All Pseudo-Gauges Are Equivalent—but Are They Thermodynamically Equivalent?

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Pseudo-gauge freedom and thermodynamics

The definition of hydrodynamic currents is not unique:

$$
\begin{aligned}
T^{\mu\nu} &\longrightarrow T^{\mu\nu} + \frac{1}{2}\partial_\lambda \left(\Phi^{\lambda[\mu\nu]} - \Phi^{\mu[\lambda\nu]} + \Phi^{\nu[\mu\lambda]} \right), \\
J^{\mu} &\longrightarrow J^{\mu} + \partial_\lambda B^{[\lambda\mu]}, \\
S^{\mu\nu\lambda} &\longrightarrow S^{\mu\nu\lambda} - \Phi^{\mu[\nu\lambda]} + \dots .
\end{aligned}
$$

- Equations of motion are unchanged.
- Conserved charges are unchanged with appropriate boundary conditions.

The physical question

Are different pseudo-gauges thermodynamically equivalent?

Our point of view: pseudo-gauge transformations (PGTs) represent a nonphysical redundancy.

Global thermodynamics is invariant

At global equilibrium,

$$\hat{\rho} = \frac{1}{Z} e^{-\hat{\Upsilon}}, \qquad Z = \mathrm{Tr}\, e^{-\hat{\Upsilon}},$$

$$\hat{\Upsilon} = b_\mu \hat{P}^\mu - \frac{1}{2}\Omega_{\alpha\beta}\hat{J}^{\alpha\beta} - \alpha\hat{N}.$$

We define the global thermodynamic potential: Becattini and Rindori, 2019

$$\boxed{\Phi := \ln Z = S - \langle\hat{\Upsilon}\rangle = S - b_\mu P^\mu + \frac{1}{2}\Omega_{\alpha\beta}J^{\alpha\beta} + \alpha N}$$

The conserved charges, and hence Z, $\hat{\rho}$, S, and Φ, are pseudo-gauge invariant.

In homogeneous equilibrium, $\Phi = PV\beta = \int_\Sigma \mathrm{d}\Sigma_\mu\, P\beta^\mu$

Two prescriptions for the potential current

Define thermodynamic-potential 4-current:

$$\ln Z = \int \mathrm{d}\Sigma_\mu \phi^\mu$$

For a charged fluid without spin:

(1) Integrate at fixed μ [Becattini and Rindori, 2019]

$$\phi^\mu_{(1)} = \int_\beta^\infty \mathrm{d}\beta' \ \left(T^{\mu\nu}u_\nu - \mu J^\mu\right)\Big|_{\mu,u}$$

(2) Integrate at fixed $\alpha = \beta\mu$ [Ambrus and Gecić, 2026]

$$\phi^\mu_{(2)} = \int_\beta^\infty \mathrm{d}\beta' \ T^{\mu\nu}u_\nu|_{\alpha,u}$$

In homogeneous equilibrium, both prescriptions agree:

$$\phi^\mu_{(1)} = \phi^\mu_{(2)} = P\beta^\mu$$

An apparent disagreement under a PGT

In inhomogeneous equilibrium, consider the PGT

$$\boxed{J'^{\mu} = J^{\mu} + \partial_{\lambda} B^{[\lambda\mu]}, \qquad T'^{\mu\nu} = T^{\mu\nu}}$$

$$\Longrightarrow \quad n' = u_{\mu} J'^{\mu} \neq u_{\mu} J^{\mu} = n, \quad \varepsilon' = u_{\mu} T'^{\mu\nu} u_{\nu} = u_{\mu} T^{\mu\nu} u_{\nu} = \varepsilon$$

As a consequence:

$$\phi'^{\mu}_{(1)} = \phi^{\mu}_{(1)} + C^{\mu}, \quad C^{\mu} = -\mu \int_{\beta}^{\infty} \mathrm{d}\beta' \; \partial_{\lambda} B^{[\lambda\mu]}\Big|_{\mu,u}, \qquad \phi'^{\mu}_{(2)} = \phi^{\mu}_{(2)}$$

$$\Longrightarrow \quad P'_{(1)} \neq P_{(1)} \equiv P, \qquad P'_{(2)} = P_{(2)} \equiv P$$

Violation of thermodynamic identities:

$$\left.\frac{\partial(\beta P'_{(2)})}{\partial\alpha}\right|_{\beta} = n \neq n', \quad \left.\frac{\partial(\beta P'_{(1)})}{\partial\beta}\right|_{\alpha} = -\varepsilon + \left.\frac{\partial(u_{\mu} C^{\mu})}{\partial\beta}\right|_{\alpha} \neq -\varepsilon'$$

We will show that this apparent disagreement comes from omitted surface terms!

Surface terms and the definition of a current

$$\int_{\Sigma} \mathrm{d}\Sigma_{\mu}\, \partial_{\lambda} \mathcal{G}^{[\lambda\mu]} = \int_{\partial\Sigma} \mathrm{d}S_{\lambda\mu}\, \mathcal{G}^{[\lambda\mu]}$$

Boundary compatibility

We assume that the boundary integral vanishes.
This is an assumption, not an automatic property.

