Entropy, Kinetic Theory and Berry Phase

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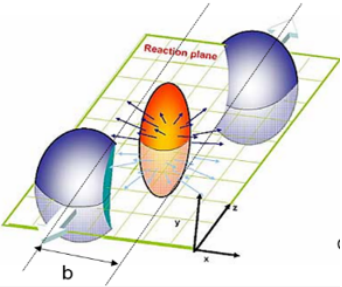


Overview

- motivation and application
- The matrix-valued Wigner function and the total Berry
- transition amplitude decomposition
- H-theorem at leading order
- fixed equation of motion and entropy production
- summary

Motivation

- In spin kinetic theory, **nonlocal collisions** convert orbital into spin angular momentum and polarize the fluid along the vorticity.
- The side jump is encoded by a matrix-valued shift (Berry curvature).
- Missing piece: a **positive, frame-independent entropy production** for the matrix-valued theory.
- Goal: entropy current + H-theorem, with *fixed* equation of motion.



Weickgenannt, Speranza, Sheng, Wang, Rischke (2020, 2021) Wagner, Weickgenannt, Rischke (2022)
Sheng, Wang, Rischke (2022)

The Collision Operator

Binary elastic collisions, molecular chaos: $\hat{\rho} = \hat{W}_1 \otimes \hat{W}_2$, $\hat{\rho} \to S\hat{\rho}S^\dagger$, $S = 1 - it$:

$$\Delta\hat{\rho} \;=\; t\hat{\rho}'\,t^\dagger \;-\; \tfrac{1}{2}\{t^\dagger t, \hat{\rho}\} \;-\; \tfrac{i}{2}[t + t^\dagger, \hat{\rho}]$$

- Three terms: **gain**, **loss**, **coherent spin precession**.
- Unitarity $\Rightarrow$ operator optical theorem: $i(t - t^\dagger) = t^\dagger t$.
- Trace over the unobserved partner and integrate over phase space:

$$\hat{C}[\hat{W}_1] = \int d\Gamma_2\, d\Gamma_1'\, d\Gamma_2'\, \delta^{(4)}(P)\, \mathrm{Tr}_2\big[\,\Delta\hat{\rho}\,\big]$$

The Matrix-Valued Wigner Function

The one-particle spin density matrix in phase space is a 2×2 Hermitian matrix:

$$\hat{W}(x,p) = \tfrac{1}{2}\Big[F(x,p)\,\mathbb{1} + A_\nu(x,p)\,n^\nu(p)\Big]$$

- F: scalar distribution; A_ν: spin polarization (axial vector).
- Positivity of the eigenvalues of $\hat{W}$ is what makes $\ln \hat{W}$ (and the entropy) well defined.
- On-shell measure: $dP = d^4p\,\delta(p^2 - m^2)$ ($\theta(p^0)$ understood).

Transition Amplitude Decomposition and Spin-Flip Model

General two-particle transition operator in the 16-element Pauli-product basis:

$$t \;=\; a_1\,\mathbb{1}\otimes\mathbb{1} \;+\; \vec{a}_2\cdot(\vec{\sigma}\otimes\mathbb{1}) \;+\; \vec{a}_3\cdot(\mathbb{1}\otimes\vec{\sigma}) \;+\; a_4^{ij}\,\sigma_i\otimes\sigma_j$$

- **Explicit two-leg spin-flip model** along collision axis $\hat{n}$ $\big(a_1 = A,\quad \vec{a}_2 = \vec{a}_3 = 0,$ $a_4^{ij} = d\,n_i n_j\big)$:

$$t \;=\; A\,I \;+\; d\,\Sigma, \qquad I \equiv \mathbb{1}_1\otimes\mathbb{1}_2, \qquad \Sigma \equiv (\hat{n}\cdot\vec{\sigma}_1)\otimes(\hat{n}\cdot\vec{\sigma}_2)$$

- Since $\Sigma^2 = I$, $\{I,\Sigma\}$ closes under the optical theorem $i(t - t^\dagger) = t^\dagger t$:

