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Nonlinear Response and Correlators of Charge Transport

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Based on:

J. Hu, arXiv:2610.38571
J. Hu, SciPost Phys. **20**, 020 (2026)
X. Zhao et al, JHEP **11** (2026) 074

Outline

- Motivation: nonlinear response and its physical relevance
- Kinetic setup and linear response
- Nonlinear response: three-point correlators
- Analytic structure and consistency limits
- Relation to Schwinger–Keldysh EFT and outlook

Backup slides: technical details, extra checks.

Motivation: Linear Response

- Linear response is encoded in two-point retarded correlators.
  [L. P. Kadanoff and P. C. Martin, Annals Phys. 24, 419 (1963)]
- The analytic structure in the complex frequency plane dictates real-time dynamics:
  - Hydrodynamic poles $\rightarrow$ long-lived collective modes.
  - Branch cuts / non-hydrodynamic singularities $\rightarrow$ fast microscopic modes.
    [P. Romatschke, Eur. Phys. J. C 76, 352 (2016)]
- In kinetic theory with energy-dependent RTA, branch cuts could also become gapless. Then non-hydrodynamic modes acquire long lifetimes, and the standard hydrodynamic picture may break down.
- Whether this separation exists or not depends on microscopic details controlling the collision spectrum; the analytic structure must be studied case by case.
  [J. Hu, SciPost Phys. **20**, 020 (2026)]
- Most studies focus on *linear* response. What about nonlinear response?

Why Nonlinear Response?

- One side of the coin : nonlinear dynamics generates and controls non-Gaussian fluctuations described by symmetrized three-point correlators, sensitive to critical phenomena and far-from-equilibrium dynamics.

  [X. An et al., Nucl. Phys. A 1017, 122343 (2022) ...]
- The other side : the leading nonlinear response is captured by retarded three-point correlators.
- In the Schwinger–Keldysh EFT, three-point functions have been computed for diffusive systems in the kinetic regime.

  [N. Abbasi and D. H. Rischke, JHEP 05, 241 (2025)]
- Direct calculations in kinetic theory, starting from the transport equation, remain less explored.

  [L. V. Delacretaz et al., Phys. Rev. Res. 4, 033131 (2022)]
- **Goal:** directly compute three-point retarded correlators of charge transport in kinetic theory.

  [J. Hu, arXiv:2610.38571]

Kinetic Setup

- Dilute classical gas of massless charged particles: $p^0 = p = |\mathbf{p}|$, $|\mathbf{v}| = 1$.
- External $U(1)$ gauge field as a source.
- Relativistic Boltzmann equation with RTA collision term:

$$p^\mu \partial_\mu f + F^{i\beta} p_\beta \frac{\partial f}{\partial p^i} = \frac{p_\mu u^\mu}{\tau_R} (f - f_{\mathrm{eq}}) .$$

- Matching condition enforces charge conservation order by order:

$$\int \frac{d^3 p}{(2\pi)^3} [f - f_{\mathrm{eq}}] = 0.$$

- Charge current: $J^\mu(x) = \int \frac{d^3 p}{(2\pi)^3} v^\mu f(x, \mathbf{p}), \quad v^\mu = (1, \mathbf{v})$.

Source Expansion

- Expand around homogeneous equilibrium: $f_0(\mathbf{p}) = \exp\left[(\mu_0 - p^0)/T_0\right]$, $u^\mu = (1, \mathbf{0})$.
- Perturb with a weak gauge field δA^μ and let the chemical potential respond: $\mu = \mu_0 + \delta\mu$.
- Expand in powers of the source:

$$f = f_0 + \delta f^{(1)} + \delta f^{(2)} + \cdots, \quad \delta\mu = \delta\mu^{(1)} + \delta\mu^{(2)} + \cdots, \quad f_{\mathrm{eq}} = f_0 \exp\left(\frac{\delta\mu}{T_0}\right)$$

- The n-th order kinetic equation:

$$p^\mu \partial_\mu \delta f^{(n)} + \delta F^{i\beta} p_\beta \frac{\partial}{\partial p^i} \delta f^{(n-1)} = -\frac{p^0}{\tau_R} \left(\delta f^{(n)} - \delta f_{\mathrm{eq}}^{(n)}\right).$$

- Lower-order distributions act as known sources for sloving the next order.
- This gives a recursive solution to arbitrary response order.

