

Four-loop non-singlet splitting functions in QCD

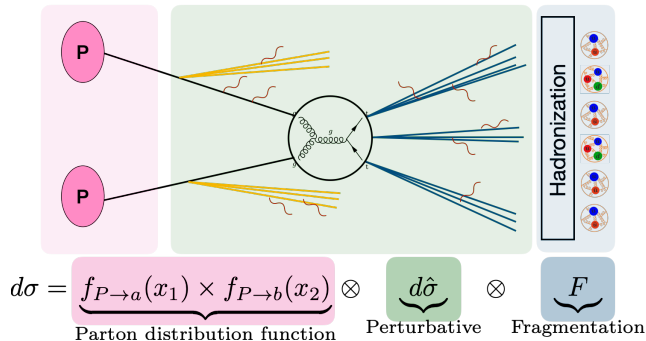
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Based on 2604.09534

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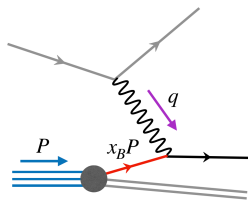


QCD factorization at hadron colliders



- Non-perturbative (n)PDFs: global fitting or LQCD; FFs: global fitting **Universal**
- DGLAP evolution of (n)PDFs, governed by the splitting functions P_{ik} **Universal**
- $\hat{\sigma}$ in perturbation theory: $\hat{\sigma} = \hat{\sigma}^{(0)} + \alpha_s \hat{\sigma}^{(1)} + \alpha_s^2 \hat{\sigma}^{(2)} + \alpha_s^3 \hat{\sigma}^{(3)} + \dots$ **Process-dependent**
 LO NLO NNLO(10%) N3LO(1%)

Deep inelastic scattering: the cleanest window on the proton



- Virtuality and Bjorken variable

$$Q^2 = -q^2, \quad x_B = \frac{-q^2}{2P \cdot q}, \quad \nu = \frac{P \cdot q}{M}$$

- Inclusive cross section in terms of structure functions

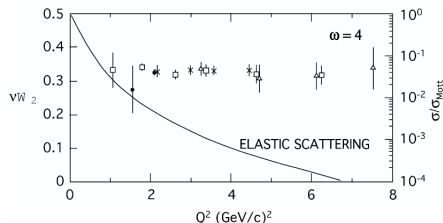
$$\frac{d^2\sigma}{dx_B dQ^2} \propto \left[\dots F_2(x_B, Q^2) + \dots F_L, F_3 \right]$$

- Q sets the **resolution**: $\lambda \sim 1/Q$

What happens to F_2 when we increase the resolution Q at fixed x_B ?

Bjorken scaling (1969): a surprisingly simple result

- SLAC-MIT data: at fixed $x = Q^2/(2M\nu)$, the structure function stops changing with Q^2



- Bjorken: scaling hypothesis from current-algebra sum rules [Bjorken,69]

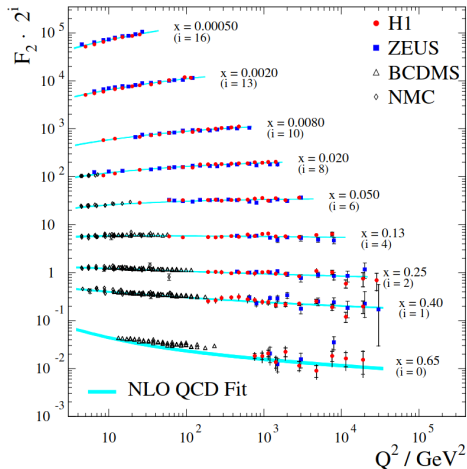
$$\nu W_2(\nu, Q^2) \xrightarrow[Q^2 \rightarrow \infty, x \text{ fixed}]{} F_2(x)$$

- Feynman's **parton model**: the proton were made of free, point-like **partons**

$$F_2(x) = \sum_i e_i^2 x f_i(x)$$

...but scaling is not exact

- More precise data showed F_2 **does** depend on Q^2
- Pattern: **rises** with Q^2 at small x , **falls** with Q^2 at large x



Why: quarks radiate gluons

- At high resolution (Q^2 large), the proton looks busier: more low- x partons
- Parton densities $f_i(x, Q^2)$ change with resolution

$$\frac{df_i(x, Q^2)}{d \ln Q^2} = \sum_j \int_x^1 \frac{d\xi}{\xi} P_{ij}(\xi) f_j(x/\xi, Q^2)$$

