Where curvature meets $\hbar$

Covariant Wigner formalism at higher order in $\hbar$

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Motivation

E. Wigner, Phys. Rev. **40**, 749-759 (1932) , "*On the Quantum Correction for Thermodynamic Equilibrium*"

Classical

$$f = F(\beta \cdot p)\,, \qquad \nabla_{(\mu}\beta_{\nu)} = 0$$

$$T^{\mu\nu}_{id} = (\varepsilon + p)u^\mu u^\nu + p g^{\mu\nu}$$

Quantum

$$T^{\mu\nu} = T^{\mu\nu}_{id} + \hbar^2 \Delta T^{\mu\nu}$$

$$\Delta T^{\mu\nu} = \left(\mathcal{O}(a^2), \mathcal{O}(\omega^2), \mathcal{O}(R)\right)$$

Buzzegoli, Grossi, Becattini, JHEP 10 (2017) 091
Becattini, Grossi, PRD (2015)

Can these corrections be computed from quantum kinetic theory?

Covariant quantum kinetic theory

In flat spacetime: two points separated by a relative coordinate y and Fourier transform with respect to their separation

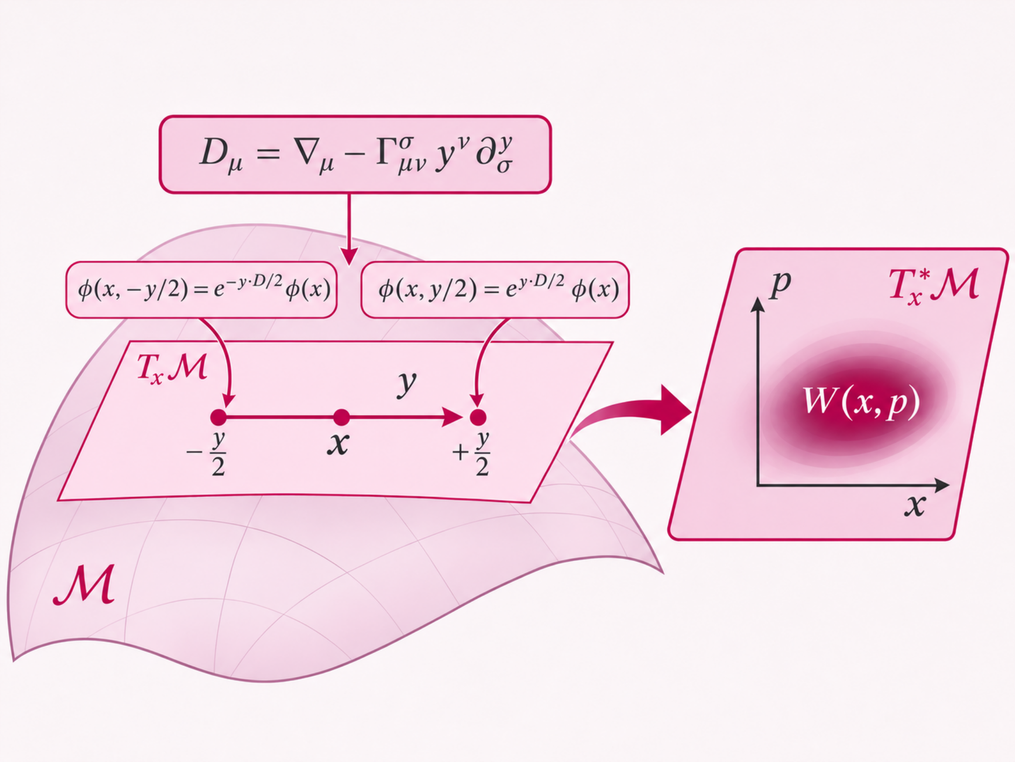
$$W(x,p) = \frac{2}{\hbar}\int d^4y \;\; e^{-ip\cdot y/\hbar} \left\langle : \phi^\star(x+\tfrac{y}{2})\,\phi(x-\tfrac{y}{2}) : \right\rangle$$



ill-defined in curved spacetime !

