

中国科学院大学
University of Chinese Academy of Sciences

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Lie–Poisson Formulation of Dissipative Spin Hydrodynamics

Lie–Poisson Formulation
arXiv:2611.22615 [hep-th]

Is Spin Slow in Hot QCD?
arXiv:2611.16684 [hep-th]

Zi-Hao Liu

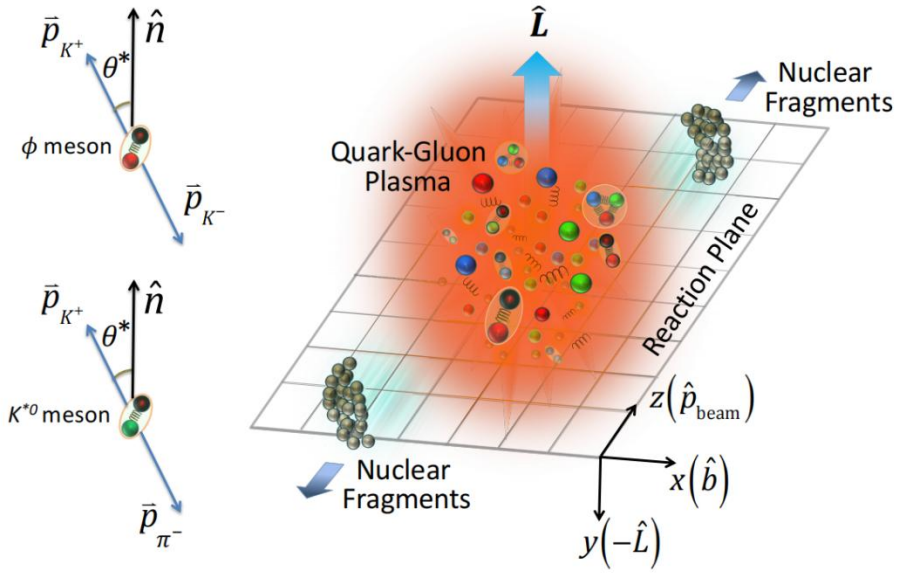
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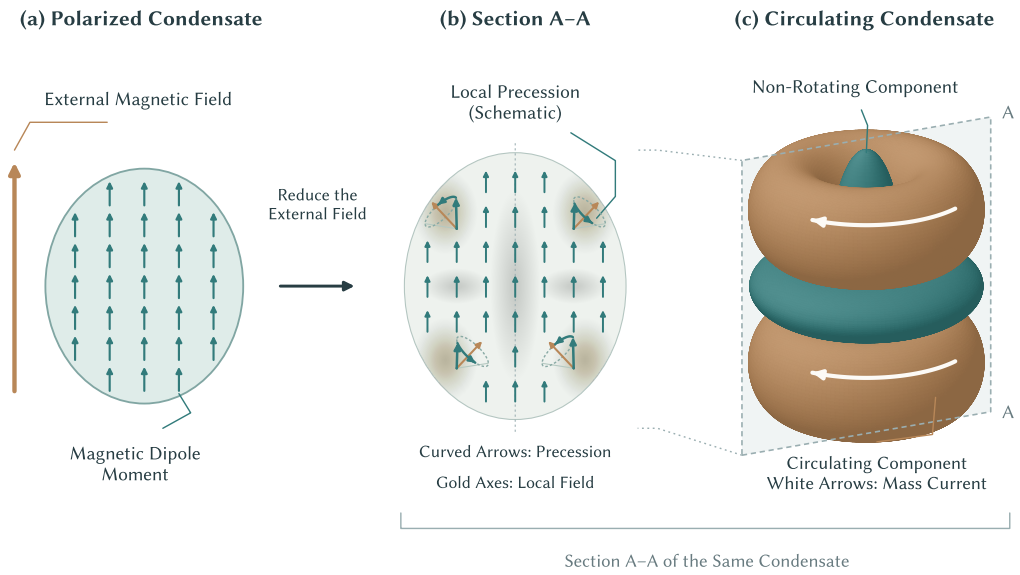
Spin in Hot Matter and Superfluids

Heavy-Ion Collisions



Orbit ⟶ Spin

Dipolar Condensates



Spin ⟶ Orbit

Two Questions: What Fixes How a Flow Acts on Spin? · When Is Spin a Slow Variable?

STAR Collaboration, Nature 614 (2023) 244 · Matsui et al., Science 391 (2026) 384

Lie–Poisson Equations Recover Ideal Hydrodynamics

- Poisson Bracket Fixes the Reversible Kinematics
- Hamiltonian Gives Velocity and Thermodynamic Forces

$$\frac{dF}{dt} = \{F, H\}_{\mathrm{LP}}, \qquad v^i = \frac{\delta H}{\delta \pi_i}$$

$$\partial_t n + \partial_i (n v^i) = 0, \qquad \partial_t s_{\mathrm{ent}} + \partial_i (s_{\mathrm{ent}} v^i) = 0$$

$$\partial_t \pi_i + \partial_j (\pi_i v^j) + \partial_i p = 0$$

n: Number Density · s_{ent}: Entropy Density · π_i: Momentum Density · p: Pressure

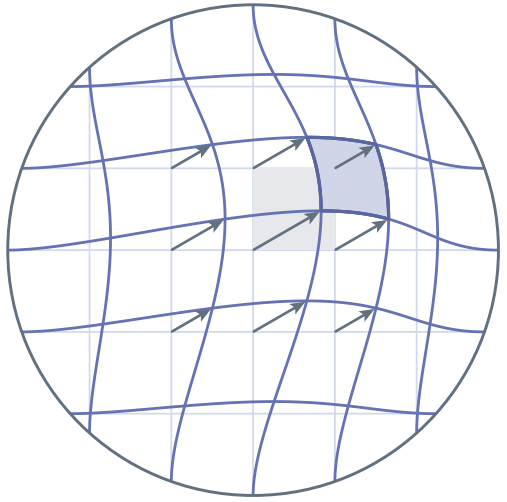
Continuity and Euler Equations Follow for Any Energy without Gradient Terms

Morrison and Greene, Phys. Rev. Lett. 45 (1980) 790

Current Algebra: Momentum Generates Diffeomorphisms

- Momentum Density Generates Spatial Diffeomorphisms

$$P[w] = \int d^3x\, w^i \pi_i, \qquad x^i \to x^i + w^i(\mathbf{x})$$



$\mathbf{x} \mapsto \mathbf{x} + w(\mathbf{x})$

- Brackets with $P[w]$: Vector-Field Algebra and Lie Derivatives

$$\{P[w], P[w']\} = -P[[w, w']], \quad [w, w']^i = w^j \partial_j w'^i - w'^j \partial_j w^i$$

$$\{n, P[w]\} = -\mathcal{L}_w n, \quad \{s_{\text{ent}}, P[w]\} = -\mathcal{L}_w s_{\text{ent}}, \quad \mathcal{L}_w n = \partial_i (n w^i)$$

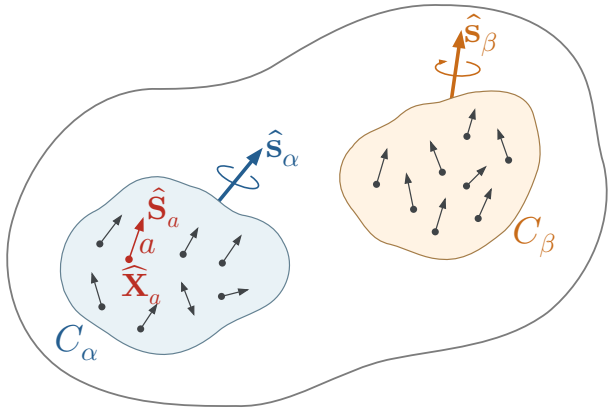
- Removing the Test Fields Gives the Local Brackets

$$\{\pi_i(\mathbf{x}), \pi_j(\mathbf{y})\} = -\pi_j(\mathbf{x}) \partial_{x^i} \delta^{(3)}(\mathbf{x} - \mathbf{y}) + \pi_i(\mathbf{y}) \partial_{y^j} \delta^{(3)}(\mathbf{x} - \mathbf{y})$$

$$\{\pi_i(\mathbf{x}), n(\mathbf{y})\} = n(\mathbf{x})\, \partial_{y^i} \delta^{(3)}(\mathbf{x} - \mathbf{y}), \qquad \{\pi_i(\mathbf{x}), s_{\text{ent}}(\mathbf{y})\} = s_{\text{ent}}(\mathbf{x})\, \partial_{y^i} \delta^{(3)}(\mathbf{x} - \mathbf{y})$$

