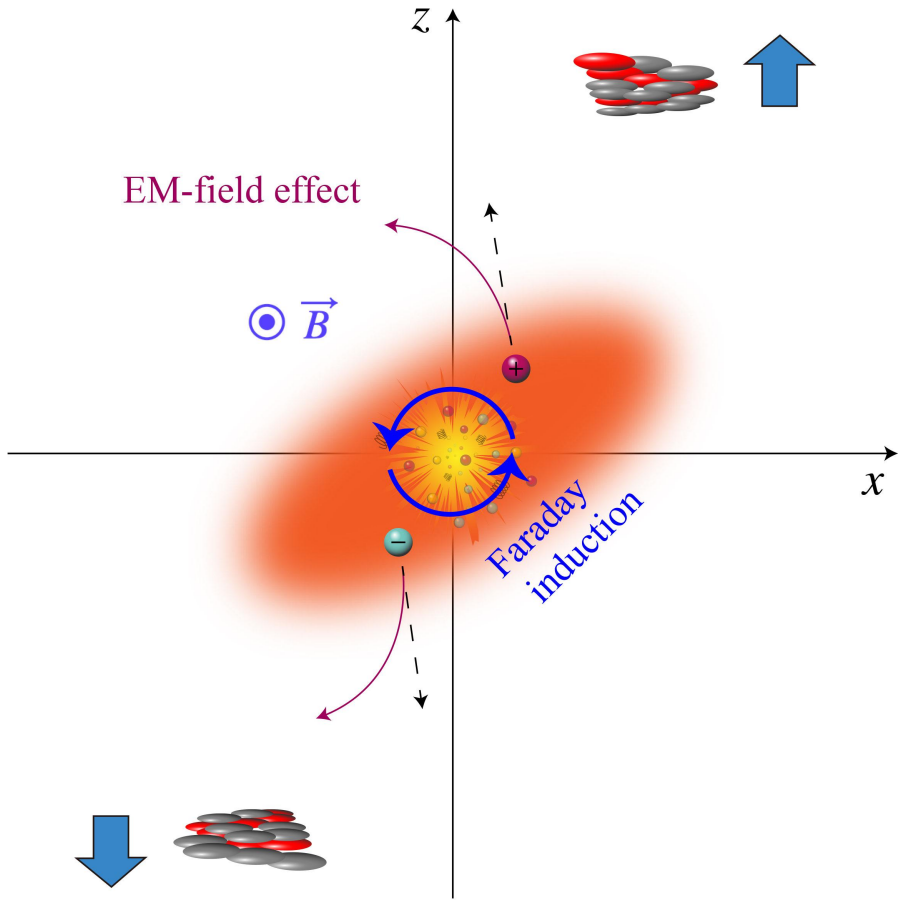


$$\mathrm{Kn} = \frac{\lambda_{\mathrm{mfp}}}{L} \ll 1$$

A small Knudsen number supports an expansion in hydrodynamic gradients.

Refs.: Ollitrault, Eur. J. Phys. 29, 275 (2008); Gale et al., Int. J. Mod. Phys. A 28, 1340011 (2013); Jaiswal & Roy, Adv. High Energy Phys. 2016, 9623034 (2016).

Strong magnetic fields in heavy-ion collisions



$$B \sim 10^{18}\,\mathrm{G}$$

A preferred spatial direction

The magnetic field separates longitudinal and transverse transport.

Transport split

Bulk	$\zeta_{\parallel},\ \zeta_{\perp},\ \zeta_{\times}$
Shear	$\eta_{\parallel},\ \eta_{\perp}$
Resistivity	$\rho_{\parallel},\ \rho_{\perp}$

Refs.: Bzdak & Skokov, Phys. Lett. B 710, 171 (2012); Deng & Huang, Phys. Rev. C 85, 044907 (2012); Hattori, Hongo & Huang, Symmetry 14, 1851 (2022).

Conservation laws and the entropy-current approach

Evolution equations

$$\partial_\mu T^{\mu\nu} = 0$$

$$\partial_\mu N^{\mu} = 0$$

$$\partial_\mu \widetilde{F}^{\mu\nu} = 0$$

Hydrodynamic variables

$$\epsilon,\ n,\ u^\mu,\ B^\mu$$

$$B^\mu = \widetilde{F}^{\mu\nu} u_\nu, \qquad u_\mu B^\mu = 0$$

Landau–Lifshitz frame

Construction principle

$$\partial_\mu \mathcal{S}^\mu \geq 0$$

Thermodynamics and transport
then characteristic modes

Refs.: Hernandez & Kovtun, JHEP 05, 001 (2017); Grozdanov et al., Phys. Rev. D 95, 096003 (2017); Hongo & Hattori, JHEP 02, 011 (2021).