- $P_{ij}(\xi)$: probability for parton j to "split" into parton i carrying momentum fraction ξ – **universal**, same kernel for every process
- **1972**: Gribov & Lipatov derive the same equation – before QCD even existed, in a simpler toy theory (QED and pseudoscalar Yukawa) [Gribov,Lipatov,72]
- **1977**: Dokshitzer and Altarelli–Parisi independently extend it to QCD [Dokshitzer,77;Altarelli,Parisi,77]
- Today called **DGLAP** (x -space): **probability** interpretation, implication for **factorization**

Another route to DGLAP

- Wilson's operator product expansion [Wilson, 69]

- $T J^\mu(z) J^\nu(0) \xrightarrow{z \rightarrow 0} \sum_j C_j^{\mu\nu}(z) \mathcal{O}_j(0)$
- Anomalous dimension governs the scale evolution of local operators
- Twist-two operators, contracted with a light-like Δ ($\Delta^2 = 0$), $O_i \equiv O_i^{\mu_1 \cdots \mu_n} \Delta_{\mu_1} \cdots \Delta_{\mu_n}$

$$O_{q,k} = \frac{i^{n-1}}{2} \bar{\psi} \not{\Delta} (\Delta \cdot D)^{n-1} \frac{\lambda_k}{2} \psi, \quad k = 3, 8, \dots, n_f^2 - 1$$

$$O_q = \frac{i^{n-1}}{2} \bar{\psi} \not{\Delta} (\Delta \cdot D)^{n-1} \psi, \quad O_g = -\frac{i^{n-2}}{2} \Delta_\mu G_{a_1, \nu}^\mu (\Delta \cdot D)_{a_1 a_2}^{n-2} \Delta_\rho G_{a_2}^{\rho\nu}$$

- First explicit one-loop anomalous dimension $\gamma^{(0)}(n)$ in QCD [Gross, Wilczek, 73, 74; Georgi, Politzer, 74]
- $\gamma(n)$: a number per n , **hard to interpret**

- Two routes converge, they are related by a Mellin transform

$$\gamma(n) = - \int_0^1 dx x^{n-1} P(x)$$

Non-singlet & singlet splitting functions

- Non-singlet case

- ▶ Flavor non-singlet

$$\frac{dT_i^\pm}{d\ln\mu} = 2P_{\text{ns}}^\pm \otimes T_i^\pm, \quad i = 3, 8, \dots, n_f^2 - 1$$

$$T_3^\pm = u^\pm - d^\pm, T_8^\pm = u^\pm + d^\pm - 2s^\pm, \dots, \quad q^\pm = q \pm \bar{q}$$

- ▶ Valence non-singlet

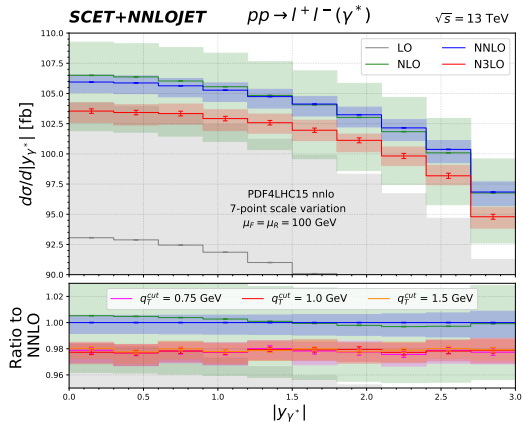
$$\frac{dq^V}{d\ln\mu} = 2P_{\text{ns}}^V \otimes q^V, \quad P_{\text{ns}}^V = P_{\text{ns}}^- + P_{\text{ns}}^s, \quad q^V = \sum_{k=1}^{n_f} q_k^-$$

- Singlet case

$$\frac{d}{d\ln\mu} \begin{pmatrix} \Sigma \\ g \end{pmatrix} = \begin{pmatrix} P_{qq} & P_{qg} \\ P_{gq} & P_{gg} \end{pmatrix} \otimes \begin{pmatrix} \Sigma \\ g \end{pmatrix}, \quad \Sigma = \sum_{k=1}^{n_f} q_k^+$$

Motivation of $P_{ij}^{(3)}$: required by high-precision physics

- (HL-)LHC will produce precise data
- Several $\hat{\sigma}$ are available at N3LO $\rightarrow$ N3LO PDF $\leftarrow P_{ij}^{(3)}$



- Drell-Yan $|y_{\gamma^*}|$ at N³LO using NNLO PDFs
- N³LO scale band lies **outside** the NNLO band
- Genuine N³LO ingredients, $P_{ij}^{(3)}$ included, are needed

[Chen, Gehrmann, Glover, Huss, Yang, Zhu, 22']

Why properly matched PDFs matter?