Kinematics Is Fixed by the Algebra, Independently of the Hamiltonian

Adding Spin to the Fluid Algebra



- A Fluid Cell Has the Sum of Its Spins
- Lie–Poisson Bracket of Expectation Values
- Spin Indices A, B, C Are Internal, Not Spatial

$$\mathfrak{su}(2) \simeq \mathfrak{so}(3): \quad [\hat{S}_A, \hat{S}_B] = i\hbar\epsilon_{ABC}\hat{S}_C, \qquad \{\langle\widehat{A}\rangle, \langle\widehat{B}\rangle\} = \frac{\langle[\widehat{A}, \widehat{B}]\rangle}{i\hbar}$$

$$\{s^A(\mathbf{x}), s^B(\mathbf{y})\} = \epsilon_{ABC}\, s^C(\mathbf{x})\, \delta^{(3)}(\mathbf{x}-\mathbf{y})$$

$$\{\pi_i(\mathbf{x}), s^A(\mathbf{y})\} = s^A(\mathbf{x})\, \partial_{y^i}\delta^{(3)}(\mathbf{x}-\mathbf{y})$$

Jacobi Identity: Unique Local First-Order Bracket, up to Local Rotations of the Spin Axes

Marsden and Ratiu, *Introduction to Mechanics and Symmetry* (1999) · Delacrétaz et al., Phys. Rev. Research 4 (2022) 033131

The Comoving View: Spin of a Fluid Element

- Spin Is Advected with the Fluid

$$\partial_t s^A + \partial_i (v^i s^A) = 2\epsilon_{ABC}\omega_B s^C, \qquad v^i = \frac{\delta H}{\delta \pi_i}, \quad 2\omega_A = \frac{\delta H}{\delta s^A}$$

- Following a Fluid Element, Spin per Particle Only Precesses

$$D_t\left(\frac{s^A}{n}\right) = 2\epsilon_{ABC}\omega_B \frac{s^C}{n}, \qquad D_t\left(\frac{s^A s^A}{n^2}\right) = 0, \qquad D_t \equiv \partial_t + v^i\partial_i$$

Spin Length per Particle $|\mathbf{s}|/n$: a Casimir Invariant for Any $H[n, s_{\text{ent}}, \boldsymbol{\pi}, \mathbf{s}]$

- Circulation Is Conserved Only Together with the Berry Phase of the Spin

$$\frac{d}{dt}\left[\oint_C \frac{\pi_i}{n}\, dx^i - \frac{|\mathbf{s}|}{n}\,\Phi_C\right] = 0$$

C: Comoving Loop of Uniform s_{ent}/n and $|\mathbf{s}|/n$

Φ_C: Solid Angle Enclosed by the Spin Direction along C

Kelvin's Theorem with Spin: Inhomogeneous Spin Precession Generates Circulation

Poincaré Algebra and Thomas Precession

- Boost Generators Must Close the Poincaré Algebra

$$J^{\mu\nu} = \int d^3x \left(x^\mu \Theta^{0\nu} - x^\nu \Theta^{0\mu} + \Sigma^{0\mu\nu} \right), \qquad \{J^{0i}, J^{0j}\} = -J^{ij}$$

$$\Theta^{0i} = \pi_i, \qquad \Sigma^{0ij} = \epsilon_{ijA}\, s^A \qquad \text{Given by the Bracket}$$

$$\Sigma^{00i} = \frac{\gamma}{\gamma+1} (\mathbf{s} \times \mathbf{v})^i, \qquad \Theta^{00} = \mathcal{H}_0 + \frac{\mathbf{s} \cdot (\mathbf{v} \times \boldsymbol{\nabla} p)}{(\varepsilon + p)(\gamma+1)} \qquad \text{Required by the Algebra}$$

$$\partial_t \mathbf{s} + \boldsymbol{\nabla} \cdot (\mathbf{v}\, \mathbf{s}) = (2\boldsymbol{\omega} + \boldsymbol{\Omega}_\mathrm{T}) \times \mathbf{s}, \qquad \boldsymbol{\Omega}_\mathrm{T} = \frac{\mathbf{v} \times \boldsymbol{\nabla} p}{(\varepsilon + p)(\gamma+1)} = \frac{\gamma^2}{\gamma+1} \dot{\mathbf{v}} \times \mathbf{v}$$

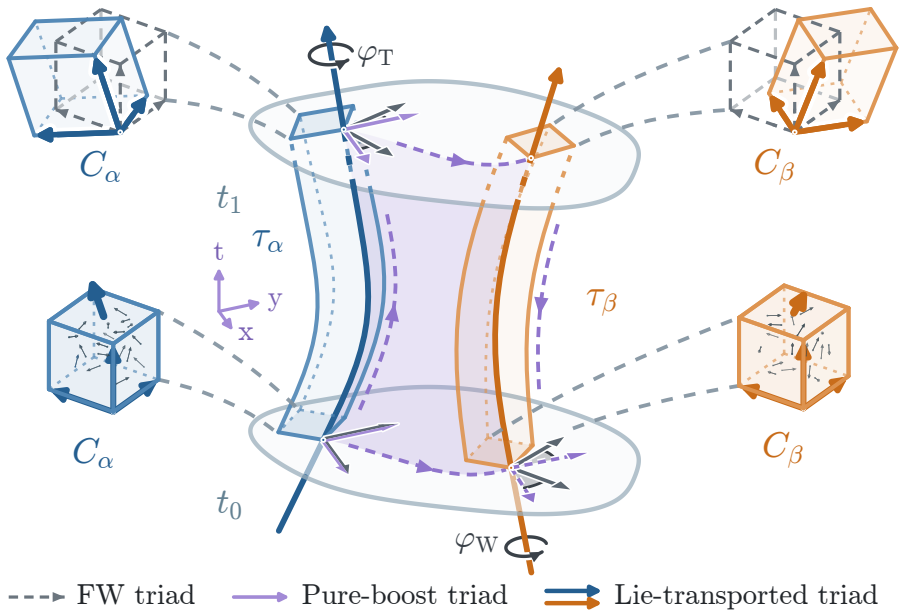
Thomas Precession: a Torque on the Spin, or a Rotation of Its Axes?

$\mathcal{H}_0$: Energy Density without Spin · $\varepsilon + p$: Rest-Frame Enthalpy Density · Thomas, Nature 117 (1926) 514

Choosing a Comoving Spatial Triad for Each Fluid Element

A Tetrad for Each Fluid Element: Four-Velocity u^μ and Spatial Triad ${e_A}^\mu$, $u_\mu {e_A}^\mu = 0$

$$s^\mu = {e_A}^\mu s^A : \qquad \Delta^\mu{}_\nu \, ds^\nu/d\tau + \theta s^\mu = \underbrace{{e_A}^\mu \left(ds^A/d\tau + \theta s^A \right)}_{\text{Spin Bracket}} + \underbrace{s^A \Delta^\mu{}_\nu \, d{e_A}^\nu/d\tau}_{\text{Motion of the Triad}}$$



Fermi–Walker	0
Pure-Boost Triad	$-\gamma\, \boldsymbol{\Omega}_{\mathrm{T}} \times \mathbf{s}$
Lie-Transported	$+\left(\boldsymbol{\sigma}\mathbf{s} + \frac{1}{3}\theta\mathbf{s} + \boldsymbol{\Omega} \times \mathbf{s}\right)$

Canonical: $ds^A/d\tau + \theta s^A = \mathcal{T}^A + \gamma \left(\boldsymbol{\Omega}_{\mathrm{T}} \times \mathbf{s}\right)^A$

$$\Longrightarrow \quad \Delta^\mu{}_\nu \, ds^\nu/d\tau + \theta s^\mu = \mathcal{T}^\mu \quad \text{Fermi–Walker}$$

The Motion of the Triad Is Not a Torque

FFWP, PRL 136 (2026) 182301: Their Shear and Expansion Terms from the Dual of a Lie-Transported Triad and a Rescaling by T^2/n

Dissipation from GENERIC

Systematic Construction: Spin–Orbit Relaxation · Viscosity · Heat and Diffusion

$$\frac{dF}{dt} = \{F, H\} + (F, \mathcal{S}), \qquad (F, G) = (G, F), \qquad (F, F) \geq 0$$

$$\underbrace{\{\mathcal{S}, H\} = 0, \qquad (H, \mathcal{S}) = 0}_{\text{Assumed, Not Derived}} \qquad \Longrightarrow \qquad \frac{dH}{dt} = 0, \qquad \frac{d\mathcal{S}}{dt} = (\mathcal{S}, \mathcal{S}) \geq 0$$