Ideal RMHD and magnetic pressure anisotropy

Ideal-order currents

$$T^{\mu\nu}_{(0)} = \epsilon u^\mu u^\nu - p_\perp \Xi^{\mu\nu} + p_\parallel b^\mu b^\nu$$

$$N^\mu_{(0)} = n u^\mu, \qquad \tilde{F}^{\mu\nu}_{(0)} = B^\mu u^\nu - B^\nu u^\mu$$

$$b^\mu = \frac{B^\mu}{B}, \qquad b^2 = -1, \qquad u_\mu b^\mu = 0$$

$$\Xi^{\mu\nu} = g^{\mu\nu} - u^\mu u^\nu + b^\mu b^\nu$$

(1)–(3)

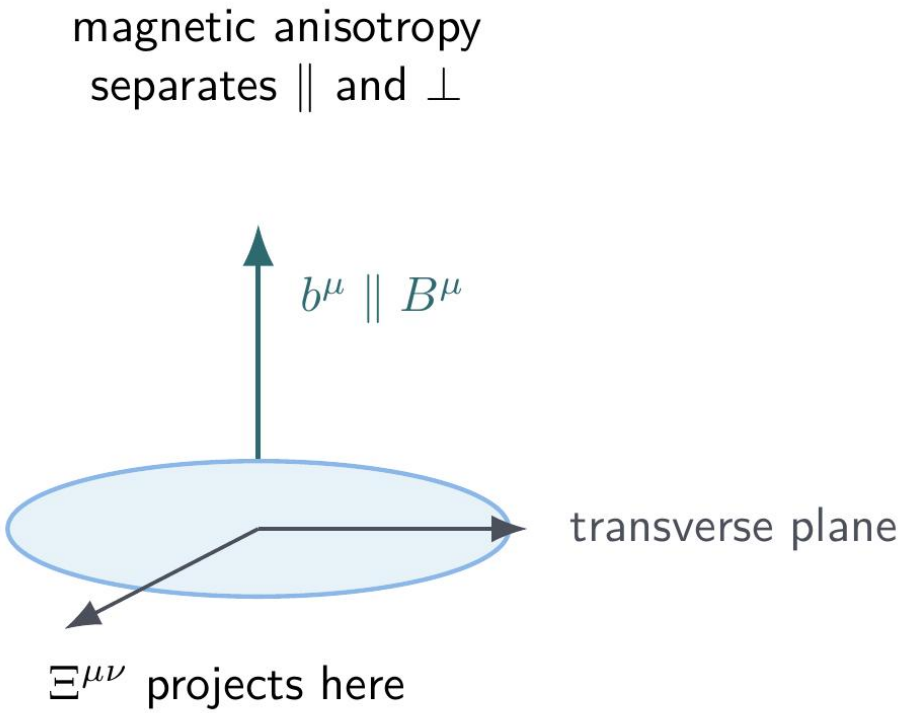
Hydrodynamic projections

$$n = N^\mu_{(0)} u_\mu, \qquad \epsilon = T^{\mu\nu}_{(0)} u_\mu u_\nu$$

$$p_\parallel = T^{\mu\nu}_{(0)} b_\mu b_\nu, \qquad p_\perp = -\tfrac{1}{2} T^{\mu\nu}_{(0)} \Xi_{\mu\nu}$$

$$B^\mu = \tilde{F}^{\mu\nu}_{(0)} u_\nu$$

(4)



Local first-law assumption

$$T\, ds = d\epsilon - \mu\, dn - H_\mu\, dB^\mu$$

(5)

Refs.: Hernandez & Kovtun, JHEP 05, 001 (2017); Fang, Hattori & Hu, Phys. Rev. D 110, 056049 (2024); Hattori, Hongo & Huang, Symmetry 14, 1851 (2022).

Ideal entropy conservation and thermodynamic closure

$$\mathcal{S}^{\mu}_{(0)} = su^{\mu}, \qquad \partial_{\mu}\mathcal{S}^{\mu}_{(0)} = 0, \qquad \beta = \frac{1}{T}$$

Reversible ideal dynamics

$$\begin{aligned}\partial_{\mu}\mathcal{S}^{\mu}_{(0)} = \beta\left(Ts - \epsilon - p_{\perp} + \mu n + B^{\mu}H_{\mu}\right)\theta \\ - \beta b^{\mu}\left[\left(p_{\perp} - p_{\parallel}\right)b^{\nu} + BH^{\nu}\right]\partial_{\mu}u_{\nu}\end{aligned} \tag{6}$$

Independent velocity gradients determine the closure relations.

Thermodynamic identity

$$Ts + \mu n + B^{\mu}H_{\mu} = \epsilon + p_{\perp} \tag{7}$$

Magnetic pressure anisotropy

$$\left(p_{\perp} - p_{\parallel}\right)b^{\nu} + BH^{\nu} = 0 \tag{8}$$

Refs.: Z. Fang, K. Hattori & J. Hu, Phys. Rev. D 110, 056049 (2024).
K. Hattori, M. Hongo & X.-G. Huang, Symmetry 14, 1851 (2022).

First-order currents and entropy current

$$T^{\mu\nu} = T^{\mu\nu}_{(0)} + T^{\mu\nu}_{(1)} = \epsilon u^\mu u^\nu - (p_\perp + \Pi_\perp)\,\Xi^{\mu\nu} + \left(p_\parallel + \Pi_\parallel\right) b^\mu b^\nu + f^\mu b^\nu + f^\nu b^\mu + \pi^{\mu\nu}_\perp \tag{9}$$

$$N^\mu = N^\mu_{(0)} + N^\mu_{(1)} = n u^\mu + \nu^\mu_\perp \tag{10}$$

$$\tilde{F}^{\mu\nu} = \tilde{F}^{\mu\nu}_{(0)} + \tilde{F}^{\mu\nu}_{(1)} = B^\mu u^\nu - B^\nu u^\mu + b^\mu \ell^\nu - b^\nu \ell^\mu + m^{\mu\nu}_\perp \tag{11}$$