Quadratic Rate Operator and Cross Sections

$$t^\dagger t \;=\; \sigma_0\,I \;+\; \sigma_\times\,\Sigma$$

$$\sigma_0 \;\equiv\; |A|^2 + |d|^2 \;=\; -2\,\mathrm{Im}\,A, \qquad \sigma_\times \;\equiv\; 2\,\mathrm{Re}(A^* d) \;=\; -2\,\mathrm{Im}\,d$$

Gain, Loss, and Coherent Sectors After Tr_2

With $\hat{W}_j = \frac{1}{2}(F_j\,\mathbb{1} + \vec{A}_j \cdot \vec{\sigma})$, $\vec{A}_j = A_j^{\parallel}\hat{n} + \vec{A}_j^{\perp}$, and $\mathrm{Tr}_2(A \otimes B) = A\,\mathrm{Tr}(B)$:

- **Gain sector** $\mathrm{Tr}_2\big[t\,\hat{\rho}'\,t^\dagger\big]$ (with $\hat{\rho}' = \hat{W}_3 \otimes \hat{W}_4$):

$$\begin{aligned}&\tfrac{1}{2}\big(\sigma_0 F_3 F_4 + \sigma_\times A_3^{\parallel} A_4^{\parallel}\big)\mathbb{1} + \tfrac{1}{2}\big(\sigma_0 F_4 A_3^{\parallel} + \sigma_\times F_3 A_4^{\parallel}\big)\hat{n}\cdot\vec{\sigma} \\ &\quad + \tfrac{1}{2}\Big[\big(|A|^2 - |d|^2\big)F_4\vec{A}_3^{\perp} + 2\,\mathrm{Im}(A^* d)\,A_4^{\parallel}(\hat{n}\times\vec{A}_3)\Big]\cdot\vec{\sigma}\end{aligned}$$

- **Loss sector** $\mathrm{Tr}_2\big[-\tfrac{1}{2}\{t^\dagger t, \hat{\rho}\}\big]$ (with $\hat{\rho} = \hat{W}_1 \otimes \hat{W}_2$):

$$-\tfrac{1}{2}\big(\sigma_0 F_1 F_2 + \sigma_\times A_1^{\parallel} A_2^{\parallel}\big)\mathbb{1} - \tfrac{1}{2}\Big[\big(\sigma_0 F_2 A_1^{\parallel} + \sigma_\times F_1 A_2^{\parallel}\big)\hat{n} + \sigma_0 F_2 \vec{A}_1^{\perp}\Big]\cdot\vec{\sigma}$$

- **Coherent sector** $\mathrm{Tr}_2\big[-\tfrac{i}{2}[t + t^\dagger, \hat{\rho}]\big]$ (scalar A drops out via $[I, \hat{\rho}] = 0$):

$$+\,\mathrm{Re}(d)\,A_2^{\parallel}(\hat{n}\times\vec{A}_1)\cdot\vec{\sigma}$$

Final Result: Decomposed Collision Kernel

Decomposing the collision operator as $\hat{C}[\hat{W}_1] = C_F[\hat{W}]\,\mathbb{1} + \vec{C}_A[\hat{W}]\cdot\vec{\sigma}$:

Scalar (Particle Number) Channel $C_F[\hat{W}]$

$$C_F[\hat{W}] \;=\; \frac{1}{2}\int d\Gamma_{234}\,\Big[\sigma_0\big(F_3F_4 - F_1F_2\big) \;+\; \sigma_\times\big(A_3^{\parallel}A_4^{\parallel} - A_1^{\parallel}A_2^{\parallel}\big)\Big]$$

