Confidence in the Neutrino Mass Hierarchy

Jarah Evslin - TPCSF, IHEP, Chinese Academy of Sciences

νFACT

IHEP, August 21st, 2013

Based on work done in collaboration with:
Emilio Ciuffoli and Xinmin Zhang

χ^2 Analysis - Wilks' Theorem

Ref: Wilks, Ann. Math. Stat. 9, No. 1 (1938), 60-62

To test a null hypothesis that a continuous variable $x = x_0$:

(1) Measure the quantities y_i and calculate the statistic

$$\chi^2(x) = \sum_i \frac{\left(y_i^{(\mathrm{measured})} - y_i^{(\mathrm{theoretical})}(x)\right)^2}{\sigma_i^2}$$

(2) Define $\overline{x}$ to be the value of x that minimizes χ^2.

(3) Define (note that this is always positive by def of $\overline{x}$):

$$\Delta\chi^2 = \chi^2(x_0) - \chi^2(\overline{x})$$

Wilks' Theorem: $\Delta\chi^2$ follows a 1 DOF χ^2 distribution

Conclusion:

The hypothesis $x = x_0$ is excluded with confidence $\sqrt{\Delta\chi^2}\ \sigma$

Wilks' Theorem Does Not Apply to the Hierarchy

Ref: Qian et al., Phys.Rev. D86 (2012) 113011 (see also Wei Wang's talk)

The neutrino mass hierarchy is *not* a continuous variable, it is a discrete variable.

In the case of the hierarchy one instead defines the statistic

$$\Delta\chi^2 = \chi^2_{(\mathrm{inv})} - \chi^2_{(\mathrm{nor})}$$

where $\chi^2_{(\mathrm{inv})}$ and $\chi^2_{(\mathrm{nor})}$ are the χ^2 statistics obtained by fitting with respect to each hierarchy, with nuisance parameters chosen separately to minimize each χ^2.

This is not the quantity described in Wilks' theorem, it is not even necessarily positive.

Therefore $\Delta\chi^2$ does not satisfy a χ^2 distribution.

χ^2 Analysis of the Hierarchy

Ref: Qian et al., Phys.Rev. D86 (2012) 113011; Ciuffoli et al., arXiv:1305.5150

Under fairly general conditions, which are satisfied by JUNO and RENO 50 for example, $\Delta\chi^2$ follows a Gaussian distribution centered at $\overline{\Delta\chi^2}$ with $\sigma = 2\sqrt{|\overline{\Delta\chi^2}|}$.

In such cases, $\overline{\Delta\chi^2}$ for the inverted hierarchy is equal to $-\overline{\Delta\chi^2}$ for the normal hierarchy to within about 10%.

What is the sensitivity to the hierarchy given the measured $\Delta\chi^2$ and the calculated $\overline{\Delta\chi^2}$?

This question can be interpreted in different ways, which has caused disagreements in the literature.

Frequentist Approach

In particle physics, frequentist statistics are the most popular, in which one defines the consistency of the data with respect to a hypothesis.

In a standard frequentist approach one makes no assumptions concerning which is the true model.

On the other hand, an optimal determination of the hierarchy uses the assumption that precisely one of the hierarchies is correct.

In this sense a frequentist analysis is suboptimal for determining the hierarchy.

The relevant question is not whether a particular hierarchy is consistent with the data, but rather *which* hierarchy is *most* consistent with the data.

I will therefore not discuss this approach.

Bayesian Approach

The question of interest in a hierarchy determination is:
Given that precisely one hierarchy is true, which hierarchy is preferred by the data?

The assumption that one hierarchy is true is easily implemented in Bayesian statistics, where one assigns a prior to each hierarchy.

Consider for example the case of a symmetric prior, in which each hierarchy has a probability of 50% of being realized.

A given experiment determines the hierarchy successfully if the true hierarchy yields a lower value of χ^2.

Probability of Success

If the true hierarchy is normal, then the experiment successfully determines the hierarchy if and only if $\Delta\chi^2 > 0$.

Recall that $\Delta\chi^2$ follows a Gaussian distribution centered at $\overline{\Delta\chi^2}$ with $\sigma = 2\sqrt{|\overline{\Delta\chi^2}|}$.