Covariant quantum kinetic theory

Fonarev, Journal of Mathematical Physics, **35** (1994)



$$W(x,p) = \frac{2}{\hbar}\int_{T_x\mathscr{M}} d^4y\,\sqrt{-g}\;e^{-ip\cdot y/\hbar}\,\left\langle : \phi^\star(x,\tfrac{y}{2})\,\phi(x,-\tfrac{y}{2}) : \right\rangle$$

Validity of semiclassical expansion

$$\underbrace{\hbar/\Delta p}_{\text{quantum scale}} \;\ll\; \underbrace{\Delta R_G}_{\text{curvature radius}}$$

massive: $\Delta p \sim m \;\Rightarrow\; \lambda_C = \hbar/m$ thermal: $\Delta p \sim T \;\Rightarrow\; \hbar/T$

The method: generalised Bopp operator

$$\widetilde{\mathscr{D}}_\mu W(x,p) = \frac{2}{\hbar}\int_{T^\star\mathscr{M}} d^4y\sqrt{-g}\,e^{-ip\cdot y/\hbar} D_\mu\left(\phi_+^\star\phi_-\right)$$

$$\widetilde{\mathscr{D}}_\mu = \nabla_\mu + \Gamma^\rho_{\mu\nu}p_\rho\partial_p^\nu \quad \text{lift to } T^\star_x\mathscr{M}$$

$$D_\mu = \nabla_\mu - \Gamma^\rho_{\mu\nu}y^\nu\partial_\rho^y \quad \text{lift to } T_x\mathscr{M}$$

$$\left(\Box - \frac{m^2}{\hbar^2} - \xi R\right)\phi(x) = 0$$



Acts on each field individually !

The method: generalised Bopp operator

Dynamics of ϕ

Field eoms $\propto \nabla\phi$

$$\xrightarrow[y^\alpha \longleftrightarrow i\hbar\,\partial_p^\alpha]{\mathscr{W}}$$

Dynamics of W

Wigner eom $\mathscr{W}\left[\phi^\star\, i\hbar\,\nabla_\mu\phi\right] = \mathscr{B}_\mu W$

Generalised Bopp Operator

$$\mathscr{B}_\mu = p_\mu + \frac{i\hbar}{2}\widetilde{\mathscr{D}}_\mu - \frac{\hbar^2}{12}\left(p_\sigma R^\sigma{}_{\alpha\beta\mu}\,\partial_p^\alpha\partial_p^\beta + R_{\alpha\mu}\,\partial_p^\alpha\right) + \mathscr{O}(\hbar^4)$$

- Tidal term at $\mathscr{O}\left(\hbar^2\right)$
- No curvature correction at cubic order
- Computable up to every order (in progress)
- Extension to spin and EM fields

$$\xrightarrow{R\to 0} \qquad \mathscr{B}_\mu^{flat} = p_\mu + \frac{i\hbar}{2}\partial_\mu$$

$$\left(\mathscr{B}^\mu\mathscr{B}_\mu + \hbar^2\mathscr{C} + m^2 + \hbar^2\xi\,\hat{R}\right)W = 0, \qquad \hat{R} = R - \frac{i\hbar}{2}(\nabla_\alpha R)\,\partial_p^\alpha$$

$$\mathscr{C} = \frac{1}{4}R^\rho{}_{\alpha\beta}{}^\sigma\,\partial_p^\alpha\partial_p^\beta \circ \Pi_\rho\Pi_\sigma - \frac{1}{2}R^\sigma{}_\alpha\,\partial_p^\alpha \circ \Pi_\sigma + \mathscr{O}(\hbar\,\nabla R), \qquad \Pi_\mu = p_\mu + \frac{i\hbar}{2}\widetilde{\mathscr{D}}_\mu$$

Constraining the Wigner function

[Winter, Phys. Rev. D 32, 1871 (1985); Fonarev, J. Math. Phys. 35 (1994)]