Spin-Polarization Channel $\vec{C}_A[\hat{W}]$

$$\begin{aligned}\vec{C}_A[\hat{W}] \;=\; \frac{1}{2}\int d\Gamma_{234}\,\Big\{&\Big[\sigma_0\big(F_4A_3^{\parallel} - F_2A_1^{\parallel}\big) + \sigma_\times\big(F_3A_4^{\parallel} - F_1A_2^{\parallel}\big)\Big]\hat{n}\\ &+\; \big(|A|^2 - |d|^2\big)F_4\vec{A}_3^{\perp} \;-\; \sigma_0F_2\vec{A}_1^{\perp}\\ &+\; 2\,\mathrm{Im}(A^*d)\,A_4^{\parallel}(\hat{n}\times\vec{A}_3) \;+\; 2\,\mathrm{Re}(d)\,A_2^{\parallel}(\hat{n}\times\vec{A}_1)\Big\}\end{aligned}$$

H-Theorem at Leading Order: Setup

Shiftless Entropy current and its divergence:

$$S^\mu(x) = -\int dP\ \mathrm{Tr}\big[p^\mu\, H(\hat{W})\big], \qquad H(\hat{W}) \equiv \hat{W}(\ln \hat{W} - 1)$$

$$\begin{aligned}\partial_\mu S^\mu &= -\int dP\ \mathrm{Tr}\big[(p\cdot\partial\hat{W})\ \ln \hat{W}\big] \\ &= -\int dP\ \mathrm{Tr}\big[C[\hat{W}]\ \ln \hat{W}\big], \qquad \text{using } p\cdot\partial\hat{W} = C[\hat{W}]\end{aligned}$$

- Matrix chain rule: $\delta\,\mathrm{Tr}(\hat{W}\ln\hat{W} - \hat{W}) = \mathrm{Tr}[\ln\hat{W}\ \delta\hat{W}]$.
- Sign convention: production $= -\,\mathrm{Tr}[C\ln\hat{W}]$; the gain−loss structure of C makes this the positive side.

H-Theorem at Leading Order

$$\partial_\mu S^\mu = \frac{1}{2}\sum_{I,J}\Big[K_{J\leftarrow I}\, p_I \ln p_I - K_{I\leftarrow J}\, p_J \ln p_I\Big]$$

Swap $I \leftrightarrow J$ in the first term and add the zero sum:

Theorem 1 (leading-order H-theorem)

$$\partial_\mu S^\mu = \frac{1}{2}\sum_{I,J} K_{I\leftarrow J}\Big[p_J \ln\frac{p_J}{p_I} - p_J + p_I\Big] \;\geq\; 0$$

by the scalar inequality $x\ln(x/y) - x + y \geq 0$ for $x, y > 0$, term by term. **Hypotheses:** ***S*-matrix unitarity and Hermiticity .**

Furst, Mendl, Spohn (2018)

When Does the Production Vanish?

Every term is non-negative, so — *after* the theorem:

Remark (vanishing condition — derived, not assumed)

$\partial_\mu S^\mu = 0$ iff $p_I = p_J$ for every pair with $K_{I \leftarrow J} \neq 0$, i.e. the eigenvalues are constant along collision-connected configurations.

- The theorem itself assumes *only* unitarity + Hermiticity; the sum rule and the change of variables do the work.
- the condition is satisfied in equilibrium such that $C[\hat{W}] = 0$.

The Total Shift Δ^{μ}_{η}

built on the Hermitian matrix structure $\epsilon^{\mu\nu\alpha\beta}\, u_\nu p_\alpha\, n_\beta(p)$

Stacking identity

$$\hat{W}_\Delta \;=\; \hat{W} + \Delta \cdot \partial \hat{W}$$

- Δ_η is Hermitian (both pieces: real prefactors on the same n-structure).
- The total Δ is the **phase space displacement** iduced by berry curvature: it dresses $\hat{W}_\Delta$, drives $\int \Delta \cdot \partial\, c$ in the equation of motion, fills the entropy slot, and displaces the collision legs — one object, four appearances.