Therefore, if the true hierarchy is normal, then the probability of successfully determining the hierarchy is

$$p_s = \frac{1}{2}\left(1 + \mathrm{erf}\left(\sqrt{\frac{|\overline{\Delta\chi^2}|}{8}}\right)\right)$$

If the true hierarchy is inverted, then $\overline{\Delta\chi^2}$ is negative and the true hierarchy is determined if $\Delta\chi^2 < 0$.

In this case the probability of success is still p_s.

Confidence in the Median Experiment

Ref: Ciuffoli et al., arXiv:1305.5150

After an experiment has been completed, the experimenter calculates $\Delta\chi^2$.

The knowledge of $\Delta\chi^2$ affects his confidence in a determination of the hierarchy.

In this case the probability that the hierarchy is determined correctly is $1/(1+e^{-\Delta\chi^2/2})$ (See Wei Wang's talk)

For example, in the median experiment the experimenter will find $|\Delta\chi^2| = \overline{\Delta\chi^2}$ and so the median probability of success is

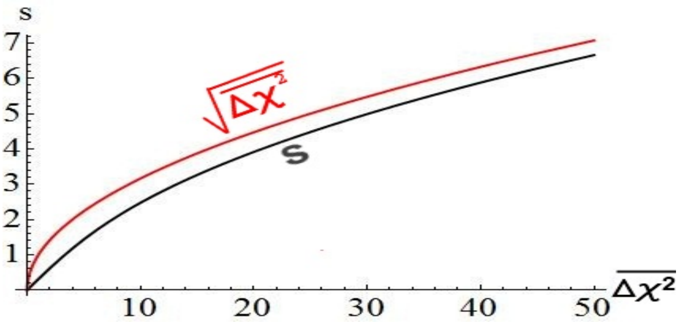
$$p(\overline{\Delta\chi^2}) = 1/(1+e^{-\overline{\Delta\chi^2}/2}).$$

One can then define the number s of σ's of confidence in the experiment as

$$p = \frac{1}{2}\left(1+\mathrm{erf}\left(\frac{s}{\sqrt{2}}\right)\right), \qquad s = \sqrt{2}\,\mathrm{erf}^{-1}\left(\frac{1-e^{-\overline{\Delta\chi^2}/2}}{1+e^{-\overline{\Delta\chi^2}/2}}\right)$$

Bayesian Analysis vs Square Root Rule

Ref: Ciuffoli et al., arXiv:1305.5150



A naive application of Wilks' theorem (red) overestimates the confidence in the median experiment (black) by about 0.5σ.

Using this curve and the value of $\overline{\Delta\chi^2}$ from GLoBES or theoretical spectra one can determine the number s of σ of confidence.

Probability of a Discovery

Ref: Ciuffoli et al., arXiv:1305.5150

Summarizing, the median experiment yields $s\sigma$ of confidence where s is

$$s = \sqrt{2}\,\mathrm{erf}^{-1}\left(\frac{1 - e^{-\overline{\Delta\chi^2}/2}}{1 + e^{-\overline{\Delta\chi^2}/2}}\right).$$

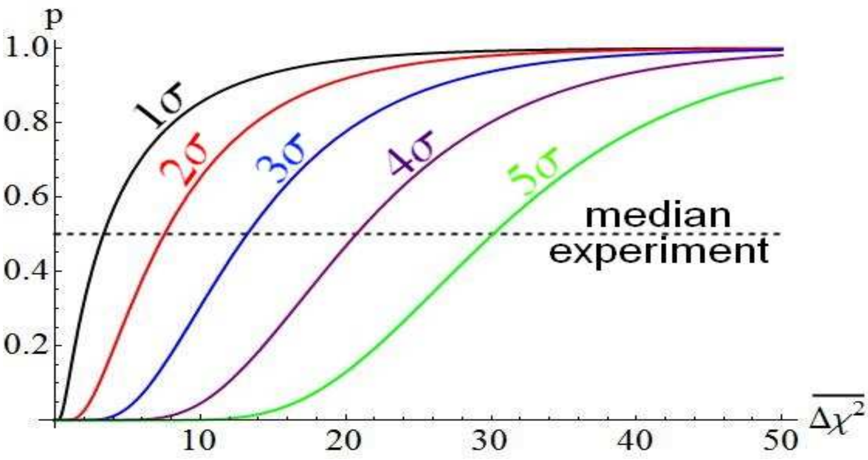
So given a calculated $\overline{\Delta\chi}^2$, what is the probability of achieving $s\sigma$ of confidence?

$$p(s) = \frac{1}{2}\left(1 + \mathrm{erf}\left(\frac{\overline{\Delta\chi^2} - \mathrm{arctanh}\left(\mathrm{erf}\left(\frac{s}{\sqrt{2}}\right)\right)}{\sqrt{8\overline{\Delta\chi^2}}}\right)\right).$$

The case $s = 5$, applied to δ, is a similar quantity to the probability of a discovery determined from simulations in Mattias Blennow's talk this morning.