Linear Response: Two-Point Correlators

- First-order solution:

$$\delta f^{(1)} = \frac{f_0}{T_0} \frac{\delta\mu^{(1)} + \tau_R \mathbf{v}\cdot\mathbf{E}}{1 + \tau_R(-i\omega + i\mathbf{k}\cdot\mathbf{v})}.$$

- Define the retarded density response: $G_R^{0,\nu}(K) = \left.\frac{\delta n(K)}{\delta A_\nu(K)}\right|_{A=0}, \qquad K = (\omega, \mathbf{k}).$
- The density two-point function has the structure

$$G_R^{0,0}(\omega,\mathbf{k}) \propto \frac{1}{1 - \frac{1}{2ik\tau_R}\ln\left(\frac{\omega - k + i/\tau_R}{\omega + k + i/\tau_R}\right)}.$$

- Logarithmic branch points at $\omega = \pm k - i/\tau_R$.
  [P. Romatschke, Eur. Phys. J. C 76, 352 (2016)]
- Hydrodynamic diffusion pole for $k\tau_R \ll 1$: $\omega_{\text{diff}} = -iDk^2, \qquad D = \tau_R/3.$

Nonlinear Response: Recursive Structure

- At order n, formal solution:

$$\delta f^{(n)} = \frac{1}{\partial_t + \mathbf{v}\cdot\nabla + 1/\tau_R}\left(\frac{1}{\tau_R}\delta f_{\rm eq}^{(n)} - \delta F^{i\beta} v_\beta \frac{\partial}{\partial p^i}\delta f^{(n-1)}\right).$$

- Focusing on $n = 2$, the fully retarded three-point response is defined by

$$G_R^{\mu,\nu\rho}(K_1, K_2) = \left.\frac{\delta^2 J^\mu(K_1 + K_2)}{\delta A^\nu(K_1)\,\delta A^\rho(K_2)}\right|_{A=0}.$$

- In the r/a basis, this is a *raa*-type correlator corresponding to 3 legs for $\{-(K_1 + K_2), K_1, K_2\}$.

Three-Point Correlators: Main Result (I)

- We analytically obtain the fully retarded density correlator $G_R^{0,00}(\omega-\omega',\mathbf{k}-\mathbf{k}';\omega',\mathbf{k}')$ at generic frequencies and momenta. Refer to [J. Hu, arXiv:2610.38571] for detailed expressions.
- Structure:

$$G_R^{0,00} = \frac{1}{1-\frac{1}{2ik\tau_R}\ln\left(\frac{\omega-k+i/\tau_R}{\omega+k+i/\tau_R}\right)} \times \Big(\sum_i C_i\, F_i(\omega-\omega',\mathbf{k}-\mathbf{k}';\omega',\mathbf{k}') + [\omega-\omega'\leftrightarrow\omega',\mathbf{k}-\mathbf{k}'\leftrightarrow\mathbf{k}']\Big),$$

- an overall denominator inherited from the linear response;
- F_i consists of two types of terms:
  - pure angular integrals I_{mnl};
  - nested combinations of I_{mnl} with two-point functions.

Three-Point Correlators: Main Result (II)

- The angular integrals I_{mnl} encode multi-leg kinematics:

$$I_{mnl} = \int \frac{d\Omega}{2\pi} \frac{(\mathbf{v}\cdot\mathbf{k})^m (\mathbf{v}\cdot\mathbf{k}')^n}{(-i\omega + i\mathbf{v}\cdot\mathbf{k} + 1/\tau_R)(-i\omega' + i\mathbf{v}\cdot\mathbf{k}' + 1/\tau_R)^l} \,.$$

- All required integrals reduce to elementary functions analytically.
- Explicit exchange symmetry between 2 and 3 external legs (a type).
- The result satisfies the three-point Ward identity identities implied by charge conservation:

$$\omega' G_R^{0,00}(\omega - \omega', \mathbf{k} - \mathbf{k}'; \omega', \mathbf{k}') + k_i' \, G_R^{0,0i}(\omega - \omega', \mathbf{k} - \mathbf{k}'; \omega', \mathbf{k}') = 0 \,.$$

Analytic Structure I: Inherited Singularities

- The nonlinear response remembers the linear-response structure.
- The overall denominator (leg 1, r type) has logarithmic branch points at

$$\omega = \pm k - \frac{i}{\tau_R}.$$

- Nested two-point functions replicate the same structure on the external legs 2 and 3:

$$\omega' = \pm k' - \frac{i}{\tau_R}, \qquad \omega - \omega' = \pm|\mathbf{k} - \mathbf{k}'| - \frac{i}{\tau_R}.$$

- So the same analytic structure governs all three external legs.
- This is a nontrivial consistency check: nonlinearity does not erase linear physics.