Real Part: **Quantum corrected Mass shell condition**

$$(p^2 + m^2)\, W = \hbar^2\, \mathscr{M}\, W + \mathcal{O}(\hbar^4)$$

$$\mathscr{M} = \frac{1}{4}\,\widetilde{\mathscr{D}}^2 + \left(\frac{1}{3} - \xi\right) R - \frac{1}{12}\, p_\sigma p^\mu R^\sigma{}_{\alpha\beta\mu}\, \partial_p^\alpha \partial_p^\beta + \frac{1}{4}\, R_{\mu\nu}\, p^\mu \partial_p^\nu$$

Imaginary Part: **Quantum corrected Vlasov equation**

$$p^\mu \widetilde{\mathscr{D}}_\mu W = \hbar^2\, \mathscr{K}\, W + \mathcal{O}\left(\hbar^4\right)$$

$$\mathscr{K} = -\frac{1}{6}\, p^\mu R^\sigma{}_{\alpha\beta\mu}\, \partial_p^\alpha \partial_p^\beta \widetilde{\mathscr{D}}_\sigma + \frac{1}{12}\, R^\sigma{}_\beta\, \partial_p^\beta \widetilde{\mathscr{D}}_\sigma + \frac{1}{24}\, p_\lambda p^\mu (\nabla_\sigma R^\lambda{}_{\alpha\beta\mu}) \partial_p^\alpha \partial_p^\beta \partial_p^\sigma - \frac{1}{24}\, p^\mu (\nabla_\mu R_{\alpha\beta})\, \partial_p^\alpha \partial_p^\beta + \frac{1}{2}\left(\xi - \frac{1}{4}\right)(\nabla_\alpha R) \partial_p^\alpha$$

Recovering the flat spacetime

Real Part:

Quantum corrected Mass shell condition

$$(p^2 + m^2)\, W = \hbar^2\, \mathscr{M}\, W + \mathcal{O}(\hbar^4)$$

$$(p^2 + m^2)W = \frac{\hbar^2}{4}\partial^2 W$$

Imaginary Part:

Quantum corrected Vlasov equation

$$p^\mu \widetilde{\mathscr{D}}_\mu W = \hbar^2\, \mathscr{K}\, W + \mathcal{O}\left(\hbar^4\right)$$

$$p^\mu \partial_\mu W = 0$$

Solving for the Wigner function

Distributional solution around $\Delta = p^2 + m^2$

$$W(x,p) = \delta(\Delta) f - \hbar^2 \delta'(\Delta) \mathscr{M} f - \frac{\hbar^2}{3} \delta''(\Delta) R_{\mu\nu} p^\mu p^\nu f + \mathcal{O}(\hbar^4)$$

$$\longrightarrow \quad p^\mu \widetilde{\mathscr{D}}_\mu f \Big|_{\Delta=0} = \hbar^2 \mathscr{K} f \Big|_{\Delta=0}$$

Global equilibrium

$$f = F(\beta \cdot p) - \hbar^2 \left[\frac{1}{12} F''' p^\mu \varpi_{\mu\nu} \alpha^\nu + F'' \left(\frac{1}{8} \varpi_{\alpha\beta} \varpi^{\alpha\beta} + \frac{1}{24} R_{\alpha\beta\beta}{}^{\alpha} \beta^\beta \right) \right] + \mathcal{O}(\hbar^4)$$

$$\varpi_{\mu\nu} = \nabla_\mu \beta_\nu$$
$$\nabla_\rho \varpi_{\mu\nu} = R_{\rho\mu\nu\sigma} \beta^\sigma$$
$$\alpha^\mu = \beta \cdot \nabla \beta^\mu$$

- ξ term drops out
- flat limit

In curved spacetime, transport fixes these $\mathcal{O}(\hbar^2)$ coefficients

Recovering hydro: stress energy tensor

$$T^{\mu\nu} = \int_p \left(p^\mu p^\nu + \hbar^2 \xi R^{\mu\nu}\right) W - \hbar^2 \left(\frac{1}{4} - \xi\right) \left(g^{\mu\nu}\Box - \nabla^\mu \nabla^\nu\right) \int_p W$$

- ξ dependence
- Equilibrium up to $\mathcal{O}(\hbar^2)$ after inserting W

Equilibrium-persisting corrections *(preliminary)*

$$T^{\mu\nu} = \mathscr{E}\, u^\mu u^\nu + \mathscr{P}\, \Delta^{\mu\nu} + \mathscr{Q}^\mu u^\nu + \mathscr{Q}^\nu u^\mu + \mathscr{T}^{\mu\nu}\,,$$