- Twice the Spin Potential, $2\omega_A$, Relaxes toward Ω_A · Opposite Orbital Torque

$$T\partial_t s_{\text{ent}} = \begin{pmatrix} 2\omega_A - \Omega_A & \sigma_{ij} \end{pmatrix} \begin{pmatrix} \lambda\,\delta_{AB} & 0 \\ 0 & 2\eta\,\delta_{ik}\delta_{jl} \end{pmatrix} \begin{pmatrix} 2\omega_B - \Omega_B \\ \sigma_{kl} \end{pmatrix} \geq 0$$

λ: Rotational Viscosity · σ_{ij}: Shear Tensor · $\boldsymbol{\Omega} = \frac{1}{2}\boldsymbol{\nabla} \times \mathbf{v}$: Vorticity

Grmela and Öttinger / Öttinger and Grmela, Phys. Rev. E 56 (1997) 6620 / 6633

A Rotation–Shear Coupling at Finite Spin Density

- Onsager–Casimir $M_{ab}(\mathbf{s}) = M_{ba}(-\mathbf{s})$: a Unique Dissipative Rotation–Shear Coupling

$$\Xi_{Aij} = \xi(\mathbf{s}^2)\left(\epsilon_{iAk}s_k s_j + \epsilon_{jAk}s_k s_i\right)$$

$$T\partial_t s_{\text{ent}} = \begin{pmatrix} 2\omega_A - \Omega_A & \sigma_{ij} \end{pmatrix} \begin{pmatrix} \lambda\,\delta_{AB} & \Xi_{Akl} \\ \Xi_{Bij} & 2\eta\,\delta_{ik}\delta_{jl} \end{pmatrix} \begin{pmatrix} 2\omega_B - \Omega_B \\ \sigma_{kl} \end{pmatrix}$$

$$= \lambda\,(2\boldsymbol{\omega} - \boldsymbol{\Omega})^2 + 2\eta\,\sigma_{ij}\sigma_{ij} + 2\Xi_{Aij}(2\omega_A - \Omega_A)\sigma_{ij} \geq 0 \quad \Longleftrightarrow \quad \xi^2|\mathbf{s}|^4 \leq \lambda\eta$$

$$\begin{pmatrix} \partial_t s^A\big|_{\text{diss}} \\ \Pi_{\langle ij\rangle} \end{pmatrix} = \begin{pmatrix} -\lambda(2\omega_A - \Omega_A) - 2\xi\,[\mathbf{s}\times(\boldsymbol{\sigma}\mathbf{s})]_A \\ -2\eta\,\sigma_{ij} - \Xi_{Bij}(2\omega_B - \Omega_B) \end{pmatrix}$$

Shear Torque

Onsager-Reciprocal Stress

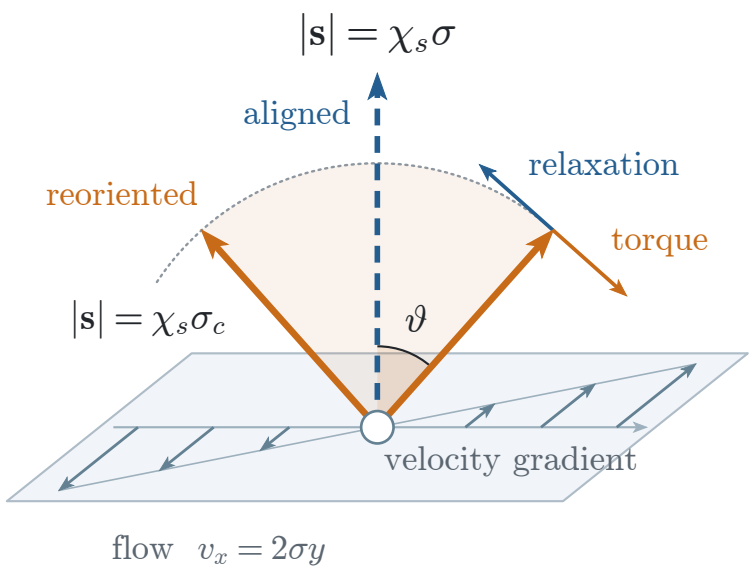
First Order for $s^A = O(1), \omega_A = O(\partial)$ · Absent in FFWP: Isotropic Dissipative Coefficients

FFWP: Fang, Fukushima, Pu and Wang, Phys. Rev. Lett. 136 (2026) 182301

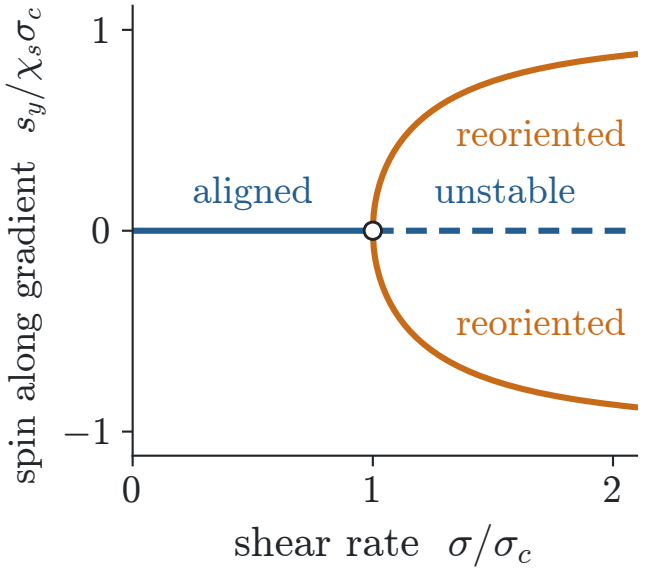
Shear-Driven Spin Reorientation

$$\dot{\mathbf{s}} = \underbrace{-\Gamma_s(\mathbf{s} - \chi_s \boldsymbol{\Omega})}_{\text{Relaxation}} \underbrace{- 2\xi\, \mathbf{s} \times (\boldsymbol{\sigma}\mathbf{s})}_{\text{Shear Torque}}, \qquad \frac{2\xi\chi_s\sigma^2}{\Gamma_s} = \frac{\sigma^2}{\sigma_c^2}$$

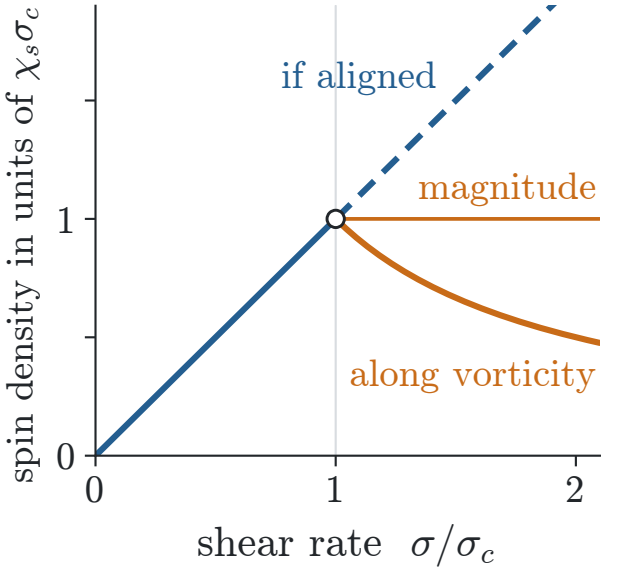
Torque against Relaxation



Pitchfork Bifurcation at $\sigma_c = \Gamma_s/\sqrt{2\xi\lambda}$



Spin Saturates at $\chi_s\sigma_c = \sqrt{\lambda/2\xi}$



χ_s: Spin Susceptibility, $2\boldsymbol{\omega} = \mathbf{s}/\chi_s \cdot \Gamma_s \equiv \lambda/\chi_s$: Spin Relaxation Rate · σ: Shear Rate

Above σ_c the Spin Reorients toward the Velocity Gradient · $\eta_{\text{eff}} = \eta - (\lambda/4)\sin^2\vartheta$

Light-Quark Spin Relaxes on the Transport Time Scale

Weakly Coupled Hot QCD · Full Leading-Order Collision Operator

- Spin Lies along the Momentum: Small-Angle Scatterings Rotate Both
- Nearly as Fast as a Vector Current, with or without Rotation
- No Slow Spin Mode: Spin Follows the Hydrodynamic Gradients

Z.-H. Liu and M. Huang, Is Spin Slow in Hot QCD?
arXiv:2611.16684 [hep-th]

Summary

- **Algebra Fixes the Reversible Kinematics**
  Conserved Spin Length · Thomas Precession · Fermi–Walker Transport

- **A Comoving Triad for Each Fluid Element Separates Torques from Axes**
  Reversible Shear and Expansion Terms: Motion of a Triad and a Rescaling, Not a Torque