Probability of a Discovery Plot

Ref: Ciuffoli et al., arXiv:1305.5150



Example: Reactor Neutrino Experiments

A nuclear reactor emits $\overline{\nu}_e$ which then oscillate.

Measuring these neutrinos, for example with inverse β decay, one can determine the electron neutrino survival probability as a function of energy E.

This survival probability depends upon the neutrino mass hierarchy and so a reactor neutrino experiment can in principle determine the mass hierarchy (Petcov and Piai, 2002).

The reactor neutrino spectrum at baselines shorter than 25 km is essentially independent of the neutrino mass hierarchy (Petcov and Piai, 2002; Choubey et al, 2003)

The observed reactor neutrino spectrum at a medium baseline manifests 1-2 oscillations on large scales and a fine structure with amplitude $\sin^2(2\theta_{13})$ of 1-3 oscillations, perturbed by 2-3 oscillations.

3 Flavor Oscillations

$$
\begin{aligned}
P_{ee} &= |\langle \nu_e | \exp\left(i \frac{\mathbf{M}^2 L}{2E} \right) | \nu_e \rangle|^2 \\
&= \sin^4(\theta_{13}) + \cos^4(\theta_{12})\cos^4(\theta_{13}) + \sin^4(\theta_{12})\cos^4(\theta_{13}) \\
&\quad + \frac{1}{2}(P_{12} + P_{13} + P_{23}) \\
P_{12} &= \sin^2(2\theta_{12})\cos^4(\theta_{13}) \cos\left(\frac{\Delta M^2_{21} L}{2E} \right) \\
P_{13} &= \cos^2(\theta_{12})\sin^2(2\theta_{13}) \cos\left(\frac{|\Delta M^2_{31}| L}{2E} \right) \\
P_{23} &= \sin^2(\theta_{12})\sin^2(2\theta_{13}) \cos\left(\frac{|\Delta M^2_{32}| L}{2E} \right)
\end{aligned}
$$

So the fine structure consists of 1-3 oscillations P_{13} perturbed by 2-3 oscillations P_{23} which have a slightly different wavenumber, leading to beats at the 1-2 wavenumber.

Reactor electron antineutrino spectrum with oscillations

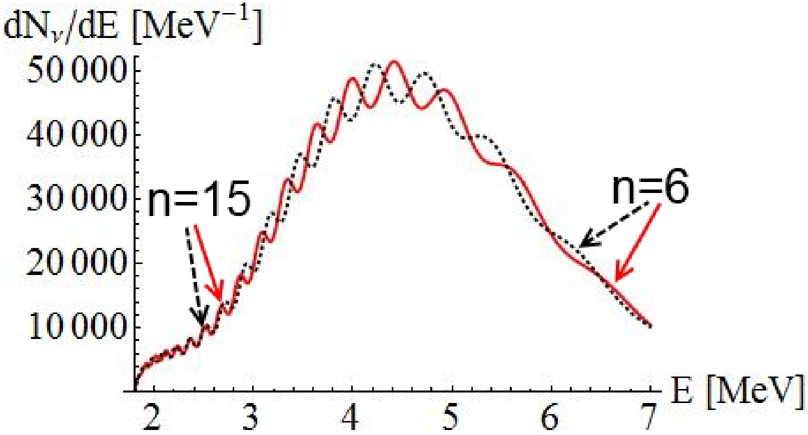
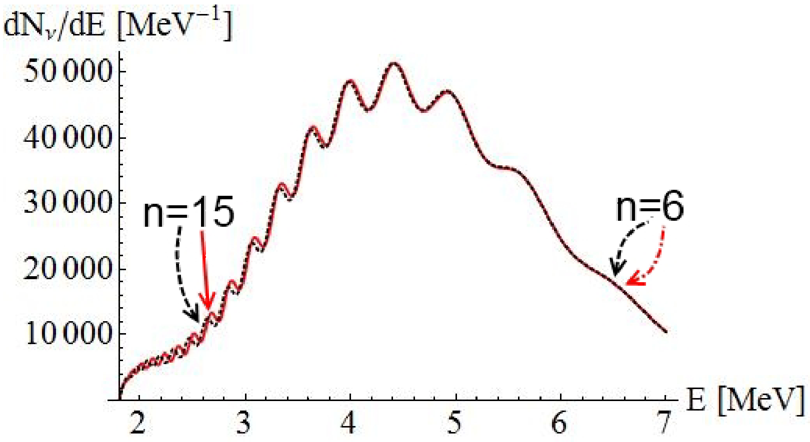