$\mathscr{E}, \mathscr{P} : a^2,\ \ \omega^2,\ \ R,\ \ R_{\alpha\beta}u^\alpha u^\beta$

$\mathscr{Q}^\mu : \epsilon^{\mu\lambda\rho\sigma} a_\lambda u_\rho \omega_\sigma,\ \ \Delta^{\mu\rho} R_{\rho\sigma} u^\sigma$

$\mathscr{T}^{\mu\nu} : a^{\langle\mu} a^{\nu\rangle},\ \ \omega^{\langle\mu}\omega^{\nu\rangle},\ \ u_\alpha R^{\alpha\langle\mu\nu\rangle\beta} u_\beta,\ \ R^{\langle\mu\nu\rangle}$

- f_1, f_2 and f_3 fixed up to $\mathcal{O}(\hbar^2)$ for any m and ξ

Wigner calculation seems to reproduce the non-dissipative second-order equilibrium structures predicted by QFT.

Spinor case *(work in progress)*

Same construction: W is now a 4×4 matrix in spinor space

$$W(x,p) = \int_{T_x\mathscr{M}} d^4y \sqrt{-g}\; e^{-ip\cdot y/\hbar} \left\langle : \bar{\psi}(x, \tfrac{y}{2}) \otimes \psi(x, -\tfrac{y}{2}) : \right\rangle$$

Spinor indices are parallel-transported with the spin connection

Generalised Bopp Operator *(preliminary!)*

$$\mathscr{B}_\mu^{(s)} W = \mathscr{B}_\mu W - \frac{\hbar^2}{8}\left[\Omega_{\alpha\mu}, \partial_p^\alpha W\right] + \frac{i\hbar^3}{48}\left\{ \nabla_\beta \Omega_{\alpha\mu}, \partial_p^\alpha \partial_p^\beta W \right\} + \mathscr{O}(\hbar^4) \qquad \Omega_{\mu\nu} = \frac{i}{4} R_{\alpha\beta\mu\nu}\sigma^{\alpha\beta},$$

$$\left(\gamma^\mu \mathscr{B}_\mu^{(s)} - m\right) W = 0$$

Mass shell + transport equation at $\mathscr{O}(\hbar^2)$

$$W = \frac{1}{4}\left[\mathscr{F} + i\gamma^5 \mathscr{P} + \gamma^\mu \mathscr{V}_\mu + \gamma^5\gamma^\mu \mathscr{A}_\mu + \frac{1}{2}\sigma^{\mu\nu}\mathscr{S}_{\mu\nu}\right]$$

Liu, Gao, Mameda, Huang, arXiv:1812.10127 ·
Liu, Mameda, Huang, Chin. Phys. C 44 (2020) 094101

Summary & outlook

Covariant Wigner function formalism at higher order in $\hbar$

* Generalised Bopp Operator $\mathscr{B}_\mu$
* Systematic method for spin 0 and spin 1/2

Quantum corrections to global equilibrium

On-shell distribution and Hilbert stress-energy tensor may reproduce non-dissipative equilibrium structures predicted by quantum field theory

Outlook: particles with spin

Extension to Dirac fields within the same geometric formalism, incorporating additional spin-curvature couplings

Back up

$$f_1 = \frac{1-6\xi}{24\pi^2\hbar}\sum_{\pm}\int_0^\infty dk\,\frac{k^2}{E_k}\,n_\pm(E_k) \qquad f_2 = \frac{m^2}{48\pi^2\hbar}\sum_{\pm}\int_0^\infty dk\,n'_\pm(E_k) \qquad f_3 = -\frac{1}{48\pi^2\hbar}\sum_{\pm}\int_0^\infty dk\,\frac{2k^2+m^2}{E_k}\,n_\pm(E_k)$$

$$m = 0: \quad f_1 = \frac{(1-6\xi)\,T^2}{72\hbar}, \quad f_2 = 0, \quad f_3 = -\frac{T^2}{72\hbar}$$