(Landau–Lifshitz frame, Charge conjugation symmetry)

Orthogonal projections

$$\begin{aligned} &\epsilon = T^{\mu\nu} u_\mu u_\nu, \quad p_\perp + \Pi_\perp = -\tfrac{1}{2} T^{\mu\nu}\Xi_{\mu\nu}, \quad p_\parallel + \Pi_\parallel = T^{\mu\nu} b_\mu b_\nu \\ &f^\mu = -\Xi^\mu_\alpha T^{\alpha\beta} b_\beta, \quad \pi^{\mu\nu}_\perp = \Xi^{\mu\nu}_{\alpha\beta} T^{\alpha\beta}, \quad n = N^\mu u_\mu, \quad \nu^\mu_\perp = \Xi^\mu_\nu N^\nu \\ &B^\mu = \tilde{F}^{\mu\nu} u_\nu, \quad \ell^\nu = -\Xi^\nu_\alpha b_\mu \tilde{F}^{\mu\alpha}, \quad m^{\mu\nu}_\perp = \Xi^{\mu[\alpha}\Xi^{\beta]\nu}\tilde{F}_{\alpha\beta} \end{aligned} \tag{12}$$

First-order entropy current

$$\mathcal{S}^\mu_{\rm NS} = \mathcal{S}^\mu_{(0)} + \beta_\nu T^{\mu\nu}_{(1)} - \alpha N^\mu_{(1)} + \mathcal{H}_\nu \tilde{F}^{\mu\nu}_{(1)} + \mathcal{O}\left(\partial^2\right) \tag{13}$$

$$\beta^\mu = \beta u^\mu, \qquad \alpha = \beta\mu, \qquad \mathcal{H}_\mu = \beta H_\mu$$

Refs.: Z. Fang, K. Hattori & J. Hu, Phys. Rev. D 110, 056049 (2024).
K. Hattori, M. Hongo & X.-G. Huang, Symmetry 14, 1851 (2022).

Entropy production and first-order constitutive relations

$$\partial_\mu \mathcal{S}^\mu_{\rm NS} = -\beta\Pi_\perp\theta_\perp - \beta\Pi_\parallel\theta_\parallel + 2f^\mu b^{(\alpha}\Xi^{\beta)}_\mu\partial_\alpha\beta_\beta + \beta\pi^{\mu\nu}_\perp\Xi^{\alpha\beta}_{\mu\nu}\partial_\alpha u_\beta$$
$$- \nu^\mu_\perp\Xi^\rho_\mu\partial_\rho\alpha + 2\ell^\mu b^{[\alpha}\Xi^{\beta]}_\mu\partial_\alpha\mathcal{H}_\beta + m^{\mu\nu}_\perp\Xi^{[\alpha}_\mu\Xi^{\beta]}_\nu\partial_\alpha\mathcal{H}_\beta \qquad (14)$$

The local second law fixes the flux–force relations.

Bulk viscosity

$$\Pi_\perp = -\zeta_\perp\theta_\perp - \zeta_\times\theta_\parallel$$

$$\Pi_\parallel = -\zeta_\parallel\theta_\parallel - \zeta'_\times\theta_\perp$$

Shear viscosity

$$f^\mu = -2\eta_\parallel b^{(\alpha}\Xi^{\beta)\mu}\partial_\alpha\beta_\beta$$

$$\pi^{\mu\nu}_\perp = 2\eta_\perp\Xi^{\mu\nu\alpha\beta}\partial_\alpha u_\beta$$

Charge diffusion

$$\nu^\mu_\perp = \kappa\Xi^{\mu\rho}\partial_\rho\alpha + 2\kappa_\times b^{[\alpha}\Xi^{\beta]\mu}\partial_\alpha\mathcal{H}_\beta$$

magnetic response

$$\ell^\mu = -2\rho_\parallel b^{[\alpha}\Xi^{\beta]\mu}\partial_\alpha\mathcal{H}_\beta - \kappa'_\times\Xi^{\mu\rho}\partial_\rho\alpha$$

$$m^{\mu\nu}_\perp = 2\rho_\perp\Xi^{\mu[\alpha}\Xi^{\beta]\nu}\partial_\alpha\mathcal{H}_\beta$$

Entropy positivity $\quad \zeta_\perp, \zeta_\parallel \geq 0, \qquad \zeta_\perp\zeta_\parallel \geq \frac{1}{4}\left(\zeta_\times + \zeta'_\times\right)^2$ (15)–(21)

Refs.: Z. Fang, K. Hattori & J. Hu, Phys. Rev. D 110, 056049 (2024).
K. Hattori, M. Hongo & X.-G. Huang, Symmetry 14, 1851 (2022).
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Second-order entropy current and relaxation equations

$$\mathcal{S}^{\mu}_{\mathrm{IS}} = \mathcal{S}^{\mu}_{\mathrm{NS}} + Q^{\mu}, \qquad u_{\mu} Q^{\mu} \leq 0$$

Quadratic terms in the dissipative fluxes supply their time derivatives.

$$\tau_i D\Phi_i + \Phi_i = \Phi_i^{\mathrm{NS}} + \text{second-order couplings}$$

$$\Phi_i \in \{\Pi_{\perp}, \Pi_{\parallel}, f^{\mu}, \nu^{\mu}_{\perp}, \ell^{\mu}, \pi^{\mu\nu}_{\perp}, m^{\mu\nu}_{\perp}\}$$

$$\tau_{\Pi_{\perp}} D\Pi_{\perp} + \Pi_{\perp} = -\zeta_{\perp}\theta_{\perp} - \zeta_{\times}\theta_{\parallel} + \cdots$$

The theory has seven relaxation times. Their ratios to transport coefficients enter the causal bounds.