Figure: Theoretical Neutrino Spectrum from 17.4 GW of reactors observed in 24 years at a 5 kton target which is 10% hydrogen, including 3 flavor oscillation, for the normal (black dotted curve) and inverted (red curve) hierarchies as seen at 40 km, fixing $|\Delta M^2_{32}|$.

Now Add a 4% Relative Shift to $|\Delta M^2_{32}|$.



As above, but fixing $\Delta M^2_{\mathrm{eff}} := \cos^2(\theta_{12})|\Delta M^2_{31}| + \sin^2(\theta_{12})|\Delta M^2_{32}|$ in both hierarchies.

The first 10 peaks alone cannot be used to determine the hierarchy, but the next 5 can.

Hierarchy from Peak Positions

Ref: Ciuffoli et al., JHEP 1303 (2013) 016 arXiv:1208.1991

The energy E_n of the nth peak satisfies

$$\frac{L}{E_n} = \frac{4\pi\hbar}{\Delta M^2_{(n)} c^3} \tag{1}$$

where we define the effective mass measured by the nth peak

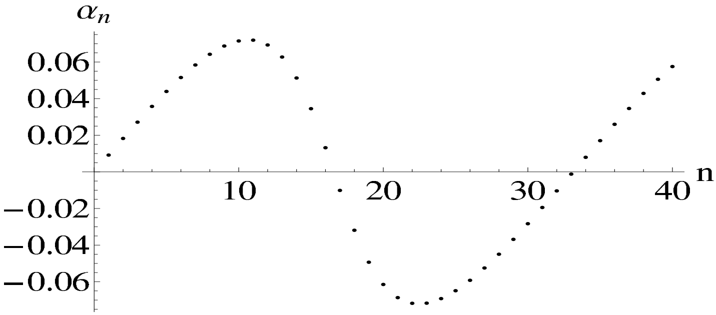
$$\Delta M^2_{(n)} = \frac{|\Delta M^2_{31}|}{1 \pm \alpha_n/n} \tag{2}$$

The $+$ (-) corresponds to the normal (inverted) hierarchy.

Strategy: You calculate α_n and measure *at least two* E_n.
Then use these Eqs. (1,2) to determine $|\Delta M^2_{31}|$ and the hierarchy.

The Function α_n

Ref: Ciuffoli et al., JHEP 1303 (2013) 016 arXiv:1208.1991



At $n << 10$, α_n/n is about 0.01, whereas $\alpha_{16} \sim 0$.

Therefore, fixing $\Delta M^2_{\rm eff}$ using a high energy peak, the energy E_{16} of the 16th peak will be 2% higher in the case of the normal hierarchy than in that of the inverted hierarchy.

Summary of how to determine the hierarchy

The mid and high energy part of the spectrum determines

$$\Delta M^2_{\text{eff}} = \cos^2(\theta_{12})|\Delta M^2_{31}| + \sin^2(\theta_{12})|\Delta M^2_{32}|$$

The low energy part determines other combinations, for example the 16th peak determines $|\Delta M^2_{31}|$

Subtracting these:

$$\Delta M^2_{\text{eff}} - |\Delta M^2_{31}| = \sin^2(\theta_{12})(|\Delta M^2_{32}| - |\Delta M^2_{31}|)$$

which is positive (negative) if the hierarchy is inverted (normal).

Note that the difference between the hierarchies is $2\sin^2(\theta_{12})\Delta M^2_{21}$ which is only 2% of $|\Delta M^2_{31}|$ - the energy must be determined very precisely.