Defining a current through

$$F := \int_{\Sigma} \mathrm{d}\Sigma_{\mu}\, f^{\mu},$$

boundary compatibility leaves the freedom to change f^{μ}:

$$f^{\mu} \longrightarrow f^{\mu} + \partial_{\lambda} \mathcal{G}^{[\lambda\mu]}$$

We may choose any convenient $\partial_{\lambda} \mathcal{G}^{[\lambda\mu]}$.

Equal global quantities do not fix local relations

Suppose two expressions for a global quantity agree:

$$\int_\Sigma \mathrm{d}\Sigma_\mu \, A^\mu = \int_\Sigma \mathrm{d}\Sigma_\mu \, B^\mu$$

This does not imply that the integrands are identical

We must retain the possibility

$$A^\mu = B^\mu + \partial_\rho \mathcal{I}^{[\rho\mu]}$$

Unlike the previous case, we do not have the freedom to discard the surface term $\partial_\rho \mathcal{I}^{[\rho\mu]}$!

Application: derive local thermodynamic relations from global ones while retaining these terms.

From the global potential to a local current

Introduce the scaled equilibrium state

$$\hat{\rho}(\lambda) = \frac{e^{-\lambda\hat{\Upsilon}}}{Z(\lambda)}, \qquad O(\lambda) = \mathrm{Tr}\left[\hat{\rho}(\lambda)\hat{O}\right]$$

The global potential can be written as

$$\Phi = \int_\Sigma \mathrm{d}\Sigma_\mu \int_1^\infty \mathrm{d}\lambda \, \left[T^{\mu\nu}(\lambda)\beta_\nu - \alpha J^\mu(\lambda)\right]$$

Comparing with $\Phi = \int_\Sigma \mathrm{d}\Sigma_\mu \, \phi^\mu$ gives

The potential current, including its ambiguity

$$\phi^\mu = \int_1^\infty \mathrm{d}\lambda \, \left[T^{\mu\nu}(\lambda)\beta_\nu - \alpha J^\mu(\lambda)\right] + \partial_\lambda \mathcal{G}_\Phi^{[\lambda\mu]}$$

Setting $\partial_\lambda \mathcal{G}_\Phi^{[\lambda\mu]} = 0$ is a particular choice.

From global to local Gibbs–Duhem relations

For a fixed domain Σ,

$$\mathbf{d}\Phi|_D = \int_\Sigma \mathrm{d}\Sigma_\mu \, [J^\mu \, \mathbf{d}\alpha - T^{\mu\nu} \, \mathbf{d}\beta_\nu]$$

But $\Phi = \int_\Sigma \mathrm{d}\Sigma_\mu \, \phi^\mu$ also gives

$$\mathbf{d}\Phi|_D = \int_\Sigma \mathrm{d}\Sigma_\mu \left[\left.\frac{\partial \phi^\mu}{\partial \alpha}\right|_{\beta_\nu} \mathbf{d}\alpha + \left.\frac{\partial \phi^\mu}{\partial \beta_\nu}\right|_{\alpha} \mathbf{d}\beta_\nu \right]$$

Retain the surface terms when comparing integrands

$$\left.\frac{\partial \phi^\mu}{\partial \alpha}\right|_{\beta_\nu} = J^\mu + \partial_\rho \mathcal{I}^{[\rho\mu]}_\alpha = J^\mu + \mathcal{I}^\mu_\alpha,$$

$$\left.\frac{\partial \phi^\mu}{\partial \beta_\nu}\right|_{\alpha} = -T^{\mu\nu} + \partial_\rho \mathcal{I}^{[\rho\mu]\nu}_\beta = -T^{\mu\nu} + \mathcal{I}^{\mu\nu}_\beta$$

$\mathbf{d}$ varies the equilibrium state; it is distinct from the spacetime differential d.

The two prescriptions with surface terms retained

(1) From the global potential [Becattini and Rindori, 2019]

$$\phi^\mu = \int_\beta^\infty \mathrm{d}\beta' \; (T^{\mu\nu}u_\nu - \mu J^\mu)\big|_{\mu,u} + \partial_\lambda \mathcal{G}_\Phi^{[\lambda\mu]}$$

(2) From the Gibbs–Duhem relation [Ambrus and Gecić, 2026]

$$\left.\frac{\partial\phi^\mu}{\partial\beta}\right|_{\alpha,u} = -T^{\mu\nu}u_\nu + \mathcal{I}_\beta^{\mu\nu}u_\nu$$

and hence

$$\phi^\mu = \int_\beta^\infty \mathrm{d}\beta' \; \left(T^{\mu\nu}u_\nu - \mathcal{I}_\beta^{\mu\nu}u_\nu\right)\Big|_{\alpha,u},$$

The surface terms must be treated consistently in both prescriptions.