- Using an $N^3\text{LO}$ $\hat{\sigma}$ with NNLO PDFs is inconsistent – how large is the error?

$$\Delta_{\text{NNLO}}^{\text{exact}} = \left| \frac{\sigma_{\text{aN}^3\text{LO-PDF}}^{\text{N}^3\text{LO}} - \sigma_{\text{NNLO-PDF}}^{\text{N}^3\text{LO}}}{\sigma_{\text{aN}^3\text{LO-PDF}}^{\text{N}^3\text{LO}}} \right|, \Delta_{\text{NNLO}}^{\text{est}} = \frac{1}{2} \left| \frac{\sigma_{\text{NNLO-PDF}}^{\text{NNLO}} - \sigma_{\text{NLO-PDF}}^{\text{NNLO}}}{\sigma_{\text{NNLO-PDF}}^{\text{NNLO}}} \right|$$

- Two $\text{aN}^3\text{LO PDF}$ sets exist (MSHT20, NNPDF4.0) based on approximated $P_{ij}^{(3)}$

channel	NNLO	aN ³ LO	shift
$gg \rightarrow H$	46.5 pb	44.9 pb	-3.3%
H VBF	4.35 pb	4.45 pb	+2.3%
HV assoc.	~0.86 pb	~0.86 pb	< 1%

$\Delta_{\text{NNLO}}^{\text{est}} = 0.9\%$ vs. **3.3%** actual ($gg \rightarrow H$)

- Naive estimate from the NLO→NNLO step **badly underestimates** the true shift

Why do we need exact $P_{ij}^{(3)}$?

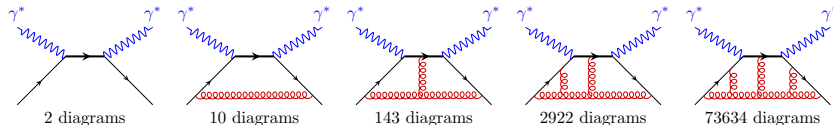
- Eliminate any remaining uncertainties, promote aN3LO PDF $\rightarrow$ N3LO PDF
- Small- x divergent terms are difficult to predict (from BFKL)
- Especially $P_{gq}^{(3)}$ has more **unknown small- x** divergent terms
 - ▶ NNPDF **assumes** a Casimir-scaling relation linking $P_{gq}^{(3)}$ to $P_{gg}^{(3)}$; MSHT does **not**
 - ▶ Result: $\sim 2\sigma$ disagreement in $P_{gq}^{(3)} \Rightarrow$ gluon PDF differs by **3–4%** for $10^{-3} \lesssim x \lesssim 10^{-1}$
 - ▶ Exact $P_{gq}^{(3)}$ is necessary to resolve the MSHT/NNPDF **tension**
- Theoretical interest:
 - ▶ To test the generalized Gribov-Lipatov reciprocity [Dokshitzer, Marchesini, Salam, 06'; Basso, Korchemsky, 06'; Mitov, Moch, Vogt, 06'; Chen, TZY, Zhu, Zhu, 20']
$$2\gamma^S(n, \mu) = 2\gamma^T(n + 2\gamma^S(n, \mu), \mu)$$
 - ▶ To study new analytic structures

Status of four-loop splitting functions

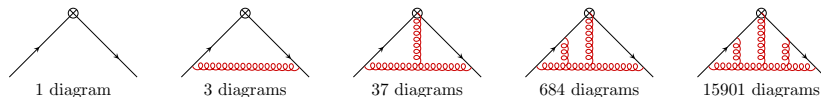
- One loop[Gross and Wilczek, 1974]
- Two loop[Furmanski, Petronzio, 1980]
- Three loop[Moch, Vogt, Vermaseren, 2004]
- Fixed moments (self-energy integrals with high numerator power, reduction by FORCER)
 - ▶ Non-singlet $\gamma_{ns}^{(3)}$ with $n \leq 16$ [Moch,Ruijl,Ueda,Vermaseren,Vogt,17']
 - ▶ Pure-singlet $\gamma_{ps}^{(3)}$, $\gamma_{qg}^{(3)}$, $\gamma_{gq}^{(3)}$ and $\gamma_{gg}^{(3)}$ with $n \leq 20$ [Falcioni,Herzog,Moch,Pelloni,A. Vogt,23', 24',25']
- Exact results with all- n dependence
 - ▶ $\gamma_{ns}^{(3)}$ with leading color[Moch,Ruijl,Ueda,Vermaseren,Vogt,17']
 - ▶ N_f^2 term for $\gamma_{qg}^{(3)}$ [Falcioni,Moch,Ruijl,Ueda,Vermaseren,Vogt,23']
 - ▶ N_f^2 for $\gamma_{ps}^{(3)}$, $N_f C_F^3$ for $\gamma_{ns}^{(3)}$ [Gehrmann,Manteuffel,Sotnikov,TZY,24']
 - ▶ N_f for $\gamma_{ns}^{(3)\pm}$ [Kniehl, Moch, Velizhanin, Vogt 25']
 - ▶ **This talk**: complete, analytic results for $\gamma_{ns}^{(3)}$