Conclusion: The high and low energy peaks are both necessary

Combining Reactor + Accelerator Disappearance Channels

Ref: Nunokawa et al., Phys. Rev. D 72, 013009 (2005)

The ν_μ disappearance channel at long baseline oscillation experiments determines the atmospheric mass difference

$$\Delta M^2_{\mathrm{atm}} = |\Delta M^2_{31}| \mp (\cos^2(\theta_{12}) - \cos(\delta)\sin(\theta_{13})\sin(2\theta_{12})\tan(\theta_{23}))\Delta M^2_{21}$$

The - (+) sign applies to the normal (inverted) hierarchy, in which case it is lower (higher) than the reactor neutrino mass differences.

Thus to determine the hierarchy it suffices to compare the atmospheric and reactor mass effective mass differences.

Quantitative Comparison

Minakata et al., Phys.Rev. D74 (2006) 053008, Ciuffoli et al. arXiv:1302.0624

For example, at medium and high energies recall that a medium baseline reactor experiment determines

$$\Delta M^2_{\text{eff}} = \cos^2(\theta_{12})|\Delta M^2_{31}| + \sin^2(\theta_{12})|\Delta M^2_{32}|$$

while at low energies, the 16th peak determines $|\Delta M^2_{31}|$.

Comparing these with the atmospheric difference

$$\Delta M^2_{\text{eff}} - \Delta M^2_{\text{atm}} = \pm(2\cos(2\theta_{12}) - \cos(\delta)\sin(\theta_{13})\sin(2\theta_{12})\tan(\theta_{23}))\Delta M^2_{21}$$

$$|\Delta M^2_{31}| - \Delta M^2_{\text{atm}} = \pm(\cos^2(\theta_{12}) - \cos(\delta)\sin(\theta_{13})\sin(2\theta_{12})\tan(\theta_{23}))\Delta M^2_{21}$$

The $\cos(\delta)$ term is always subdominant.

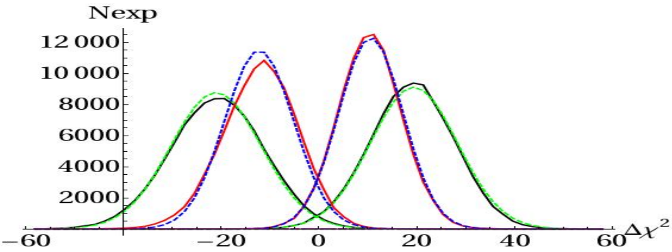
The signs of these differences provide two additional indicators of the hierarchy.

The greater $\cos(\delta)$, the smaller the mass difference and so the weaker the hierarchy signal.

Reactor and Accelerator when $\cos(\delta) = 0$

Ref: Ciuffoli et al., arXiv:1305.5150

Combining MINOS' 4% (upgraded NOνA's 1%) determination of $\Delta M^2_{\rm atm}$ with 6 years of JUNO, out of 50,000 simulations per hierarchy we obtained the purple (green) distribution of $\Delta\chi^2$

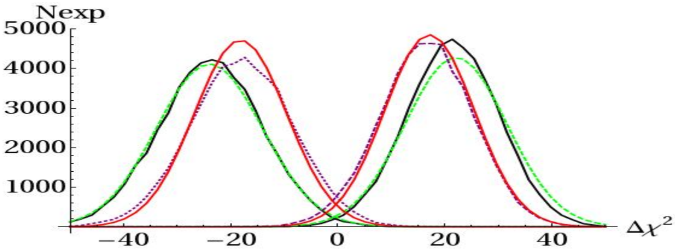


$\overline{\Delta\chi^2} \sim 11$ (20) for JUNO with MINOS (NOνA) yielding 2.6σ (3.9σ) of confidence at the median experiment

Reactor and Accelerator when $\cos(\delta) = \pm 1$

Ref: Ciuffoli et al., arXiv:1305.5150

NOνA with 6 years of JUNO using $\delta = 0$ and $\delta = \pi$



At $\delta = 0$ (π) we find $\overline{\Delta\chi^2} = 17$ (22) yielding 3.5σ (4.2σ)

In each case simulation results are overlaid with the approximate Gaussian distribution of $\Delta\chi^2$, showing excellent agreement

Conclusion: In the case of JUNO and RENO 50, it is *not* necessary to run simulations to obtain the confidence, it suffices to use the Asimov datasets and the formulae above

Towards $\cos(\delta)$

Ref: Ciuffoli et al., arXiv:1302.0624 (to appear in PRD)

NOνA and T2K have some sensitivity to sin(δ) as it determines the difference between ν_e and $\overline{\nu_e}$ appearance in the ν and $\overline{\nu}$ modes.