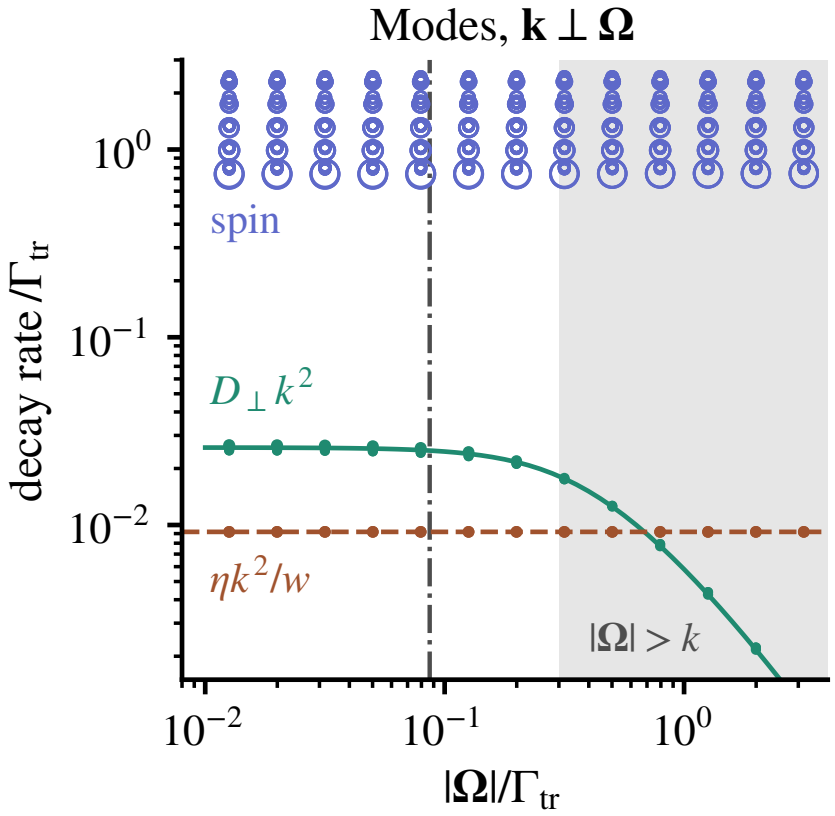
- **Finite Spin Density Adds a Rotation–Shear Torque**
  Reorientation above a Critical Shear Rate · Lower Shear Viscosity

- **Microscopic Lifetimes Set the Regime**
  Weak Coupling: Light-Quark Spin Relaxes on the Transport Scale

  Real QGP: Stronger Coupling May Likewise Leave No Slow Spin Mode

Rotation Does Not Create a Slow Spin Mode

Coriolis Precession Preserves the Collisional Damping Rates



- Uniform Spin in the Co-Rotating Frame

$$\omega = \pm 2|\mathbf{\Omega}| - i\lambda_n \qquad (k = 0)$$

- Slow Modes: Little Off-Equilibrium Spin
- Spin Follows Hydrodynamic Gradients

  Vorticity · Axial-Charge Gradients

Left: $k = 0.3\Gamma_{\rm tr}$; Incompressible Model Extrapolated beyond $2|\mathbf{\Omega}| = k/\sqrt{3}$

Notation and Time Derivatives

- Internal Spin, Spatial Components and Entropy

$$s^A, \quad s^\mu = e^\mu_A s^A, \quad s_{\text{ent}}, \quad \mathcal{S} = \int d^3x\, s_{\text{ent}}$$

- Equal-Time and Proper Densities

$$(n, s_{\text{ent}}, s^A)_{\text{lab}} = \gamma\,(n, s_{\text{ent}}, s^A), \qquad \omega_{A,\text{lab}} = \omega_A/\gamma, \qquad \gamma = u^0$$

- Comoving and Proper-Time Derivatives

$$D_t = \partial_t + v^i\partial_i, \qquad \dot{s}^A = \gamma D_t s^A, \qquad \dot{s}^\mu = u^\nu \nabla_\nu s^\mu$$

Internal Indices: A, B, C · Spatial: i, j · Spacetime: Greek

From a Local Hamiltonian to the Euler Equation

$$H = \int d^3x \left[\frac{\pi_i \pi_i}{2mn} + \varepsilon_0(n, s_{\text{ent}}) \right]$$

$$v^i = \frac{\pi_i}{mn}, \qquad p = n\,\partial_n \varepsilon_0 + s_{\text{ent}}\,\partial_{s_{\text{ent}}} \varepsilon_0 - \varepsilon_0$$

$$-\pi_j \partial_i v^j - n\,\partial_i \frac{\delta H}{\delta n} - s_{\text{ent}}\,\partial_i \frac{\delta H}{\delta s_{\text{ent}}} = -\partial_i p$$

Kinetic Terms Cancel · The Remaining Force Is the Pressure Gradient

Cartesian Variables · Constant Particle Mass · No Gradient Energy

Cell Normalization and the Spin Algebra

$$[\hat{S}_A, \hat{S}_B] = i\hbar\epsilon_{ABC}\hat{S}_C$$

$$s_\alpha^A = \frac{1}{V_\alpha}\left\langle \sum_{a\in C_\alpha} \hat{S}_{aA} \right\rangle$$

$$\{s_\alpha^A, s_\beta^B\} = \delta_{\alpha\beta}\,\frac{\epsilon_{ABC}s_\alpha^C}{V_\alpha}$$

One Inverse Cell Volume · Spatial Continuum Limit

The Full Fluid–Spin Poisson Bracket

- Chain Rule with the Local Brackets of Slides 4 and 5

$$\{F,G\} = \sum_{a,b} \int d^3x\, d^3y\, \frac{\delta F}{\delta a(\mathbf{x})}\, \{a(\mathbf{x}), b(\mathbf{y})\}\, \frac{\delta G}{\delta b(\mathbf{y})}, \qquad a, b \in \{\pi_i, n, s_{\mathrm{ent}}, s^A\}$$

- Integrate the Derivatives of $\delta^{(3)}(\mathbf{x}-\mathbf{y})$ by Parts; Boundary Terms Vanish

$$\{F,G\} = \int d^3x \left[\pi_i \left(\frac{\delta G}{\delta \pi_j} \partial_j \frac{\delta F}{\delta \pi_i} - \frac{\delta F}{\delta \pi_j} \partial_j \frac{\delta G}{\delta \pi_i} \right) + \epsilon_{ABC}\, s^C \frac{\delta F}{\delta s^A} \frac{\delta G}{\delta s^B} \right.$$
$$\left. + \sum_{a \in \{n, s_{\mathrm{ent}}, s^A\}} a \left(\frac{\delta G}{\delta \pi_j} \partial_j \frac{\delta F}{\delta a} - \frac{\delta F}{\delta \pi_j} \partial_j \frac{\delta G}{\delta a} \right) \right]$$

With $dF/dt = \{F, H\}$: Advection of the Densities and Rotation of the Spin

Basis Geometry and Transport Data

$$s^{\mu} = e_{A}^{\ \mu} s^{A}, \qquad G_{AB} = -e_{A} \cdot e_{B}$$

$$\mathcal{R}^{\mu}{}_{\nu} = \Delta^{\mu}{}_{\lambda} (\mathcal{L}_{u} e_{A})^{\lambda} e^{A}{}_{\nu}$$

$$C_{\mu\nu} = -\sqrt{\det G}\, \delta_{AB} e^{A}{}_{\mu} e^{B}{}_{\nu}$$

$$\Omega_{\mathrm{e}}^{\mu} = \Omega^{\mu} + \tfrac{1}{2} \varepsilon^{\mu\nu\rho\sigma} u_{\nu} \mathcal{R}_{\sigma\rho}$$

Spin Motion in a General Basis

$$
\begin{aligned}
\dot{s}^{\mu} = & -u^{\mu} a_{\nu} s^{\nu} + \tfrac{1}{3}(-2\theta + \mathcal{R}^{\alpha}{}_{\alpha}) s^{\mu} \\
& + (\sigma^{\mu}{}_{\nu} + g^{\mu\alpha} \mathcal{R}_{\langle \alpha\nu \rangle}) s^{\nu} \\
& - \varepsilon^{\mu\nu\rho\sigma} u_{\nu} \Omega_{\mathrm{e}\rho} s_{\sigma} \\
& - 2\varepsilon^{\mu\nu\rho\sigma} u_{\nu} \omega_{\rho} C_{\sigma\lambda} s^{\lambda}
\end{aligned}
$$

Projected Lie: $\mathcal{R} = 0$ Fermi–Walker: $G = \mathbf{1},\ \Omega_{\mathrm{e}} = 0$