Linearization and mode decomposition

Uniform equilibrium

$$u_0^\mu = (1,0,0,0), \quad B_0^\mu = (0,0,0,B), \quad \boldsymbol{k} = (k\sin\theta, 0, k\cos\theta)$$

Perturbations around equilibrium

$$\Phi = \Phi_0 + \delta\Phi, \qquad \Phi \in \{\epsilon, n, u^\mu, B^\mu\}$$

$$\Psi = \delta\Psi, \quad \Psi \in \{\Pi_\perp, \Pi_\parallel, f^\mu, \nu_\perp^\mu, \ell^\mu, \pi_\perp^{\mu\nu}, m_\perp^{\mu\nu}\}, \quad \Psi_0 = 0 \tag{22}$$

Linear order and Fourier modes

$$\delta X \propto e^{-i\omega t + i\boldsymbol{k}\cdot\boldsymbol{x}}, \qquad M(\omega, \boldsymbol{k})\,\delta X = 0$$

Decoupling assumptions

(conformal equation and vanished vetor coupling)

$$c_s^2 = \frac{1}{3}, \qquad \frac{\partial p}{\partial n} = 0, \qquad \delta\alpha = \frac{\delta n}{\bar{n}}$$

$$n_0 = 0, \qquad \kappa_\times = \kappa'_\times = 0$$

$$\boldsymbol{k}\cdot\delta\boldsymbol{B} = 0 \quad \Longleftrightarrow \quad \sin\theta\,\delta B_x + \cos\theta\,\delta B_z = 0$$

Physical mode sectors

$$\det M = (\det A)(\det B)(\det C) = 0$$

Magnetosonic Alfvén Charge diffusion

$$9 + 6 + 3 = 18 \text{ physical modes}$$

Magnetosonic modes: small-momentum solutions

Fast and slow propagating pairs

$$\omega_{\mathrm{fast/slow},\pm} = \pm\frac{k}{\sqrt{2}}\sqrt{\mathcal{V} \pm \sqrt{\mathcal{V}^2 - 4c_s^2 v_A^2 \cos^2\theta}} + \mathcal{O}(k^2)$$

$$\mathcal{V} = c_s^2 + v_A^2 - c_s^2 v_A^2 \sin^2\theta, \qquad v_A^2 = \frac{B^2}{\mu_m h}, \qquad h = \epsilon + p_\perp$$

Five relaxation modes

$$\omega_\ell = -\frac{i}{\tau_\ell} + i\frac{h v_A^2 \rho_\parallel}{B^2}k^2 + \mathcal{O}(k^4)$$

$$\omega_{\Pi_\perp} = -\frac{i}{\tau_{\Pi_\perp}} + i\frac{\zeta_\perp \sin^2\theta}{h}k^2 + \mathcal{O}(k^4)$$

$$\omega_\pi = -\frac{i}{\tau_\pi} + i\frac{\eta_\perp \sin^2\theta}{h}k^2 + \mathcal{O}(k^4)$$

$$\omega_{\Pi_\parallel} = -\frac{i}{\tau_{\Pi_\parallel}} + i\frac{\zeta_\parallel \cos^2\theta}{h(1-v_A^2)}k^2 + \mathcal{O}(k^4)$$

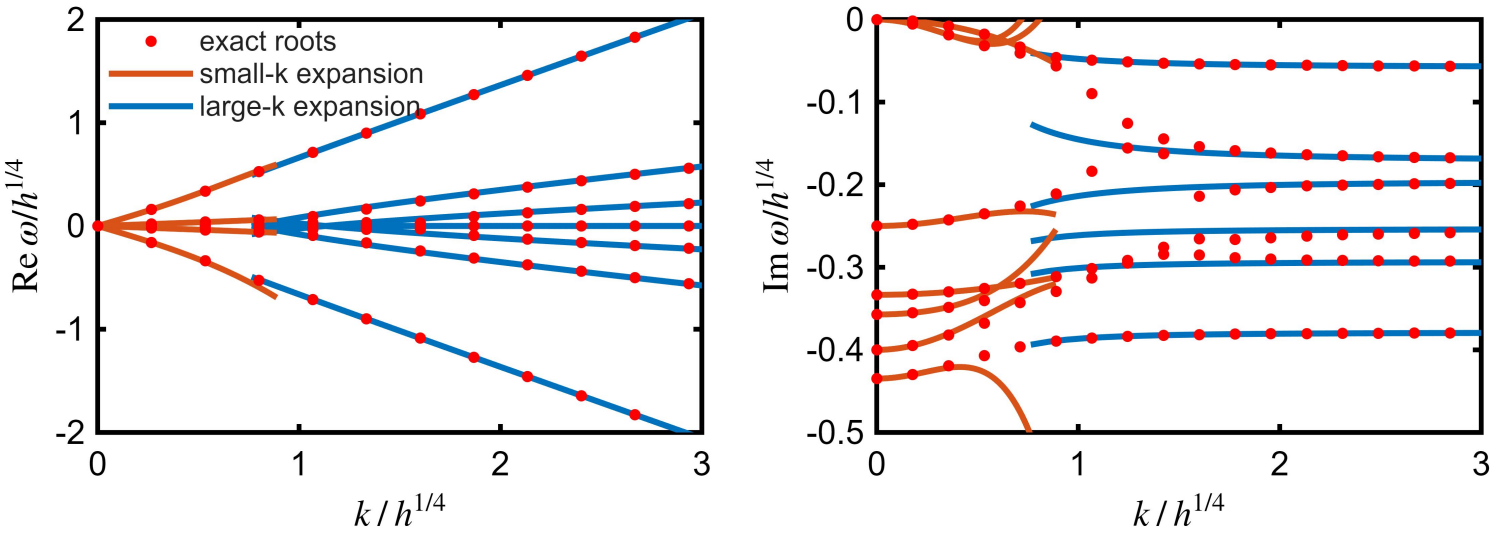
$$\omega_f = -\frac{i}{\tau_f} + i\frac{\eta_f}{h}\left(\cos^2\theta + \frac{\sin^2\theta}{1-v_A^2}\right)k^2 + \mathcal{O}(k^4)$$