However due to a severe degeneracy with θ_{13} and θ_{23}, NOνA and T2K cannot distinguish δ from $\pi - \delta$.

In principle, the differences between the reactor and atmospheric effective masses depend upon $\cos(\delta)$ and so can break this degeneracy: For $\delta = 0$ (π)

$$\frac{\Delta M^2_{\mathrm{atm}} - \Delta M^2_{\mathrm{eff}}}{\Delta M^2_{\mathrm{eff}}} \sim 0.8\% \ (1.7\%)$$

To determine the hierarchy, $\Delta M^2_{\mathrm{eff}}$ and $|\Delta M^2_{31}|$ must both be measured to within 0.5%. As $\Delta M^2_{\mathrm{eff}}$ is easier to determine, its measurement will be better. The limiting factor will then be the measurement of $\Delta M^2_{\mathrm{atm}}$ at long baseline accelerator experiments.

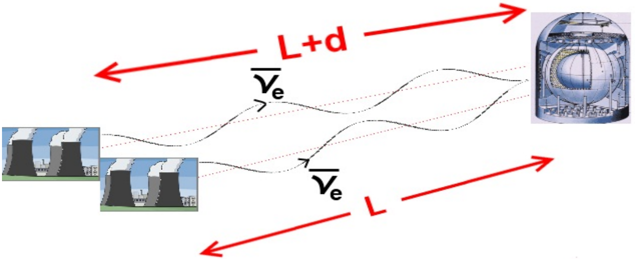
Interference

Ref: Ciuffoli et al., JHEP 1303 (2013) 016 arXiv:1208.1991

Reactors within a complex are often separated by of order 1 km.

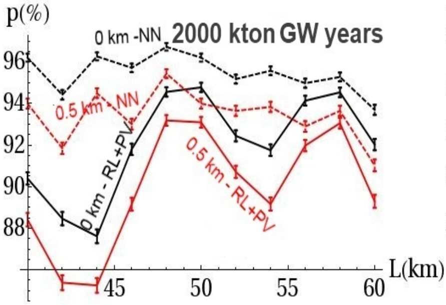
For example the reactors within the Hanbit complex lie along a line which is 1.3 km long.

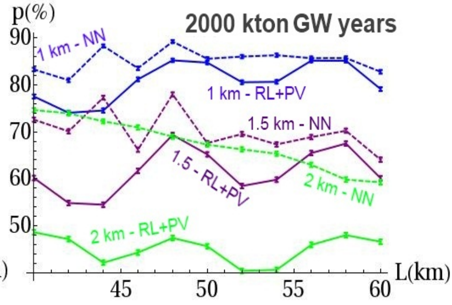
This means that neutrinos from different reactors travel different distances, and for small energies (2.5 MeV) neutrinos from 1 reactor will be at their 1-3 maximum while neutrinos from another are at their minimum, erasing the 1-3 oscillation signal



Interference as a Function of Baseline Difference

Ref: Ciuffoli et al., arXiv:1302.0624 (to appear in PRD)



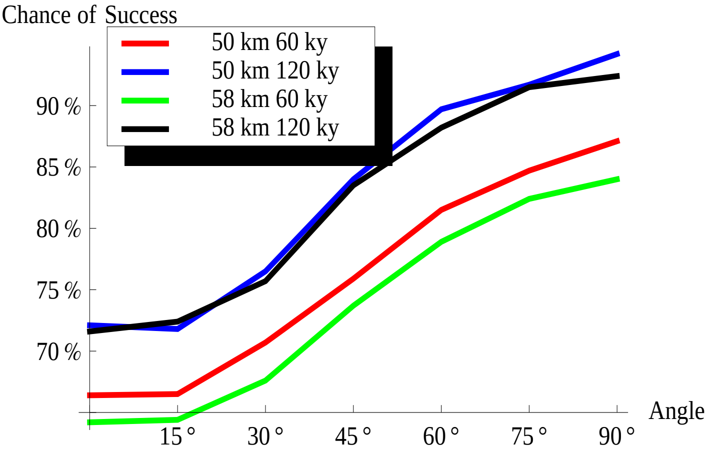


Here are the probabilities of successfully determining the hierarchy for experiments using two neutrino sources at different distances.