Physical Basis Prescription and Passive Re-Expression Are Distinct

Boosts of a Perfect Fluid

- Without Spin: the Dirac–Schwinger Condition

$$\{K_i, H\} + P_i = \int d^3x \, [\pi_i - (\mathcal{H}_0 + p) v_i] = 0 \qquad \Longrightarrow \qquad \boldsymbol{\pi} = (\varepsilon + p)\gamma^2 \mathbf{v}$$

$$\{K_i, K_j\} + \epsilon_{ijk} J_k = \epsilon_{ijA} \int d^3x \, s^A \qquad \text{An Advected Spin Is Not Rotated}$$

- A Boost: Lorentz Transformation at a Point plus a Tilt of the Slice

$$\left\{\phi, -\int d^3x \, (\boldsymbol{\beta} \cdot \mathbf{x}) \, \mathcal{H}_0 \right\} = \delta_{\boldsymbol{\beta}} \phi - (\boldsymbol{\beta} \cdot \mathbf{x}) \, \partial_t \phi, \qquad \delta_{\boldsymbol{\beta}} (n, s_{\text{ent}}, \mathbf{s}) = (\boldsymbol{\beta} \cdot \mathbf{v}) \, (n, s_{\text{ent}}, \mathbf{s})$$

$$\delta_{\boldsymbol{\beta}} \boldsymbol{\pi} = (\varepsilon + p)\gamma^2 [\boldsymbol{\beta} + \mathbf{v} \, (\boldsymbol{\beta} \cdot \mathbf{v})], \qquad \delta_{\boldsymbol{\beta}} \mathbf{v} = \boldsymbol{\beta} - \mathbf{v} \, (\boldsymbol{\beta} \cdot \mathbf{v}), \qquad \delta_{\boldsymbol{\beta}} \boldsymbol{\nabla} p = -\boldsymbol{\beta} \, \partial_t p$$

$K_i = J^{0i} \cdot \boldsymbol{\beta}$: Boost Velocity · n/γ, $\varepsilon + p$ and p Are Invariant · Schwinger, Phys. Rev. 130 (1963) 800

Closing the Poincaré Algebra at First Order in the Spin

- Two Boosts Must Commute into the Full Rotation, Spin Included

$$\Sigma^{00i} = w\,(\mathbf{s}\times\mathbf{v})^i : \qquad 1-2w+w^2\mathbf{v}^2=0, \quad \gamma\,\partial_\gamma w = w-w^2 \quad\Longrightarrow\quad w=\frac{\gamma}{\gamma+1}$$

- The Boost Charge Must Be Conserved: the Acceleration Brings In $\partial_t p$

$$\mathbf{s}\times D_t\Big(\frac{\gamma\,\mathbf{v}}{\gamma+1}\Big) \ni -\frac{\partial_t p\;\mathbf{s}\times\mathbf{v}}{(\varepsilon+p)(\gamma+1)}, \qquad \delta_{\boldsymbol{\beta}}\boldsymbol{\nabla} p = -\boldsymbol{\beta}\,\partial_t p$$

$$\Longrightarrow \qquad \Theta^{00} = \mathcal{H}_0 + \boldsymbol{\Delta}\cdot\boldsymbol{\nabla} p, \qquad \boldsymbol{\Delta} = \frac{\mathbf{s}\times\mathbf{v}}{(\varepsilon+p)(\gamma+1)}$$

Both Conditions Hold Pointwise: Closure at $O(\mathbf{s})$ to All Orders in Gradients

Gradient-Free Energies Fail at $O(\mathbf{s})$ whenever $\boldsymbol{\nabla} p \neq 0$

Wigner Rotation and Centroids of a Fluid Element

$$\{\mathbf{s}, \boldsymbol{\beta}\cdot\mathbf{K}\} = (\boldsymbol{\beta}\cdot\mathbf{v})\,\mathbf{s} + \frac{\gamma}{\gamma+1}(\mathbf{v}\times\boldsymbol{\beta})\times\mathbf{s} - (\boldsymbol{\beta}\cdot\mathbf{x})\,\partial_t\mathbf{s} \qquad \text{Wigner Rotation}$$

Centroid	Shift	$\{x^i, x^j\}$	Boost $\mathbf{K}$
Newton–Wigner	0	0	$-\mathbf{x}E + \dfrac{\mathbf{S}\times\mathbf{q}}{E+m}$
Covariant	$\dfrac{\mathbf{S}\times\mathbf{q}}{m(E+m)}$	$\dfrac{\epsilon_{ijk}W_k}{m^2E}$	$-\mathbf{x}_{\mathrm{c}}E + \dfrac{\mathbf{S}\times\mathbf{q}}{m}$
Centre of Energy	$-\dfrac{\mathbf{S}\times\mathbf{q}}{E(E+m)}$	$-\dfrac{\epsilon_{ijk}W_k}{E^3}$	$-\mathbf{x}_{\mathrm{E}}E$

Canonical Densities Count Newton–Wigner Positions; the Pressure Acts at the Covariant Centroids

Per Particle: $\mathbf{q} = \boldsymbol{\pi}/n$, $E = (\varepsilon+p)\gamma^2/n$, $m = (\varepsilon+p)\gamma/n$, $\mathbf{S} = \mathbf{s}/n\cdot W^\mu$: Pauli–Lubański Vector · Pryce (1948)

Fermi–Walker Transport of the Canonical Spin

$$s^{\mu} = L(u)^{\mu}{}_{A}\, s^{A}, \quad s^{A} = \frac{s^{A}_{\mathrm{lab}}}{\gamma}: \qquad s^{0} = \mathbf{v}\cdot\mathbf{s}_{\mathrm{lab}}, \qquad s^{i} = \frac{s^{i}_{\mathrm{lab}}}{\gamma} + \frac{\gamma\, v^{i}(\mathbf{v}\cdot\mathbf{s}_{\mathrm{lab}})}{\gamma+1}$$

- Bracket with the Thomas Term, per Unit Proper Time

$$ds^{A}/d\tau + \theta s^{A} = \mathcal{T}^{A} + \gamma\,(\boldsymbol{\Omega}_{\mathrm{T}}\times\mathbf{s})^{A}, \qquad \boldsymbol{\Omega}_{\mathrm{T}} = \frac{\mathbf{v}\times\boldsymbol{\nabla}p}{(\varepsilon+p)(\gamma+1)}$$

- Motion of the Pure-Boost Triad $L(u)$ Relative to Fermi–Walker Transport

$$s^{A}\Delta^{\mu}{}_{\nu}\, de_{A}{}^{\nu}/d\tau = e_{A}{}^{\mu}\left(\frac{\mathbf{u}\times\mathbf{a}}{1+u^{0}}\times\mathbf{s}\right)^{A} = -\gamma\, e_{A}{}^{\mu}\,(\boldsymbol{\Omega}_{\mathrm{T}}\times\mathbf{s})^{A}$$

Sum: $\Delta^{\mu}{}_{\nu}\, ds^{\nu}/d\tau + \theta s^{\mu} = \mathcal{T}^{\mu}$ · The Thomas Terms Cancel · Neither Is a Torque

Fermi–Walker Transport Composes Infinitesimal Boosts; the Single Boost $L(u)$ Differs by the Thomas Rotation

Motion of the Triad in the Spin Equation

$$\Delta^{\mu}{}_{\nu}\, ds^{\nu}/d\tau + \theta s^{\mu} = e_{A}{}^{\mu}\left(ds^{A}/d\tau + \theta s^{A}\right) + e_{B}{}^{\mu}\,\Gamma_{\tau}{}^{B}{}_{A} s^{A}, \qquad \Gamma_{\tau}{}^{B}{}_{A} = e^{B}{}_{\mu}\, de_{A}{}^{\mu}/d\tau$$

Triad	Motion of the Triad $\boldsymbol{\Gamma}_{\tau}\mathbf{s}$
Fermi–Walker	0
Pure-Boost Triad $L(u)$ of the Canonical Variables	$-\gamma\,\boldsymbol{\Omega}_{\mathrm{T}} \times \mathbf{s}$
Orthonormal, Rotating at $\boldsymbol{\Omega}_{\mathrm{e}}$	$+\boldsymbol{\Omega}_{\mathrm{e}} \times \mathbf{s}$
Lie-Transported (Comoving Line Elements)	$+\left(\boldsymbol{\sigma}\mathbf{s} + \frac{1}{3}\theta\mathbf{s} + \boldsymbol{\Omega} \times \mathbf{s}\right)$
Dual of a Lie-Transported Triad	$-\left(\boldsymbol{\sigma}\mathbf{s} + \frac{1}{3}\theta\mathbf{s} - \boldsymbol{\Omega} \times \mathbf{s}\right)$

Components with $ds^{A}/d\tau + \theta s^{A} = \mathcal{T}^{A}$ Give $\mathcal{T}^{\mu}$ plus the Motion of the Triad, Which Is Not a Torque

A Given Vector in Another Triad: $ds'^{A}/d\tau + \theta s'^{A} = \mathcal{T}'^{A} - \Gamma'_{\tau}{}^{A}{}_{B} s'^{B}$ · Stationarity and Stability Are Stated in the Fermi–Walker Triad

Matching the Spin Equation of Fang et al.