DIS method vs OME method

- Forward DIS (gauge invariant)



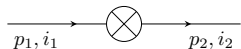
- Partonic off-shell OME (fewer diagrams, easier integrals)



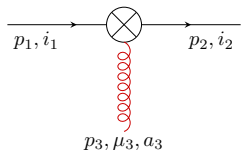
- Off-shell OMEs are **not gauge invariant**, physical operators **mix** with gauge-variant (GV) operators
- Non-singlet: problematic contribution cancel in $q - \bar{q}$
- Singlet: conceptually involved, need to find **new, unknown GV contri.**

Feynman rules for the operator vertices

- Functional variation of O_q produces vertices with an arbitrary number of gluons



$$\longrightarrow \frac{1}{2} \not{\Delta} \delta_{i_1 i_2} (\Delta \cdot p_1)^{n-1}$$



$$\longrightarrow \frac{g_s}{2} \Delta^{\mu_3} T_{i_2 i_1}^{a_3} \not{\Delta} \sum_{j_1=0}^{n-2} (-\Delta \cdot p_2)^{j_1} (\Delta \cdot p_1)^{n-2-j_1}$$

2 gluons, 3 gluons, $\dots$

- The red factors are non-standard: they carry the spin n and are not ordinary vertices

Tracing parameter to retain all- n dependence

- Non-standard terms appearing in the Feynman rules
- Two ways out: set n to fixed numbers (fixed moments), or keep n symbolic (all- n)
- Retain all- n dependence?
 - ▶ Sum non-standard terms into linear propagators by a tracing parameter t [Ablinger, Bluemlein, Hasselhuhn, Klein, Schneiderm, Wissbrock, 12'],

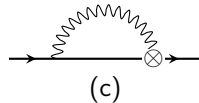
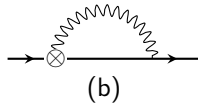
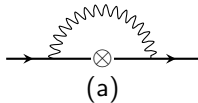
$$(\Delta \cdot p)^{n-1} \rightarrow \sum_{n=1}^{\infty} t^n (\Delta \cdot p)^{n-1} = \frac{t}{1 - t \Delta \cdot p}, \quad \sum_{j=0}^{n-2} (\Delta \cdot p_1)^{n-2-j} (\Delta \cdot p_2)^j \rightarrow \dots$$

- ▶ IBP reduction with respect to linear propagator
- ▶ Parameter- t space $\rightarrow$ n -space by re-expanding t to higher powers, for example

$$\frac{1}{1-t} \longrightarrow 1, \quad \log(1-t) \longrightarrow -\frac{1}{n}, \quad \frac{\log(1-t)}{1-t} \longrightarrow -S_1(n) = -\sum_{j=1}^n \frac{1}{j}$$

A worked example: the one-loop non-singlet OME

- Three diagrams contribute to $\langle q(p) | O_q | q(p) \rangle^{(1)}$



- The bare result from t -space calculation (IBP and differential equation)

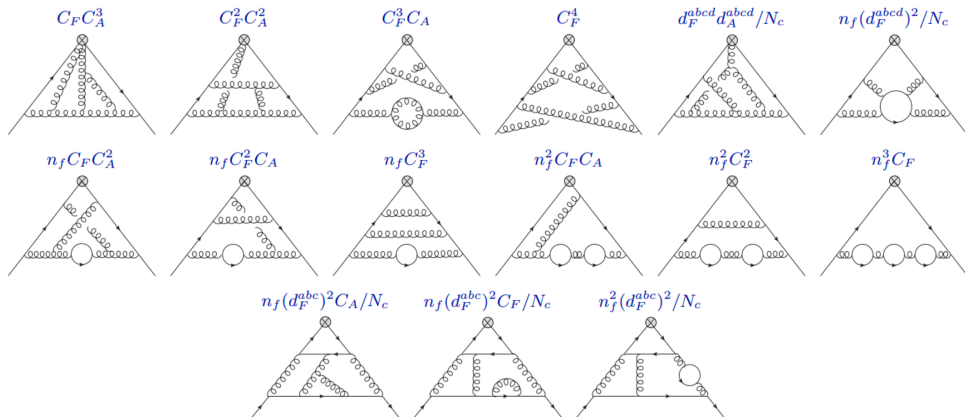
$$\langle q(p) | O_q | q(p) \rangle^{(1)} \Big|_{\text{pole}} = -\frac{C_F}{\epsilon} \left[c_0(\xi) + c_1(\xi) \frac{1}{1-t} + \left(c_2 + c_3 \frac{1}{1-t} \right) \log(1-t) \right]$$