Four hydrodynamic branches and five relaxation branches account for all nine physical modes.

Magnetosonic modes: large momentum and exact roots

$$\mathbf{d}_A x^3 + \mathbf{c}_A x^2 + \mathbf{b}_A x + \mathbf{a}_A = 0, \qquad x_i = v_i^2, \quad i = 1, 2, 3$$

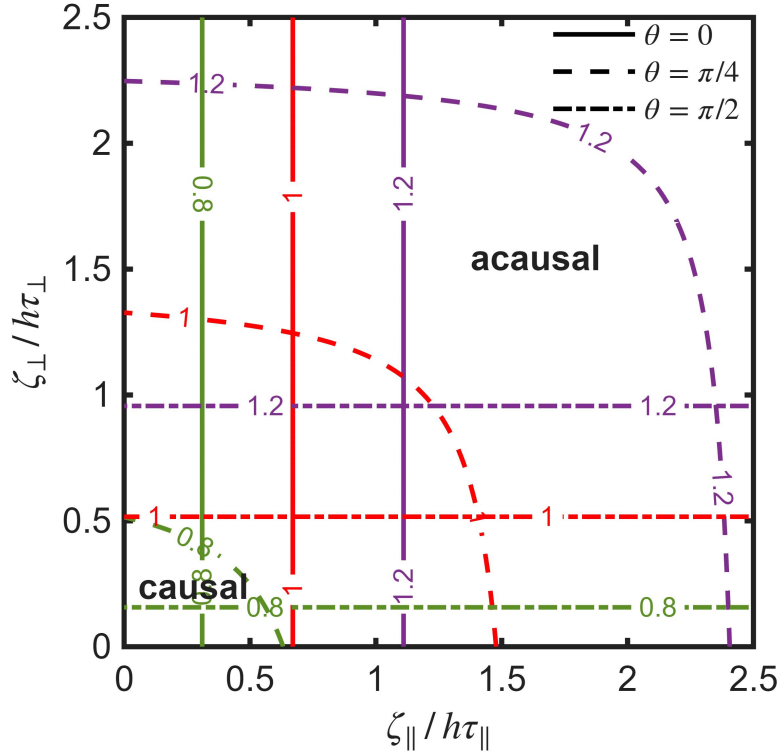
$$\omega_{i,\pm} = \pm\sqrt{x_i}\, k + \mathcal{O}(k^0), \qquad \omega_{7\ldots9} = \omega'_{A,0} + \mathcal{O}(k^{-2})$$

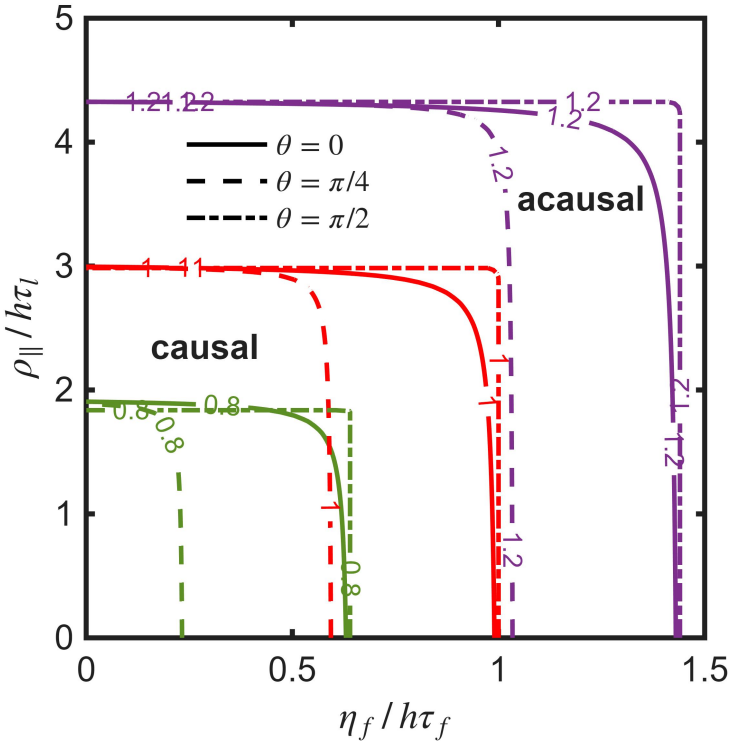


Six branches propagate at large momentum.