- Fermi–Walker Components in the Dual of a Lie-Transported Triad, Rescaled

$$s'^{\mu} = f\,\bar{E}_A{}^{\mu} S^A, \qquad f = f_*\,\beta^{-2} n^{-1}, \qquad \beta = 1/T$$

$$\dot{\mathbf{s}}' + \theta \mathbf{s}' = f\,\bar{\mathbf{E}}_A \mathcal{T}^A + (\ln f)^{\cdot}\,\mathbf{s}' - \boldsymbol{\sigma}\mathbf{s}' - \tfrac{1}{3}\theta\mathbf{s}' + \boldsymbol{\Omega} \times \mathbf{s}'$$

- Shear, Expansion and Vorticity Terms of the FFWP Equation

$$\dot{\mathbf{s}}' = -\boldsymbol{\sigma}\mathbf{s}' - \left[\tfrac{1}{3} + 2\left(\frac{\partial p}{\partial \varepsilon}\right)_{n,s^A}\right]\theta\,\mathbf{s}' + \boldsymbol{\Omega} \times \mathbf{s}' + f\,\bar{\mathbf{E}}_A \mathcal{T}^A + O(\partial^2)$$

Neither the Spin per Particle $S^A S^A/n^2$ nor the Entropy Changes

S^A: Fermi–Walker Components · $\bar{E}_A$: Dual of a Lie-Transported Triad · Counting $s^A = O(1), \omega_A = O(\partial)$

Rotational Dissipative Bracket

- Combine Spin, Orbital and Entropy Variations

$$\Omega_A = \tfrac{1}{2}\epsilon_{Aij}\partial_i v_j$$

$$\mathcal{D}_A^{\mathrm{rot}}[F] = \frac{\delta F}{\delta s^A} - \tfrac{1}{2}\epsilon_{Aij}\partial_i \frac{\delta F}{\delta \pi_j} - \frac{2\omega_A - \Omega_A}{T}\frac{\delta F}{\delta s_{\mathrm{ent}}}$$

- A Positive Square Gives the Dissipative Channel

$$(F, G)_{\mathrm{rot}} = \int d^3x\, T\lambda\, \mathcal{D}_A^{\mathrm{rot}}[F]\mathcal{D}_A^{\mathrm{rot}}[G]$$

Positive Temperature · Nonnegative Rotational Viscosity

Conserved Generators and Orbital Torque

$$\mathcal{D}_A^{\mathrm{rot}}[H] = \mathcal{D}_A^{\mathrm{rot}}[J_B] = \mathcal{D}_A^{\mathrm{rot}}[N] = \mathcal{D}_A^{\mathrm{rot}}[P_i] = 0$$

$$\mathcal{D}_A^{\mathrm{rot}}[\mathcal{S}] = -\frac{2\omega_A - \Omega_A}{T}$$

$$\Pi_{ij}^{\mathrm{rot}} = \tfrac{1}{2}\epsilon_{Aij}\lambda(2\omega_A - \Omega_A)$$

$$\partial_t \pi_j\big|_{\mathrm{rot}} = -\partial_i \Pi_{ij}^{\mathrm{rot}}$$

Equal and Opposite Spin and Orbital Torques

Onsager Matrix at Finite Spin Density

- Forces Odd under Time Reversal: $M_{ab}(\mathbf{s}) = M_{ba}(-\mathbf{s})$ · Helicity m about $\hat{\mathbf{s}}$

m	Dissipative, Even in $\mathbf{s}$	Reversible, Odd in $\mathbf{s}$
0	$\lambda_{\parallel},\ \eta_0,\ \zeta,\ \zeta_{\times}$	$(2\omega-\Omega)_{\parallel}$–$\sigma_{\parallel}$, $(2\omega-\Omega)_{\parallel}$–$\partial_k v_k$
±1	$\lambda_{\perp},\ \eta_1,\ \xi$	$(2\boldsymbol{\omega}-\boldsymbol{\Omega})_{\perp}$–$\hat{\mathbf{s}}\times(2\boldsymbol{\omega}-\boldsymbol{\Omega})_{\perp}$, $(2\boldsymbol{\omega}-\boldsymbol{\Omega})_{\perp}$–$(\boldsymbol{\sigma}\hat{\mathbf{s}})_{\perp}$, Hall Viscosity
±2	η_2	Hall Viscosity

- Three Rotation–Shear Tensors, Contracted with the Shear Tensor

$$N^{(1)}{:}\,\boldsymbol{\sigma} = 2\,\boldsymbol{\sigma}\mathbf{s}, \qquad N^{(2)}{:}\,\boldsymbol{\sigma} = 2\,\mathbf{s}\times(\boldsymbol{\sigma}\mathbf{s}), \qquad N^{(3)}{:}\,\boldsymbol{\sigma} = \mathbf{s}\,(\mathbf{s}\cdot\boldsymbol{\sigma}\mathbf{s})$$

Stretching $N^{(1)}$ and Length Change $N^{(3)}$ Are Odd in $\mathbf{s}$ · Only the Rotation $N^{(2)}$ Is Dissipative

Cross Bracket and Entropy Production

$$(F,G)_{\text{cross}} = \int d^3x\, T \Xi_{Aij} \Big(\mathcal{D}^{\text{rot}}_A[F]\, \mathcal{D}^{\text{sh}}_{ij}[G] + \mathcal{D}^{\text{sh}}_{ij}[F]\, \mathcal{D}^{\text{rot}}_A[G] \Big)$$

$$\mathcal{D}^{\text{sh}}_{ij}[F] = \tfrac{1}{2} \left(\partial_i \frac{\delta F}{\delta \pi_j} + \partial_j \frac{\delta F}{\delta \pi_i} \right) - \tfrac{1}{3} \delta_{ij}\, \partial_k \frac{\delta F}{\delta \pi_k} - \frac{\sigma_{ij}}{T} \frac{\delta F}{\delta s_{\text{ent}}}$$

- Antisymmetric Part Returns Angular Momentum · Symmetric Part Onsager-Reciprocal

$$T \partial_t s_{\text{ent}} = \lambda \left| 2\boldsymbol{\omega} - \boldsymbol{\Omega} + \frac{\Xi : \sigma}{\lambda} \right|^2 + \sigma_{ij} \left(2\eta\, \delta_{ik}\delta_{jl} - \frac{\Xi_{Aij}\Xi_{Akl}}{\lambda} \right) \sigma_{kl}$$

Positivity: $\lambda, \eta \geq 0$ and $\xi^2 |\mathbf{s}|^4 \leq \lambda\eta$

ξ Can Be Traded for an Anisotropic Viscosity and a Shifted Relaxation Target · No Field Redefinition Removes It

General Flows and General Coefficients

- Spin on a Principal Axis of the Strain Rate

$$(\mu + \Gamma_s)^2 = 4\xi^2 \chi_s^2 \mathbf{\Omega}^2 \, (\sigma_3 - \sigma_2)(\sigma_1 - \sigma_3)$$

Unstable Only on the Intermediate Axis, as for a Rigid Body

- Planar Flows $\mathbf{v} = 2\sigma(y, \beta x, 0)$: Control Parameter $|1 - \beta^2| \, \sigma^2/\sigma_c^2$