- Wave-function and operator renormalization give one-loop result

$$\gamma_{qq}^{(0)}(n) = C_F \left[4S_1(n) - 3 - \frac{2}{n(n+1)} \right], P_{qq}^{(0)}(x) = 2 C_F \left[\frac{1+x^2}{(1-x)_+} + \frac{3}{2} \delta(1-x) \right]$$

The machinery will be pushed to **four** loops

Sample four-loop Feynman diagrams



$\mathcal{O}(16k)$ diagrams at four-loop order

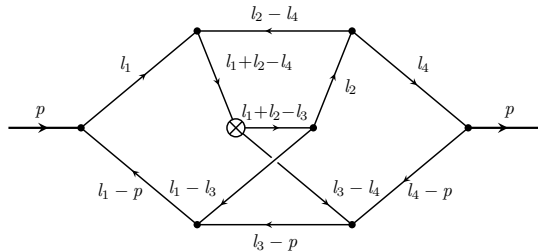
Technical aspects

- Raw integrand
 - ▶ Raw integrand generated by QGRAF,FORM
 - ▶ 549 top topologies with 13 denominators
 - ▶ 3GB integrand with 3M integrals
- IBP reduction ($\mathcal{O}(6000)$ masters)[Chetyrkin, Tkachov 81', Laporta 00']
 - ▶ **Finite field sampling** + function reconstruction[Manteuffel, Schabinger 14'; Peraro 15']
 - ▶ **Custom optimization** of IBP equation generation/selection[Driesse, Jakobsen, Mogul, Plefka, Sauer, Usovitsch 24'; Guan, Liu, Ma, Wu 24'; Bern, Herrmann, Roiban, Ruf, Smirnov, Smirnov, Zeng 24'; von Hippel, Wilhelm 25'; Song, TZY, Cao, Luo, Zhu 25'; Lange, Usovitsch, Wu 25']
 - ▶ Divide big system into smaller systems by **spanning cuts/sectors**[Larsen, Zhang 15'; Guan, Liu, Ma, Wu,24']
- Differential equation
 - ▶ Derive DE[Kotikov 91'; Gehrmann,Remiddi,99'] with respect to t

Complexity of IBP reduction

- 11 quadratic propagators; in t -space the operator vertex supplies two linear ones

$$\frac{1}{1 - t \Delta \cdot (l_1 + l_2 - l_3)} \frac{1}{1 - t \Delta \cdot (l_1 + l_2 - l_4)} \implies 11 + 2 = 13 \text{ denominators}$$



2204

non-zero sectors

49

spanning cuts/sectors

2090

master integrals

28

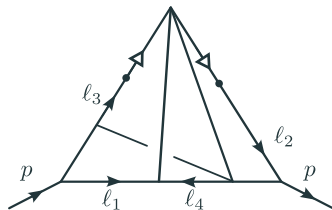
magic relations

- Spanning cuts/sectors split the IBP system into 49 independent subsystems

Elliptic geometry from DE

- Analyze the leading singularity on maximal cut
- Elliptic geometry, for example
 - A non-planar sector with 9 normal propagators and 2 linear propagators
 - 4 master integrals on maximal cut
 - ϵ -form achieved by ansatz method of [Bogner, Müller-Stach, Weinzierl, 19']
 - Complete elliptic integrals

$$F(t) = \int_0^1 d\tau \frac{1}{\sqrt{(1-\tau^2)(1-t^2\tau^2)}}$$



Hints from four-loop $\mathcal{N} = 4$ results: **only Harmonic Sums**
Bypass the complexities: series expansion & reconstruct the A.D.