Magnetosonic causality: angular constraints

$$v_{\mathrm{ms,max}} = \max_{i=1,2,3} \sqrt{x_i} \leq 1$$





Red contours mark the luminal boundary at the sampled directions.

Alfvén modes: small-momentum solutions

$$\omega_\pm = \pm v_A \cos\theta\, k - i\Gamma(\theta)k^2 + \mathcal{O}(k^3)$$

$$\Gamma(\theta) = \frac{(B^2\eta_f + h^2 v_A^2 \rho_\parallel)\cos^2\theta + (B^2\eta_\perp + h^2 v_A^2 \rho_\perp)\sin^2\theta}{2B^2 h}$$

Four non-hydrodynamic relaxation modes

$$\omega_\ell = -\frac{i}{\tau_\ell} + i\frac{h v_A^2 \rho_\parallel \cos^2\theta}{B^2}k^2 + \mathcal{O}(k^4)$$

$$\omega_m = -\frac{i}{\tau_m} + i\frac{h v_A^2 \rho_\perp \sin^2\theta}{B^2}k^2 + \mathcal{O}(k^4)$$

$$\omega_f = -\frac{i}{\tau_f} + i\frac{\eta_f \cos^2\theta}{h}k^2 + \mathcal{O}(k^4)$$

$$\omega_\pi = -\frac{i}{\tau_\pi} + i\frac{\eta_\perp \sin^2\theta}{h}k^2 + \mathcal{O}(k^4)$$

The wave speed vanishes toward perpendicular propagation. The ordinary expansion then becomes non-uniform.

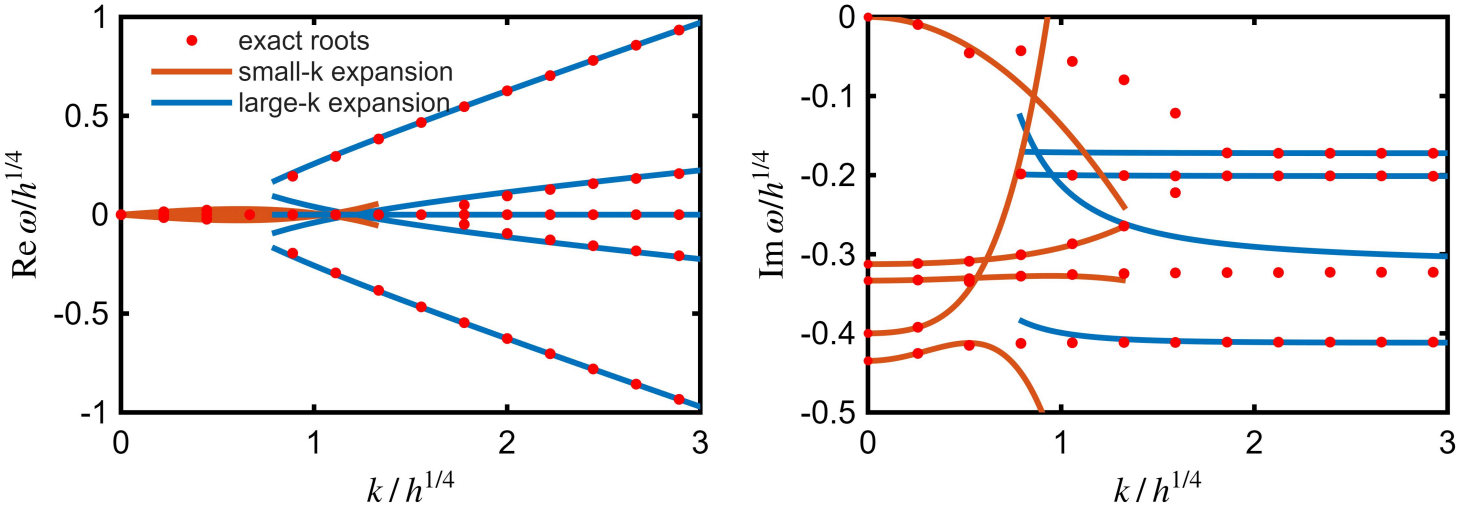
Alfvén modes: explicit large-momentum solutions

$$\omega_{1\ldots4} = \pm k\sqrt{\frac{-\mathbf{b}_B \pm \sqrt{\Delta_B}}{2\mathbf{c}_B}} + \mathcal{O}(k^0), \qquad \mathbf{c}_B = h\tau_\ell\tau_m\tau_f\tau_\pi$$

Two finite relaxation poles

$$\omega_5 = -i\frac{\rho_\parallel \cos^2\theta + \rho_\perp \sin^2\theta}{\rho_\parallel \tau_m \cos^2\theta + \rho_\perp \tau_\ell \sin^2\theta} + \mathcal{O}(k^{-1})$$

$$\omega_6 = -i\frac{\eta_f \cos^2\theta + \eta_\perp \sin^2\theta}{\eta_f \tau_\pi \cos^2\theta + \eta_\perp \tau_f \sin^2\theta} + \mathcal{O}(k^{-1})$$



The short-wavelength speeds depend on dissipative transport and relaxation, as well as the ideal Alfvén speed.