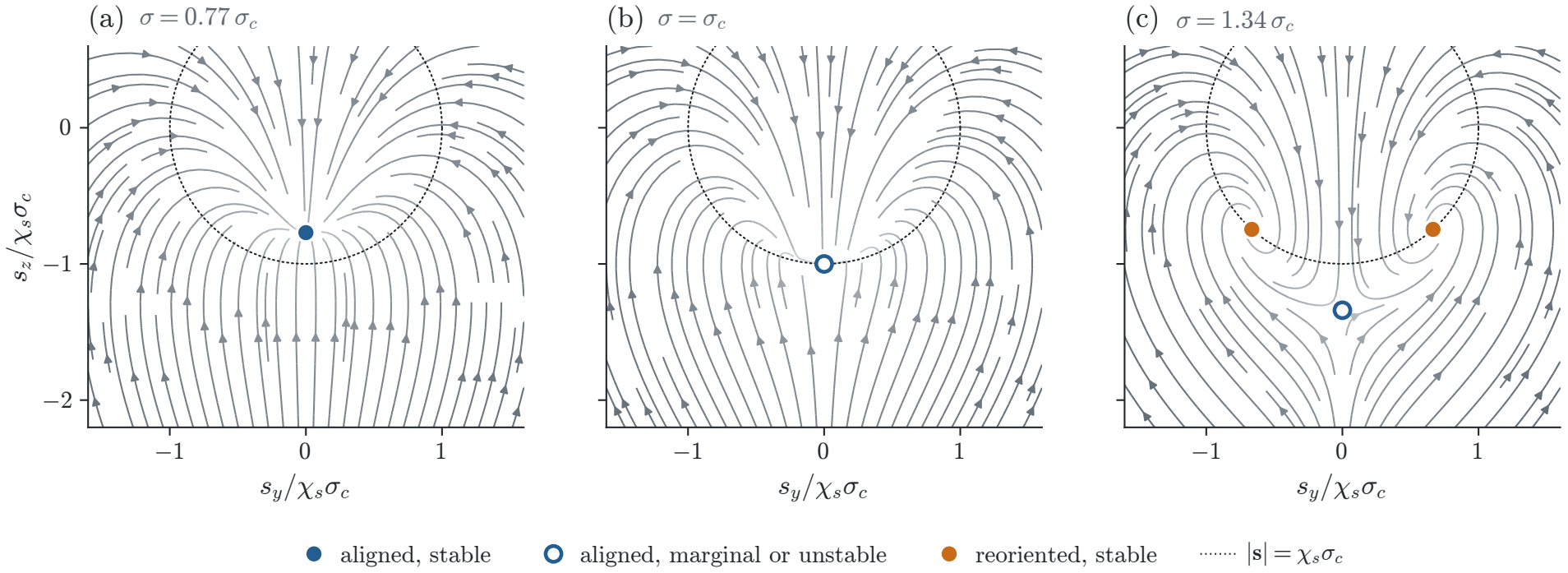
- General Coefficient Functions and a Reversible Precession

$$2\xi(\mathbf{s}^2)\,\mathbf{s}^2 = \lambda_\perp(\mathbf{s}^2) \;\text{ at }\; \mathbf{s} = \chi_s \mathbf{\Omega}, \qquad \sigma^4 > \sigma_c^4 \left(1 + \frac{\Omega_p^2}{\Gamma_s^2}\right)$$

The Threshold Involves neither χ_s nor Γ_s · Compatible with Positivity if $\lambda_\perp \leq 4\eta$

Uniaxial Strain: No Instability · Three-Dimensional Flows: an Imperfect Pitchfork

Phase Portraits and Linearized Rates



Aligned State: $\Gamma_s(\sigma^2/\sigma_c^2 - 1),\ -\Gamma_s(1 + \sigma^2/\sigma_c^2),\ -\Gamma_s$

Reoriented States: $-2\Gamma_s,\ \frac{1}{2}\Gamma_s\left(-1 \pm \sqrt{9 - 8\sigma^2/\sigma_c^2}\right)$

Normal Form $\dot{Y} = \Gamma_s[(\sigma^2/\sigma_c^2 - 1)Y - (\sigma^2/\sigma_c^2)Y^3]$, $Y = s_y/\chi_s\sigma_c$ · Monotonic Decay up to $\sigma^2 = 9\sigma_c^2/8$, Oscillatory beyond

Covariant Stress and Energy Flux

$$\Theta^{\mu\nu} = \varepsilon u^\mu u^\nu - p\Delta^{\mu\nu} + 2\eta\sigma^{\mu\nu} + \zeta\theta\Delta^{\mu\nu}$$
$$+ q_\varepsilon^\mu u^\nu - \frac{\lambda}{2}\varepsilon^{\mu\nu\rho\sigma}u_\rho(2\omega_\sigma - \Omega_\sigma)$$

$$q_\varepsilon^\mu = S^{\mu\nu}a_\nu - u_\nu\nabla_\lambda\Sigma_\perp^{\lambda\mu\nu}$$

$$u_\mu\Theta^{\mu\nu} = \varepsilon u^\nu, \qquad \eta, \zeta, \lambda \geq 0$$

Velocity Convention Fixed by the Energy Projection

Dissipative Currents and Entropy

$$j_n^\mu = L_n \Delta^{\mu\nu} \nabla_\nu (\beta\mu) + \frac{n}{\varepsilon + p} q_\varepsilon^\mu$$

$$j_{\mathrm{s}}^{\lambda\mu} = L_s \Delta^{\lambda\kappa} \Delta^{\mu\nu} \nabla_\kappa (2\beta\omega_\nu) + \frac{s^\mu}{\varepsilon + p} q_\varepsilon^\lambda$$

$$s_{\mathrm{ent}}^\mu = s_{\mathrm{ent}} u^\mu + \beta q_\varepsilon^\mu - \beta\mu j_n^\mu + 2\beta\omega_\nu j_{\mathrm{s}}^{\mu\nu}$$

$$\nabla_\mu s_{\mathrm{ent}}^\mu = \mathcal{Q} + O(\partial^3), \qquad \mathcal{Q} \geq 0, \quad L_n, L_s \geq 0$$

Nonnegative Quadratic Entropy Production at the Retained Order

Covariant Form of the Rotation–Shear Coupling

$$\mathcal{T}^{\mu} = -2\xi\,\varepsilon^{\mu\nu\rho\sigma} u_{\nu} s_{\rho}\, \sigma_{\sigma\lambda} s^{\lambda}, \qquad Y^{\mu} = \varepsilon^{\mu\nu\rho\sigma} u_{\nu} (2\omega_{\rho} - \Omega_{\rho}) s_{\sigma}$$

$$\Delta^{\mu}{}_{\nu} \dot{s}^{\nu} + \theta s^{\mu} = -\lambda (2\omega^{\mu} - \Omega^{\mu}) - \mathcal{T}^{\mu} - \Delta^{\mu}{}_{\nu} \nabla_{\lambda} j_{\mathrm{s}}^{\lambda\nu}$$

$$\Theta^{\mu\nu}_{\mathrm{cross}} = -\tfrac{1}{2} \varepsilon^{\mu\nu\rho\sigma} u_{\rho} \mathcal{T}_{\sigma} + \xi \left(Y^{\mu} s^{\nu} + s^{\mu} Y^{\nu} \right)$$

Angular-Momentum Balance $\nabla_{\lambda} \Sigma^{\lambda\mu\nu} = -2\Theta^{[\mu\nu]}$ Is Preserved

Entropy Production Gains $-2\beta(2\omega_{\mu} - \Omega_{\mu})\mathcal{T}^{\mu}$ · Non-Negative for $\xi^2 (s_{\mu} s^{\mu})^2 \leq \lambda\eta$

Isotropic Rest Energy · Fermi–Walker Transport · First Order in Gradients

Beyond the Spin Density: Spinor Condensates

- Retain Relative Phases in the Full Internal State

$$\mathfrak{su}(2) \subset \mathfrak{u}(3), \qquad n\rho_{mm'} = \psi_m \psi^*_{m'}$$

ψ_m: Spin-1 Components · ρ: Normalized Internal Density Matrix

- Nematic Texture Supports Relative Spin Flow

$$\partial_t s^A + \partial_i (s^A v_i + j_{\mathrm{s}}^{iA}) = 0$$

Local Spin Length Can Change · Total Spin Is Conserved

Spin-1 Contact Model · No Zeeman or Dipolar Coupling

A Larger Internal Observable Algebra

- The Condensate Retains Spin and Nematic Order

$$\psi = \sqrt{n}\, e^{i\varphi} z, \qquad \rho = zz^\dagger, \qquad s^A = \hbar n \, \mathrm{tr}(F_A \rho)$$