Master solution from series expansion of DE

- DE with respect to the tracing parameter t :

$$d\vec{I}(t, \epsilon)/dt = A(t, \epsilon)\vec{I}(t, \epsilon)$$

- Largest DE from a top topology: $\mathcal{O}(2000)$ masters, $\mathcal{O}(700\text{MB})$ size
- Boundary condition: 28 four-loop self-energy master integrals in the $t \rightarrow 0$ limit [Baikov, Chetyrkin 10'; Lee, Smirnov, Smirnov 11']

▶ Example: $\int \frac{1}{(1-t\Delta \cdot l_1)(1-t\Delta \cdot l_2)} \times \frac{1}{l_1^2(l_1-l_2)^2\cdots} \rightarrow \int \frac{1}{l_1^2(l_1-l_2)^2\cdots}$

- Derive **recursion relations** for each master

$$I(t, \epsilon) = \sum_{m=m_{\min}}^{m_{\max}} \sum_{n=0}^{\infty} a_{m,n} \epsilon^m t^n, \quad a_{m,n} = (\#1)a_{m,n-1} + (\#2)a_{m-1,n} + \cdots$$

- Compute each master to $\mathcal{O}(t^{4000})$ and enough ϵ order

Reconstruct all- n A.D. from fixed moments

- Derive recursion relations for OME to obtain $\mathcal{O}(4000)$ moments
- Renormalization: $\langle q(p) | O_{\text{ns}} | q(p) \rangle^{\text{R}} = Z_\psi Z_{\text{ns}} \langle q(p) | O_{\text{ns}} | q(p) \rangle^{\text{B}}$
- $Z_{\text{ns}}^{(4)} = \frac{1}{4\epsilon} \gamma_{\text{ns}}^{(3)} + \frac{1}{\epsilon^2} \left(-\frac{1}{4} \beta_0 \gamma_{\text{ns}}^{(2)} + \dots \right) + \frac{1}{\epsilon^3} (\dots) + \frac{1}{\epsilon^4} (\dots)$
- Generic ansatz:

$$\gamma_{\text{ns}}^{(3)}(n) = \sum_a \sum_{k=0}^7 \sum_{w=0}^{7-k} c_{akw} D_a^k S_w(n), \quad D_a^k = (n+a)^{-k}$$

$S_w(n)$ is HSs of weight w

We choose: $a = \{0, 1\}$ for $\gamma_{\text{ns}}^{(3)\pm}$, $\{0, 1, -1, 2, -2\}$ for $\gamma_{\text{ns}}^{(3)V}$

- Successfully reconstructed $\gamma_{\text{ns}}^{(3)}$ with **all- n dependence** using $\mathcal{O}(4000)$ moments

Analytic results for $\gamma_{ns}^{(3)}$

- Analytic results with all- n dependence via HSs to weight 7

$$\text{Example: } S_{-6,1}(n) = \sum_{\tau_2=1}^n \frac{(-1)^{\tau_2}}{\tau_2^6} \sum_{\tau_1=1}^{\tau_2} \frac{1}{\tau_1}$$

- Non- n_f terms for $\gamma_{ns}^{(3)\pm}$, n_f terms for $\gamma_{ns}^{(3)s}$ **New**

```
((1 + (-1)^n)*color[ca^*cf])*(373793/648 + 2662/(27*n^4) - 50006/(81*n^3) +  
146482/(81*n^2) - 283147/(81*n) - 2662/(27*(1 + n)^4) +  
50006/(81*(1 + n)^3) - 146482/(81*(1 + n)^2) + 283147/(81*(1 + n)) -  
320*S[-7, n] + (-1600/3 - 832/(3*n) + 832/(3*(1 + n)))*S[-6, n] +  
(-2728/9 + 1000/(3*n^2) - 6920/(9*n) + 1000/(3*(1 + n)^2) +  
6920/(9*(1 + n)))*S[-5, n] + (2728/9 - 1000/(3*n^2) + 6920/(9*n) -  
1000/(3*(1 + n)^2) - 6920/(9*(1 + n)))*S[5, n] +  
(1600/3 + 832/(3*n) - 832/(3*(1 + n)))*S[6, n] + 320*S[7, n] +  
1216*S[-6, 1, n] - (4160*S[-5, -2, n])/3 +  
(1456/3 + 3808/(3*n) - 3808/(3*(1 + n)))*S[-5, 1, n] +  
(2048*S[-5, 2, n])/3 + 384*S[-4, -3, n] + (-368 - 1152/n + 1152/(1 + n))*  
S[-4, -2, n] + (-18320/9 + 6112/(3*n^2) - 1824/n + 6112/(3*(1 + n)^2) +
```

- Agree with partially known results

- n_f terms for $\gamma_{ns}^{(3)\pm}$ [Gehrmann, Manteuffel, Sotnikov, TZY, 24'; Kniehl, Moch, Velizhanin, Vogt 25']
- $n \leq 16$ for $\gamma_{ns}^{(3)}$ [Moch, Ruijl, Ueda, Vermaseren, Vogt, 17']