Beyond Leading Order: Two-Particle Consistency

Two-particle probabilities ρ_I; master equation for the two-particle entropy:

$$\partial S = \frac{1}{2}\sum_{I,J}\Big[K_{J\leftarrow I}\,\rho_I \ln \rho_I - K_{I\leftarrow J}\,\rho_J \ln \rho_I\Big], \qquad \hat{\rho} = \hat{W}_1 \otimes \hat{W}_2$$

- Demand: a *closed one-particle* equation must reproduce this identically.
- The naive equation $p \cdot \partial \hat{W} = C[\hat{W}(x+\Delta)]$ **fails**: expanding the shifted kernel, the gradient bookkeeping on the loss legs does not close (mismatch $\propto \int \Delta \cdot \partial\, c$).
- The consistent equation carries the compensating nonlocal term.

Proposition 2 (EOM selection by two-particle consistency)

$$p \cdot \partial \hat{W}(x,p) \;=\; C[\hat{W}](x,p) \;+\; \int dP_2\, dP_3\, dP_4\; \Delta^\mu \partial_\mu\, c[\hat{W}](x,p)$$

- Equivalently, with the dressed distribution $\hat{W}_\Delta \equiv \hat{W} + \Delta^\mu \partial_\mu \hat{W}$ and Δ independent of x: $p \cdot \partial\, \hat{W}_\Delta = C[\hat{W}_\Delta] + O(\hbar^2)$ — exact at $O(\hbar)$.
- All four legs shifted (finite-shift form) $\Leftrightarrow$ dressed streaming form: same statement, two coordinate choices.
- **Provenance:** Δ— the displacement in $\int \Delta \cdot \partial c$ is by construction the same one that dresses $\hat{W}_\Delta$ and fills the entropy slot.

Entropy Current with the Shift Correction

The naive current paired with the dressed collision term leaves a mismatch $\propto \mathrm{Tr}[\Delta \cdot \partial C\, H'(\hat{W})]$. The fix — a correction slot in the current:

$$S^{\mu}(x) = -\int dP\ \mathrm{Tr}\big[p^{\mu}\, H(\hat{W}_{\Delta})\big] \;-\; \int dP\, dP_2\, dP_3\, dP_4\ \mathrm{Tr}\big[\Delta^{\mu}\, c[\hat{W}]\ H'(\hat{W}_{\Delta})\big] \;+\; \partial_{\nu} S^{\mu\nu},$$

Theorem 3 (production identity)

With the EOM of Prop. 2, exactly at $O(\hbar)$:

$$\partial_{\mu} S^{\mu} = -\int dP\ \mathrm{Tr}\big[C[\hat{W}_{\Delta}]\ \ln \hat{W}_{\Delta}\big]$$

- Proof: chain rule on the first slot; the second slot supplies exactly the $\int \Delta \cdot \partial c$ bookkeeping; mismatch cancels term by term.
- $S^{\mu\nu} = -S^{\nu\mu}$: identically conserved (magnetization-type); carries no production.

Applications

- **Spin hydrodynamics:** covariant entropy current with manifestly positive divergence — the input for moment methods and entropy-current analyses; transport coefficients from the linearized $C[\hat{W}_\Delta]$ near the stationary state.
- **Heavy-ion phenomenology:** Λ polarization; the vanishing condition as the freeze-out criterion for spin; equilibration time scales from the linearized operator.
- **Chiral media:** anomaly-driven corrections
- **quantum computation**: using a projective form of the transition amplitude allows for controlled spin switches.

Summary

- $t = A\,I + d\,\Sigma$: the optical theorem fixes σ_0 and $\sigma_\times$; the kernel splits into number and spin channels (longitudinal transfer, transverse flip, precession).
- The total shift is Δ^μ
- Two-particle consistency *fixes* the equation of motion: the fully shift-dressed form; the naive equation fails.
- Entropy current with the $\Delta\, c\, H'$ slot: production positive definite and Lorentz invariant; the naive equation violates the second law at $O(\hbar)$.
- Additional research: helicity-resolved decay rates in chiral media — the anomaly acts as a chiral filter; polarized cascades approach total polarization (Hansen–Tuchin).
- Applications (spin hydrodynamics, heavy-ion phenomenology)

Additional Research I: Spin-Dependent Decay Rates in Chiral Media

The chiral anomaly as an emergent Chern–Simons term $\mathcal{L}_{CS} = -\frac{1}{2} C_A \theta \, F_{\mu\nu} \tilde{F}^{\mu\nu}$, with chiral conductivity $b_0 = C_A \, \partial_t \theta$:

- Anomaly-dressed photon dispersion $k^2 = -\lambda b_0 \, |\vec{k}|$: for $b_0 > 0$ the $\lambda = +1$ mode is **space-like** (chiral Cherenkov radiation), the $\lambda = -1$ mode **time-like** (decays via pair production).