$$O_A = \psi^\dagger A \psi, \qquad n\rho \in \mathfrak{u}(2F+1)^*$$

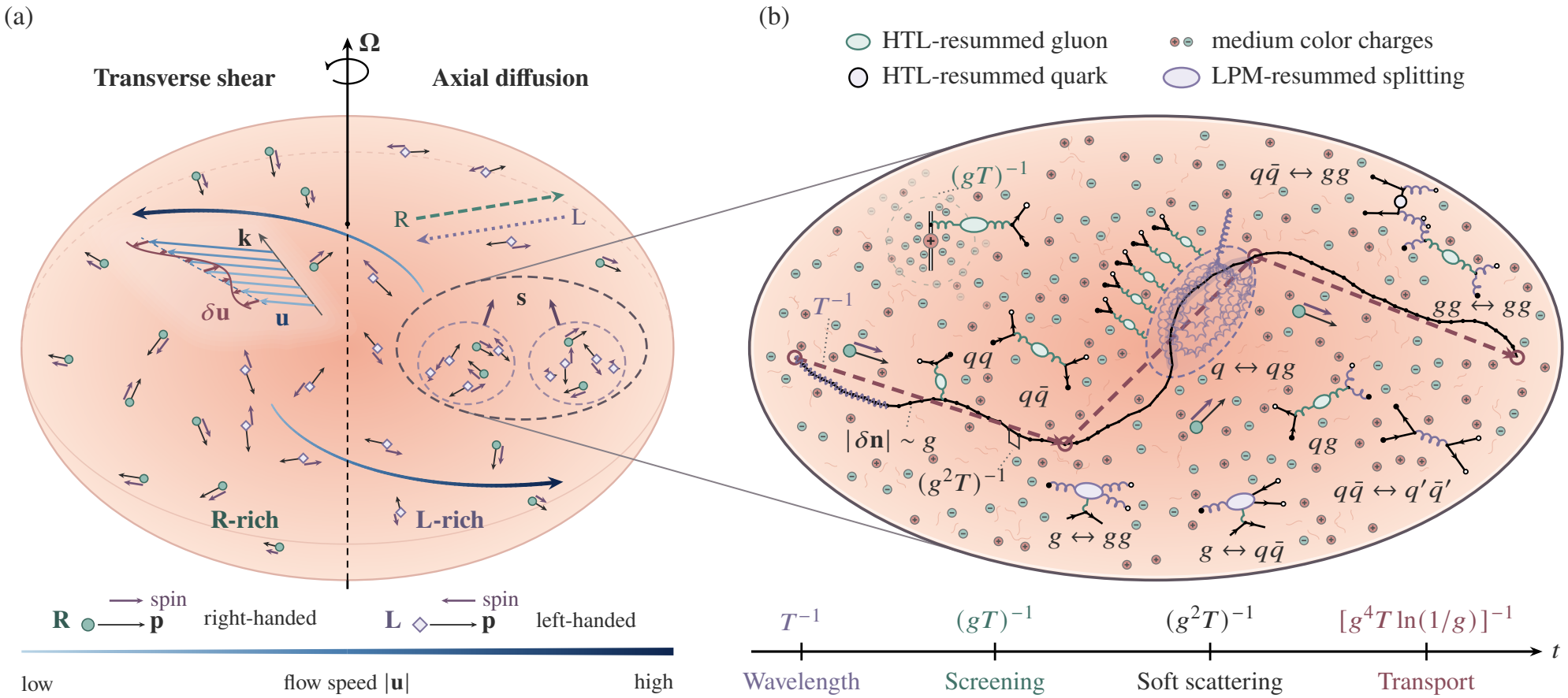
$$\{O_A(\mathbf{x}), O_B(\mathbf{y})\} = \frac{n(\mathbf{x})}{i\hbar} \mathrm{tr}(\rho(\mathbf{x})[A,B]) \delta^{(3)}(\mathbf{x}-\mathbf{y})$$

- Texture Supports Coherent Relative Spin Currents

  Local GP Equivalence: Phase, Berry Connection and Circulation Constraints

Inside a Fluid Element of the Quark–Gluon Plasma

Is Spin a Slow Variable in Hot QCD? Compare Its Lifetime with $\gamma_{\rm tr}^{-1}$ at Weak Coupling



$\gamma_{\rm tr} \equiv g^4 T \ln(1/g)$ · Arnold, Moore and Yaffe, JHEP 01 (2003) 030; JHEP 05 (2003) 051 · Schematic, Not to Scale

Computing Light-Quark Spin Relaxation

An Independent Spin Variable Requires Decay Rates $\Gamma \ll \gamma_{\rm tr}$

- Evolve the Polarization with the Linearized Leading-Order Collision Operator

$$\delta f_a = f_a^{(0)}\,\bar{f}_a^{(0)}\,\Phi_a, \qquad \bar{f}_a^{(0)} = 1 \mp f_a^{(0)}, \qquad \partial_t \Phi = -C\Phi$$

- Momentum Deflection Rotates the Spin

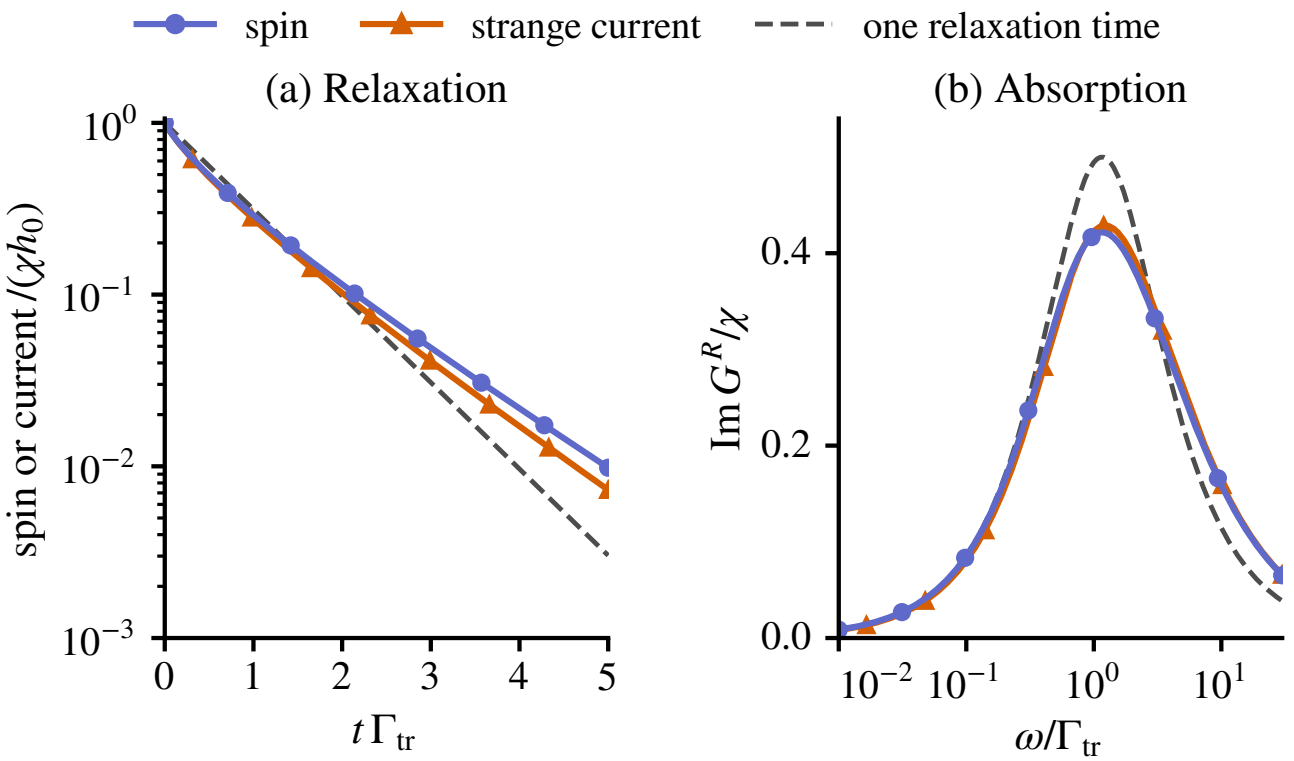
$$j_q^i = j_{\bar{q}}^i = \frac{p_i}{2p}\,\sigma_3, \qquad j_g^i = 0$$

$$\delta s^i = \tfrac{1}{2}\delta J_5^i = (j^i, \Phi) \equiv \sum_a d_{c,a} \int \frac{d^3p}{(2\pi)^3}\, f_a^{(0)}\,\bar{f}_a^{(0)}\,\mathrm{tr}_2\big(j_a^i \Phi_a\big)$$

Helicity Can Be Conserved while Spatial Spin Relaxes at $\gamma_{\rm tr} = g^4 T \ln(1/g)$

Spin Relaxes on the Transport Time Scale

$$R(t) \equiv \frac{\delta s(t)}{\delta s(0)} = \frac{(j, e^{-Ct} j)}{(j, j)}$$



- Comparable Relaxation Rates

$$\Gamma_{\text{eff}} \simeq 0.93\,\Gamma_{\text{tr}}$$

$$\tau \simeq 0.86/\Gamma_{\text{tr}}$$

- Several Kinetic Modes

Γ_{tr}: Effective Rate of the Strange Vector Current · $g = 10^{-3}$, $m_{0s} = 0.3gT$ · 40 Cubic B-Splines per Channel

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