Analytic Structure II: Mixed Singularities

- Genuinely new kinematics arises from angular integrals with more than one streaming denominator.
- Simplest example:

$$I_{001} = \int \frac{d\Omega}{2\pi} \frac{1}{(a + i\mathbf{v}\cdot\mathbf{k})(b + i\mathbf{v}\cdot\mathbf{k}')}, \qquad a = \frac{1}{\tau_R} - i\omega, \quad b = \frac{1}{\tau_R} - i\omega'.$$

- Feynman parametrization reduces it to a quadratic denominator. Square-root branch points occur when

$$a^2 k'^2 - 2ab\,\mathbf{k}\cdot\mathbf{k}' + b^2 k^2 + k^2 k'^2 - (\mathbf{k}\cdot\mathbf{k}')^2 = 0.$$

- This mixes frequencies and momenta of different legs.
- Physical picture: a Landau-type singularity — two streaming denominators resonate for the same particle direction.
- The pinch singularity generates a branch point in the three-point function.

Consistency Limits

- **Vanishing limits:**
  - $\mathbf{k}' \to 0$: $G_R^{0,00} \to 0$;
  - $\mathbf{k} - \mathbf{k}' \to 0$: $G_R^{0,00} \to 0$;
  - $\mathbf{k} \to 0$: $G_R^{0,00} \to 0$; nontrivial.
- **Thermodynamic insertion:** take the external leg static first, then homogeneous:

$$\lim_{\mathbf{k}'\to 0} \lim_{\omega'\to 0} G_R^{0,00}(\omega - \omega', \mathbf{k} - \mathbf{k}'; \omega', \mathbf{k}') = -\frac{1}{T_0} G_R^{0,0}(\omega, \mathbf{k}) = -\frac{\partial}{\partial \mu_0} G_R^{0,0}(\omega, \mathbf{k}).$$

- Physical meaning: a zero-frequency, zero-momentum density insertion measures the dependence of the lower-point response on the background chemical potential.
- This is a highly nontrivial check of the nonlinear calculation.

Collisionless Limit and Enlarged Symmetry

- Consider the limit $\tau_R \to \infty$ (collisionless Vlasov limit).
- The difference between exchanging legs 1 and 2 vanishes away from resonances:

$$\lim_{\tau_R \to \infty} \left[G_R^{0,00}(\omega - \omega', \mathbf{k} - \mathbf{k}'; \omega', \mathbf{k}') - G_R^{0,00}(-\omega, -\mathbf{k}; \omega', \mathbf{k}') \right] \to 0, \quad \text{away from resonance.}$$

- Physical interpretation:
  - No dissipation $\Rightarrow$ no arrow of time.
  - The distinction between r and a fields becomes blurred.
  - The r-a basis was introduced to describe dissipative phenomena.
  - So in the collisionless limit, an enlarged permutation symmetry emerges.
    [L. V. Delacretaz et al., Phys. Rev. Res. 4, 033131 (2022)]
- At resonances, $i/\tau_R \to i\epsilon$ is needed to select the causal retarded response.

Relation to Schwinger–Keldysh EFT

- **Linear level:** same RTA setup. The quadratic SK action reproduces the retarded two-point correlator and also contains fluctuation information.
  [N. Abbasi and D. H. Rischke, JHEP 05, 241 (2025)]
- **Nonlinear level:** the two setups differ in how nonlinearity is introduced.
  - In the SK EFT, transport is made nonlinear via $\tau_R \to \tau_R(n)$; the resulting G^{rrr} is controlled by the independent coefficient τ_R'.
  - In the present work, τ_R is kept constant; the second-order response is generated by repeated coupling to A together with Boltzmann thermodynamics.
- Importantly, τ_R' is also naturally and directly accessible here: promoting $\tau_R \to \tau_R(n)$ in our framework immediately generates the τ_R' contribution. The present G^{raa} does not probe τ_R' only because τ_R is constant.
- **Fluctuations:** the Boltzmann equation has no a-type source, so we obtain G^{raa}. The SK EFT can in principle compute all types, though only G^{rrr} was explicitly given.
- **Comparison:** the three-point FDT does not map G^{raa} directly to G^{rrr}.

Summary and Outlook

Summary

- Computed nonlinear response and fully retarded three-point correlators of charge transport in RTA kinetic theory.
- Verified Ward identity identities and nontrivial consistency limits.
- Collisionless limit exhibits an enlarged permutation symmetry.
- Compared with SK EFT.

Outlook

- Energy-momentum transport (source $g_{\mu\nu}$).
- More realistic collision operators.
- Multiple conserved charges, background fields.
- Spin transport (source spin connection).