Spin-Resolved Differential Decay Rate (QED: $C = e^2 = 4\pi\alpha$; QCD: $C = g^2 C_F$)

$$\frac{dW^{s,s',\lambda}}{d^3k} = \frac{C}{32\pi^2 \, E \, E' \, \omega} \left| M^{s,s',\lambda} \right|^2 \delta\big(E - E' - \omega\big)$$

work in progress based on Hansen–Tuchin, (2024)

Additional Research II: The Exact Squared Matrix Element

spin decomposition $s^\mu = h\,\frac{m}{E}p^\mu + s_T^\mu - h\,\frac{m}{2E}n^\mu$ (h: handedness; s_T^μ: transverse spin) splits the rate into exact mass orders $\left|M\right|^2 = \left|M\right|^2_{(0)} + \Delta\left|M\right|^2_{(m^1)} + \Delta\left|M\right|^2_{(m^2)}$:

- **Massless** $\mathcal{O}(m^0)$: helicity conserved along the fermion line through the $(1 + hh')$ structure; transverse spin enters via $(\vec{s}_T \cdot \vec{s}'_T)$ and $(\hat{k}\cdot\vec{s}_T)(\hat{k}\cdot\vec{s}'_T)$.
- **Mass-ordered**: $\mathcal{O}(m^1)$ generates handedness–transverse cross-talk; $\mathcal{O}(m^2)$ solely governs handedness-flipping transitions ($h = -h'$).
- **Anomaly shift**: enters through the dressed momentum $|\vec{k}| \simeq \omega + \frac{1}{2}\lambda b_0$.

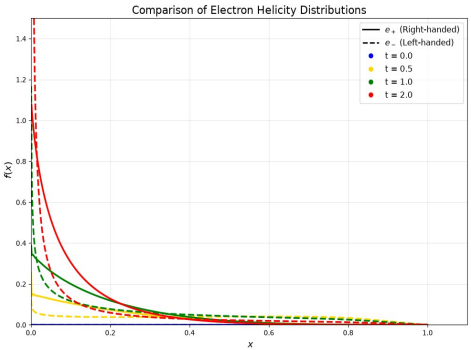
Additional Research III: Chiral Filtering in Polarized Cascades

- Helicity-resolved cascade equations for $e_h,\ \bar{e}_h,\ \gamma^\lambda$ with $\tau^{-1} = \alpha b_0/4$: for $b_0 > 0$, γ^+ is preferentially radiated and γ^- destroyed by pair production.
- Diagonal radiation kernel:

$$P_{e_h e_h}(x) = \frac{1 + x^2 - h(1 - x^2)}{1 - x} + \frac{3}{2}(1+h)\,\delta(1-x)$$

— left-handed electrons are retained more efficiently.

Electron helicity distributions



An example of such a decay: e^+ (right-handed) vs. e^- (left-handed) at $t = 0,\ 0.5,\ 1,\ 2$.

work in progress based on Hansen–Tuchin, (2025)

Plans and Opportunities

- Complete and publish our study of Entropy and my study of polarized cascades
- apply and explore the decomposition of the transition amplitude in specific situations
- find a more solid foundation for the fixed boltzmann equation
- expand my study of polarized cascades to include massive correction, and or non homogeneous chiral media