Alfvén causality: an explicit analytical bound

$$v_{B,\max}^2 = \frac{-\mathbf{b}_B + \sqrt{\Delta_B}}{2\mathbf{c}_B} \leq 1$$

Transport and relaxation coefficients

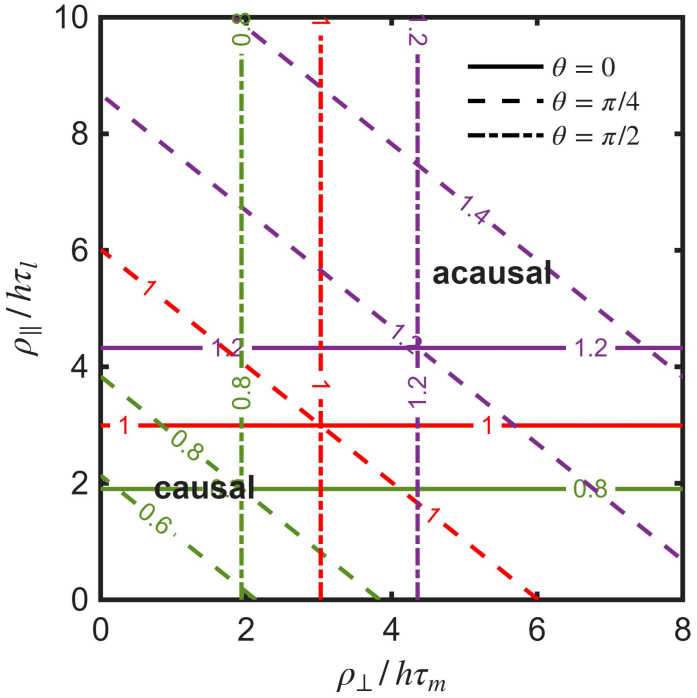
$$\mathcal{R}_B = \rho_{\parallel}\tau_m \cos^2\theta + \rho_{\perp}\tau_\ell \sin^2\theta$$

$$\mathcal{S}_B = \eta_f\tau_\pi \cos^2\theta + \eta_{\perp}\tau_f \sin^2\theta$$

$$\mathbf{a}_B = \frac{hv_A^2}{B^2}\mathcal{R}_B\mathcal{S}_B, \qquad \mathbf{c}_B = h\tau_\ell\tau_m\tau_f\tau_\pi$$

$$\mathbf{b}_B = -\tau_\ell\tau_m\left(\mathcal{S}_B + hv_A^2\tau_f\tau_\pi\cos^2\theta\right) - \frac{h^2v_A^2}{B^2}\tau_f\tau_\pi\mathcal{R}_B$$

$$\Delta_B = \mathbf{b}_B^2 - 4\mathbf{a}_B\mathbf{c}_B$$



$$-\mathbf{b}_B + \sqrt{\Delta_B} \leq 2h\tau_\ell\tau_m\tau_f\tau_\pi$$

Charge diffusion: small-momentum and exact solutions

$$D_\theta = \frac{\kappa \sin^2\theta}{\bar{n}}$$

Long-wavelength expansion

$$\omega_{\mathrm{diff}} = -iD_\theta k^2 - iD_\theta^2 \tau_\nu k^4 - 2iD_\theta^3 \tau_\nu^2 k^6 + \mathcal{O}(k^8)$$

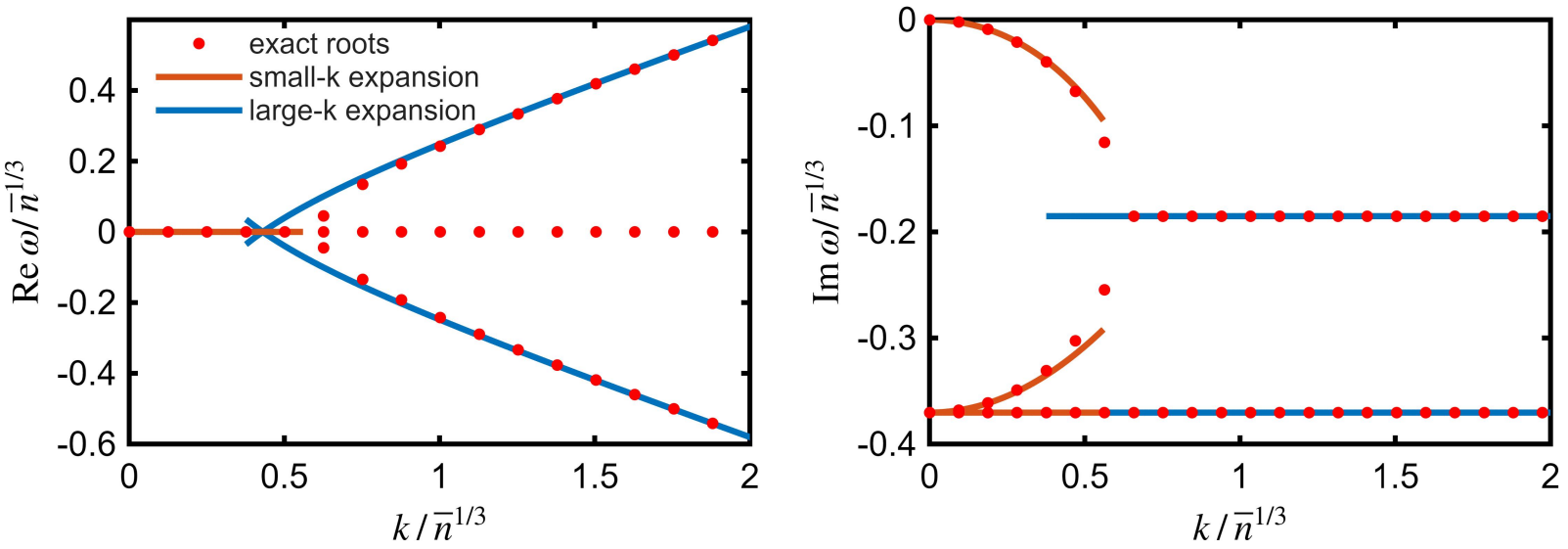
$$\omega_{\mathrm{rel},1} = -\frac{i}{\tau_\nu}$$