Analytic results for $P_{\text{ns}}^{(3)}$

- Splitting functions $P_{\text{ns}}^{(3)}$ by inverse Mellin transformation of $\gamma_{\text{ns}}^{(3)}$

$$P_{\text{ns}}^{(3)}(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} x^{-n} \gamma_{\text{ns}}^{(3)}(n) dn$$

► By HarmonicSums[Ablinger,09',12']

- $P_{\text{ns}}^{(3)}$ via HPLs to weight 6, constants to weight 7 ($\zeta_7, \zeta_3\zeta_4, \zeta_2\zeta_5$)
- Example expressions for $C_A^3 C_F$ of $P_{\text{ns}}^{(3)+}$

```
delta[1 - x]*(-2085/4 + 3220*zeta[3]^2 + 2167*zeta[4] +  
  zeta[3]*(-3260 + 128*zeta[4]) - 976*zeta[5] +  
  zeta[2]*(1167 - (1988*zeta[3])/3 + 2064*zeta[5]) + (79297*zeta[6])/18 -  
  10920*zeta[7])) + color[ca^3*cf]*(198869/81 - (283147*x)/81 +  
  (-1344 - 656/(3*(-1 + x)) - 144*x + 1456/(3*(1 + x)))*HPL[{5}, x] +  
  (1328 + 1024/(-1 + x) - 192*x - 1216/(1 + x))*HPL[{6}, x] +  
  (2656/3 + 4160/(3*(-1 + x)) + 560*x - 320/(3*(1 + x)))*HPL[{-5, 0}, x] +  
  (-2704/3 - 368/(-1 + x) - (592*x)/3 - 304/(1 + x))*HPL[{-4, 0}, x] +  
  (-4640/3 - 512/(3*(-1 + x)) + 2048*x + 7424/(3*(1 + x)))*HPL[{-4, 2}, x] +  
  (8080/3 + 17024/(9*(-1 + x)) - (14896*x)/9 + 32/(3*(1 + x)))*  
  HPL[{-3, 2}, x] + (-1456/3 + 7744/(3*(-1 + x)) + 2608*x +  
  6080/(3*(1 + x)))*HPL[{-3, 3}, x] +
```

Results in $x \rightarrow 0$ and $x \rightarrow 1$ limit

- $x \rightarrow 0$ limit

$$P_{\text{ns}}^{(3)+} = \frac{C_F^4}{9} \log^6 x + C_F^3 \left(\frac{22C_A}{9} - \frac{4n_f}{9} - \frac{4C_F}{3} \right) \log^5 x + \log^4 x \left(-48\zeta_2 \frac{d_R^{abcd} d_A^{abcd}}{N_R} + \dots \right)$$

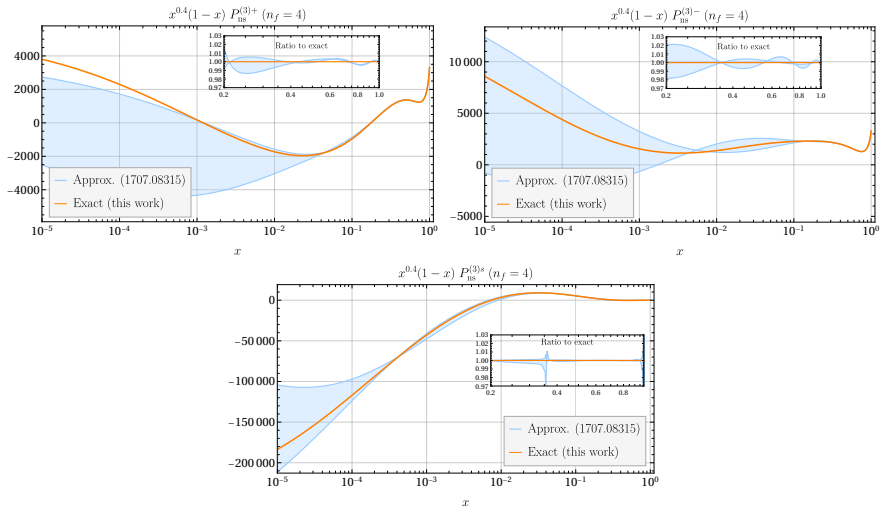
- ▶ Quartic color structure starts to contribute at $\log^4 x$, **deviation** at sub-LC with predictions of resummation [Davies, Kom, Moch, Vogt, 22']