Backup: Two-Point Function Details

- Self-consistency equation for the first-order density:

$$\delta n^{(1)}(\omega, \mathbf{k}) = \chi \frac{\int \frac{d\Omega}{4\pi} \frac{\tau_R \mathbf{v}\cdot\mathbf{E}}{1+\tau_R(-i\omega+i\mathbf{k}\cdot\mathbf{v})}}{1 - \frac{1}{2ik\tau_R} \ln\left(\frac{\omega-k+i/\tau_R}{\omega+k+i/\tau_R}\right)}.$$

- Susceptibility: $\chi = \dfrac{\partial n_0}{\partial \mu_0} = \dfrac{T_0^2}{\pi^2} e^{\mu_0/T_0}$.
- Explicit retarded correlators:

$$G_R^{0,0}(\omega, \mathbf{k}) = -\chi \frac{1 + (i\omega - \frac{1}{\tau_R})\frac{1}{2ik} \ln\left(\frac{\omega-k+i/\tau_R}{\omega+k+i/\tau_R}\right)}{1 - \frac{1}{2ik\tau_R} \ln\left(\frac{\omega-k+i/\tau_R}{\omega+k+i/\tau_R}\right)},$$

$$G_R^{0,i}(\omega, \mathbf{k}) = \frac{\omega\chi k^i}{k^2} \frac{1 + (i\omega - \frac{1}{\tau_R})\frac{1}{2ik} \ln\left(\frac{\omega-k+i/\tau_R}{\omega+k+i/\tau_R}\right)}{1 - \frac{1}{2ik\tau_R} \ln\left(\frac{\omega-k+i/\tau_R}{\omega+k+i/\tau_R}\right)}.$$

Backup: Angular Integrals

- Master integrals:

$$I_{mnl} = \int \frac{d\Omega}{2\pi} \frac{(\mathbf{v}\cdot\mathbf{k})^m (\mathbf{v}\cdot\mathbf{k}')^n}{(a + i\mathbf{v}\cdot\mathbf{k})(b + i\mathbf{v}\cdot\mathbf{k}')^l}, \quad a = \frac{1}{\tau_R} - i\omega, \quad b = \frac{1}{\tau_R} - i\omega'.$$

- Example:

$$M_{011} = M_{022} = \frac{1}{ik} \ln\left(\frac{a+ik}{a-ik}\right), \qquad M_{102} = \frac{2}{b^2 + k'^2}.$$

- The mixed integral I_{001} is evaluated by Feynman parametrization:

$$I_{001} = -\frac{4\left[\arctan\left(\frac{D_1}{\sqrt{-D_1^2+4D_0D_2}}\right) - \arctan\left(\frac{D_1+2D_2}{\sqrt{-D_1^2+4D_0D_2}}\right)\right]}{\sqrt{-D_1^2+4D_0D_2}}.$$

Backup: Three-Point Function Structure

- The density three-point function has the schematic form

$$G_R^{0,00} = \frac{1}{1 - \frac{1}{2ik\tau_R} \ln\left(\frac{\omega - k + i/\tau_R}{\omega + k + i/\tau_R}\right)} \times \sum_n C_n \, F_n(\omega, \omega', \mathbf{k}, \mathbf{k}') \,,$$

- where F_n are combinations of angular integrals and nested two-point functions.
- The final term enforces exchange symmetry between the two source legs:

$$(\omega - \omega', \mathbf{k} - \mathbf{k}') \leftrightarrow (\omega', \mathbf{k}').$$

- This symmetry is a direct consequence of the functional-derivative definition:

$$G_R^{\mu,\nu\rho}(K_1, K_2) = \left. \frac{\delta^2 J^\mu(K_1 + K_2)}{\delta A^\nu(K_1) \delta A^\rho(K_2)} \right|_{A=0} .$$

Backup: Thermodynamic Insertion Relation

- Static and homogeneous limits do not commute:
  - a constant A^0 shifts the thermodynamic chemical potential;
  - a purely dynamical zero-momentum potential is pure gauge.
- Taking the source leg static first, then homogeneous:

$$\lim_{\mathbf{k}'\to 0}\lim_{\omega'\to 0} G_R^{0,00}(\omega-\omega',\mathbf{k}-\mathbf{k}';\omega',\mathbf{k}') = -\frac{1}{T_0}G_R^{0,0}(\omega,\mathbf{k}) = -\frac{\partial}{\partial\mu_0}G_R^{0,0}(\omega,\mathbf{k}).$$

- This is the expected thermodynamic insertion relation.
- It provides a nontrivial consistency check on the nonlinear calculation.
- It follows from the Boltzmann-gas dependence $\chi \propto e^{\mu_0/T_0}$.

Backup: Summary of Limits

Limit	Behavior
$\mathbf{k}' \to 0$	$G_R^{0,00} \to 0$ (charge conservation)
$\mathbf{k} - \mathbf{k}' \to 0$	$G_R^{0,00} \to 0$ (charge conservation)
$\mathbf{k} \to 0$	$G_R^{0,00} \to 0$ (nontrivial, proven analytically)
$\omega' \to 0$, then $\mathbf{k}' \to 0$	thermodynamic insertion relation
$\tau_R \to \infty$	enlarged permutation symmetry away from resonance

- At resonances, $1/\tau_R \to 0^+$ selects the causal retarded response.
- Away from resonances, $1/\tau_R$ can be set exactly to zero, recovering the collisionless Vlasov equation.