$$\omega_{\mathrm{rel},2} = -\frac{i}{\tau_\nu} + iD_\theta k^2 + iD_\theta^2 \tau_\nu k^4 + 2iD_\theta^3 \tau_\nu^2 k^6 + \mathcal{O}(k^8)$$

Charge diffusion: propagating large-momentum roots

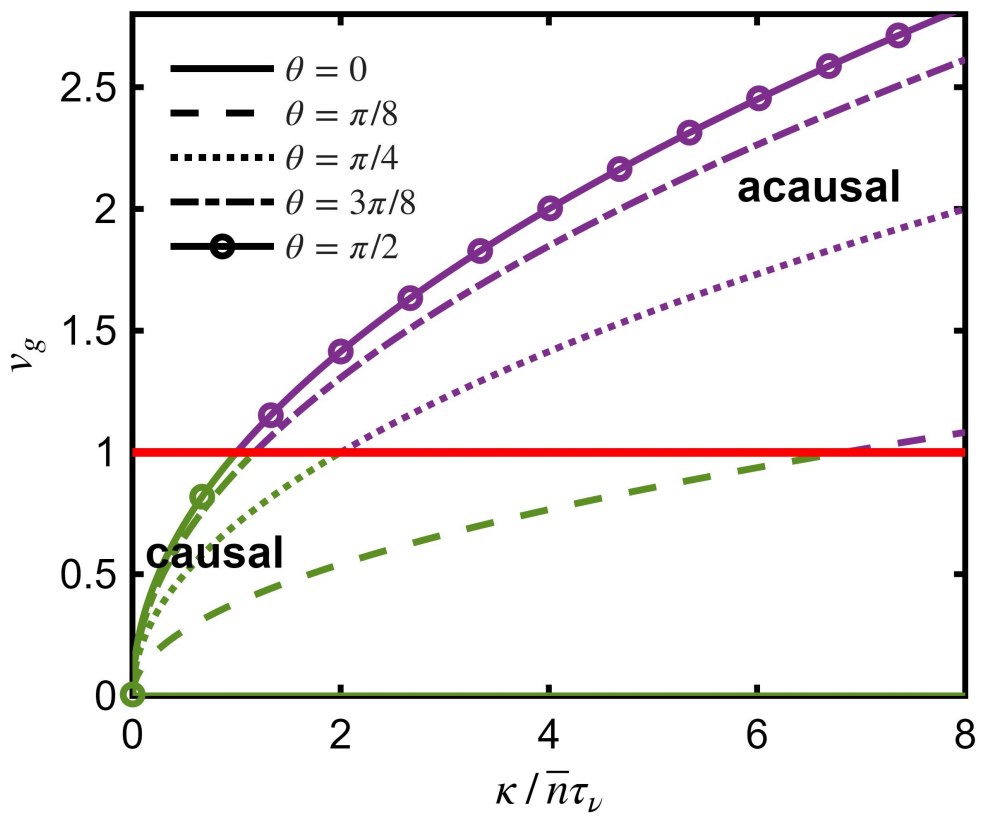
$$\omega_\pm = \pm v_C k - \frac{i}{2\tau_\nu} \mp \frac{1}{8 v_C \tau_\nu^2 k} + \mathcal{O}(k^{-2}), \qquad \omega_3 = -\frac{i}{\tau_\nu}$$

$$v_C^2 = \frac{D_\theta}{\tau_\nu} = \frac{\kappa \sin^2\theta}{\bar{n}\tau_\nu}$$



The diffusive mode and one relaxation mode form a propagating pair. The third root stays purely damped.

Charge-sector causality and the all-angle bound



At a fixed propagation angle

$$v_C^2 = \frac{\kappa \sin^2\theta}{\bar{n}\tau_\nu} \leq 1$$

$$\kappa \sin^2\theta \leq \bar{n}\tau_\nu$$

For every direction

$$\boxed{\kappa \leq \bar{n}\tau_\nu}$$

Perpendicular propagation sets the strongest restriction. The numerical luminal crossings agree with the bound.

Conclusions

Relativistic second-order MHD

We discuss second-order constitutive relations using entropy-current analysis.
They include anisotropic transport and finite relaxation of dissipative currents.

Two momentum limits with different physical roles

Small momentum identifies collective waves and damping.
Large momentum determines characteristic speeds and finite poles.

Explicit analytical bounds with numerical checks

$$-\mathrm{b}_B + \sqrt{\Delta_B} \leq 2h\tau_\ell\tau_m\tau_f\tau_\pi$$

$$\kappa \sin^2\theta \leq \bar{n}\tau_\nu, \qquad \text{all directions:} \quad \kappa \leq \bar{n}\tau_\nu$$

Key references

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Thanks for your attention!