- $x \rightarrow 1$ limit

$$P_{\text{ns}}^{(3)\pm} = A_4 \left[\frac{1}{1-x} \right]_+ + B_4 \delta(1-x) + C_4 \log(1-x) + D_4 - A_4 + \mathcal{O}(1-x),$$

- ▶ Confirm the four-loop cusp anomalous dimension A_4 [Henn, Korchemsky, Mistlberger 19'; Manteuffel, Panzer, Schabinger 20']
- ▶ **Analytic** virtual A.D. B_4 , agree with numerical res. [Das, Mach, Vogt 19']
- ▶ B_4 is the last missing piece to derive **analytic** rapidity A.D., agree well with numerical res. [Moult, Zhu, Zhu 21'; Duhr, Mistlberger, Vita 21']

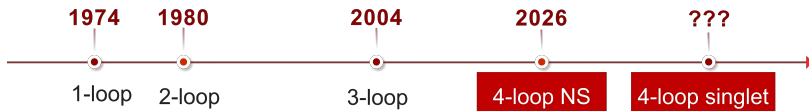
Comparison with approximations



Approximations from [Moch,Ruijl,Ueda,Vermaseren,Vogt,17'], good agreement except small- x region

Summary and Outlook

- Timeline of splitting functions in QCD



- Four-loop non-singlet $\gamma_{\text{ns}}^{(3)\pm, V}$ **Done**
 - $\mathcal{O}(10 \text{ TB})$ intermediate data $\rightarrow \mathcal{O}(100 \text{ KB})$ final analytic result: a reduction of 10^8
 - New** quartic colour structure $d_F^{abcd} d_A^{abcd} / N_c$ enters at a_s^4 , in **disagreement** with small- x resummation predictions [Davies, Kom, Moch, Vogt, 22']
- Four-loop singlet **In progress**
 - First step $\gamma_{qq} = \gamma_{\text{ns}}^{(3)+} + \gamma_{\text{ps}}^{(3)}$: **result done**, in preparation
 - $\gamma_{qg}^{(3)}$, $\gamma_{gq}^{(3)}$, $\gamma_{gg}^{(3)}$ in progress
- Our results will be used by MSHT/NNPDF/CTEQ in their next global fits

Thank you very much for your attention!

Extra material

One loop: IBP, differential equation, boundary condition

- In t -space the operator vertices are ordinary **linear propagators**

$$\frac{\not{\Delta}}{2} \delta_{i_1 i_2} \frac{t}{1 - t \Delta \cdot p_1}, \quad \frac{g_s \not{\Delta}}{2} \Delta^{\mu_3} T_{i_2 i_1}^{a_3} \frac{t^2}{(1 - t \Delta \cdot p_1)(1 - t \Delta \cdot p_2)}$$

- IBP reduction $\rightarrow$ only **two** master integrals are needed (all others finite)

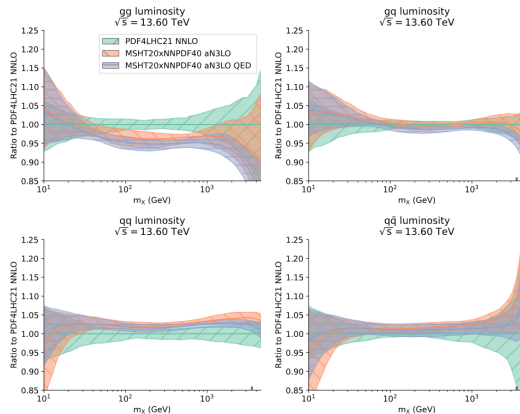
$$I_1 = \int \frac{d^d l}{i\pi^{d/2}} \frac{1}{(l-p)^2 l^2}, \quad I_2 = \int \frac{d^d l}{i\pi^{d/2}} \frac{1}{(l-p)^2 l^2 (1 - t \Delta \cdot l)}$$

- Differential equation with respect to the tracing parameter t

$$\frac{dI_1}{dt} = 0, \quad \frac{dI_2}{dt} = d_1(t, \epsilon) I_1 + d_2(t, \epsilon) I_2$$

- Boundary condition at $t \rightarrow 0$: the linear propagator collapses, $I_2|_{t=0} = I_1$
- Solution in terms of **harmonic polylogarithms** of t
- **This is the whole four-loop workflow in the non-singlet sector** – only the complexity increases

The aN3LO PDF effect is large



[Cridge, Harland-Lang, McGowan, Thorne, Ball, et al., 24']

- NNLO $\rightarrow$ aN³LO shift is **larger** than the PDF uncertainty in gg