Covariance and Properties

- frame-difference piece $\Delta_{\eta_0,\eta}$ *only*; χ_{η_0} is static
- $\hat{W}_\Delta{}' = D\,\hat{W}_\Delta\,D^\dagger + O(\hbar^2)$
- D boost two by two matrices

object	transform
$\hat{W}$	carries boost dressing
$\Delta^\mu_{\eta_0,\eta}$	cocycle; $D\Lambda^\mu_\nu(\cdot)D^\dagger$
$\chi^\mu_{\eta_0}$	static
Δ^μ_η	total (physical)
$\hat{W}_\Delta$	$D\,(\cdot)\,D^\dagger$
$C[\hat{W}_\Delta]$, $H'(\hat{W}_\Delta)$	$D\,(\cdot)\,D^\dagger$
dP	invariant

Lorentz Covariance of the Entropy Production

Theorem 5 (frame invariance)

$\partial_{\mu'} S^{\mu'}(x') = \partial_\mu S^\mu(x)$: the *sign* of the production is frame-independent.

Three steps, one display each:

1. $\hat{W}_\Delta \to D\,\hat{W}_\Delta\,D^\dagger$ (stacking lemma: cocycle on $\Delta_{\eta_0,\eta}$; χ_{η_0} static),
2. $C[\hat{W}_\Delta] \to D\,C[\hat{W}_\Delta]\,D^\dagger$ (the D's can be pulled out of the kernal),
3. $H'(\hat{W}_\Delta) \to D\,H'(\hat{W}_\Delta)\,D^\dagger$,
4. cyclicity of Tr removes the rest.

Physical Interpretation of the Collision Channels

Each directional component relative to $\hat{n}$ carries a distinct physical role:

- **Number & Longitudinal Spin ($\parallel \hat{n}$):** Both scalar ($|A|^2$) and spin-flip ($|d|^2$) scattering conserve longitudinal spin via σ_0, while the interference cross section $\sigma_\times = 2\,\mathrm{Re}(A^* d)$ transfers longitudinal polarization between collision partners.
- **Transverse Dissipative Spin ($\perp \hat{n}$):** Conjugation by Σ acts as a π-rotation about $\hat{n}$ that flips incoming transverse polarization ($\vec{A}_3^{\perp} \to -\vec{A}_3^{\perp}$), yielding $\left(|A|^2 - |d|^2\right) F_4 \vec{A}_3^{\perp}$ in the gain term against $-\sigma_0 F_2 \vec{A}_1^{\perp}$ in the loss term.
- **Reversible Spin Precession ($\hat{n} \times \vec{A}$):** The coherent commutator is purely traceless and independent of A, driving mean-field Larmor precession $\propto \mathrm{Re}(d)\, A_2^{\parallel}(\hat{n} \times \vec{A}_1)$ without producing entropy.

H-Theorem at Leading Order: Proof I

Symmetrize under $1 \leftrightarrow 2$, using $\ln \hat{\rho} = \ln \hat{W}_1 \otimes \mathbb{1}_2 + \mathbb{1}_1 \otimes \ln \hat{W}_2$ and $d\mu \equiv d\Gamma_1 d\Gamma_2 d\Gamma_1' d\Gamma_2' \, \delta^{(4)}(P)$:

$$\partial_\mu S^\mu = \tfrac{1}{2} \int d\mu \, \mathrm{Tr}_{12} \big[\, t^\dagger t \, \hat{\rho} \ln \hat{\rho} \; - \; t \hat{\rho}' t^\dagger \ln \hat{\rho} \, \big]$$

- The precession term vanishes identically: $\mathrm{Tr}_{12} \big[[t + t^\dagger, \hat{\rho}] \ln \hat{\rho} \big] = 0$ (cyclicity + $[\hat{\rho}, \ln \hat{\rho}] = 0$).
- Spin frames precess during the collision: $[\hat{\rho}, \hat{\rho}\,'] \neq 0 \Rightarrow$ diagonalize in *distinct* eigenbases,

$$\hat{\rho} = \sum_I p_I \, |I\rangle\langle I|, \qquad \hat{\rho}\,' = \sum_J p_J \, |J\rangle\langle J|, \qquad W_{I \leftarrow J} = |\langle I|t|J\rangle|^2$$

- All quantum interference is absorbed into $W_{I \leftarrow J}$.