Consistency when starting from prescription (1)

Choose $\partial_\lambda \mathcal{G}_\Phi^{[\lambda\mu]} = 0$. Under our current PGT,

$$\phi'^{\mu}_{(1)} = \phi^{\mu}_{(1)} + C^\mu, \qquad T'^{\mu\nu} = T^{\mu\nu}.$$

Although $T^{\mu\nu}$ is unchanged, in general

$$\left.\frac{\partial C^\mu}{\partial \beta_\nu}\right|_\alpha \neq 0$$

The surface term in the energy relation needs to be chosen as

$$\mathcal{I}'^{\mu\nu}_\beta = \left.\frac{\partial C^\mu}{\partial \beta_\nu}\right|_\alpha$$

$$\Longrightarrow \quad \phi'^{\mu}_{(2)} = \int_\beta^\infty \mathrm{d}\beta' \left(T^{\mu\nu}u_\nu - \mathcal{I}'^{\mu\nu}_\beta u_\nu\right)_{\alpha,u} = \phi^\mu_{(2)} + C^\mu = \phi'^{\mu}_{(1)}$$

Consistency when starting from prescription (2)

Choose $\mathcal{I}^{\mu\nu}_{\beta} = 0$. Since $T^{\mu\nu}$ is unchanged,

$$\phi'^{\mu}_{(2)} = \phi^{\mu}_{(2)} = \phi^{\mu},$$

but the charge current acquires a surface term:

$$\left.\frac{\partial \phi'^{\mu}_{(2)}}{\partial \alpha}\right|_{\beta_\nu} = \left.\frac{\partial \phi^{\mu}}{\partial \alpha}\right|_{\beta_\nu} = J^{\mu} = J'^{\mu} + \mathcal{I}'^{\mu}_{\alpha}$$

$$\Longrightarrow \quad \mathcal{I}'^{\mu}_{\alpha} = J^{\mu} - J'^{\mu} = -\partial_\lambda B^{[\lambda\mu]}$$

The surface term in prescription (1) needs to be chosen as

$$\partial_\lambda \mathcal{G}^{[\lambda\mu]}_{\Phi} = -\partial_\lambda C^{[\lambda\mu]} = -C^{\mu}$$

$$\Longrightarrow \quad \boxed{\phi'^{\mu}_{(1)} = \phi^{\mu} + C^{\mu} - C^{\mu} = \phi'^{\mu}_{(2)}}$$

Implication for spin hydrodynamics

The same considerations apply when spin is included.
In the Belinfante pseudo-gauge,

$$S_B^{\alpha\beta} = 0,$$

yet the pressure can depend on thermal vorticity:

$$\left.\frac{\partial P_B}{\partial \Omega_{\alpha\beta}}\right|_{\alpha,\beta,u} \neq 0$$

Thus, the naive identification

$$\left.\frac{\partial P_B}{\partial \Omega_{\alpha\beta}}\right|_{\alpha,\beta,u} \propto S_B^{\alpha\beta}$$

appears inconsistent.

The missing contribution

The PGT induces a change in P_B that can be written as a surface term.
Its contribution must be retained in the local thermodynamic relation.

Summary

- Thermodynamic relations for global quantities are invariant under boundary-compatible PGTs.
- Local thermodynamic relations may require surface terms when derived from global identities.
- Once these terms are retained, the two prescriptions for the thermodynamic-potential current can be reconciled.
- Local thermodynamic conjugates need not coincide with the hydrodynamic currents: their difference can be a surface term.
- Maxwell relations apply to the corrected conjugates. Dropping the corrections can produce apparent violations.

Main conclusion

Apparent violations of naive local thermodynamic relations do not, by themselves, select a preferred pseudo-gauge.

Backup

Corrected Maxwell relations

Define the local thermodynamic conjugates

$$\mathcal{K}^{\mu}_{\alpha} := \left.\frac{\partial \phi^{\mu}}{\partial \alpha}\right|_{\beta_\nu} = J^{\mu} + \mathcal{I}^{\mu}_{\alpha}, \qquad \mathcal{K}^{\mu\nu}_{\beta} := \left.\frac{\partial \phi^{\mu}}{\partial \beta_\nu}\right|_{\alpha} = -T^{\mu\nu} + \mathcal{I}^{\mu\nu}_{\beta}.$$

Smoothness of ϕ^{μ} implies

$$\boxed{\left.\frac{\partial (J^{\mu} + \mathcal{I}^{\mu}_{\alpha})}{\partial \beta_\nu}\right|_{\alpha} = \left.\frac{\partial (-T^{\mu\nu} + \mathcal{I}^{\mu\nu}_{\beta})}{\partial \alpha}\right|_{\beta_\rho}.}$$

With spin, include $\Omega_{\rho\sigma}$ among the sources.

Belinfante implication

A vanishing spin tensor does not require a vanishing thermodynamic conjugate to $\Omega_{\rho\sigma}$: the surface correction can remain nonzero.