It is clear that if the distances differ by more than about 500 meters, the confidence in the hierarchy diminishes substantially.

Interference and Angle with respect to Reactor Complex

Ref: Ciuffoli et al., JHEP 1212 (2012) 004 arXiv:1209.2227

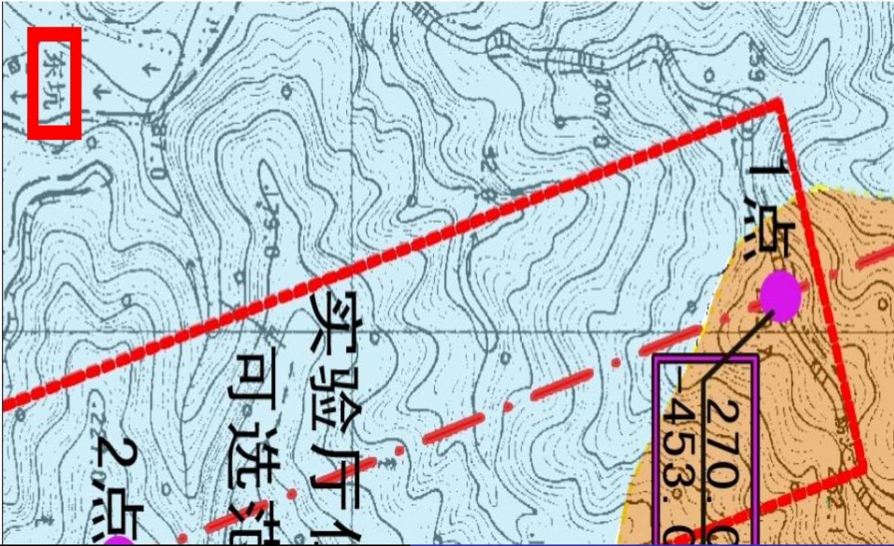


Interference and Sites for JUNO

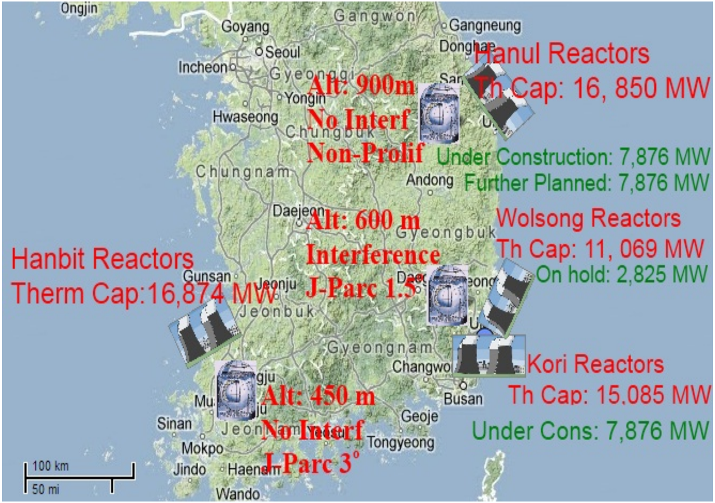
Ref: Ciuffoli et al., JHEP 1212 (2012) 004 arXiv:1209.2227

Chosen Site for Daya Bay II: DongKeng

Ref: Seminar by Yifang Wang, Paris, April 29, 2013



Sites for RENO 50



JUNO vs RENO 50

	JUNO	R50 @ Hanbit	@Hanul
baseline	52.1-52.8 km	47.4 km	51.5 km
target mass	20 ktons	18 ktons	18 ktons
detector shape	Spherical	Cylindrical	Cylindrical
interference	700 m	Negligible	Negligible
background reactor dist	210 km	238 km	133-153 km
initial rock overburden	600 m	450 m	900 m
current thermal cap.	0	16.87 GW	16.85 GW
cap. under construction	20.8 GW	0	7.88 GW
total planned cap.	35.8 GW	16.9 GW	32.6 GW
near det. baseline	17 km	23 km	32 km
near det. elevation	750m	480m	900m
near detector now?	no	RENO	no

Detector's Unknown Nonlinear Energy Response

Ref: Parke et al, Nucl.Phys.Proc.Suppl. 188 (2009) 115-117

Recall that the hierarchy signal is a 2% relative shift in the peak energies.

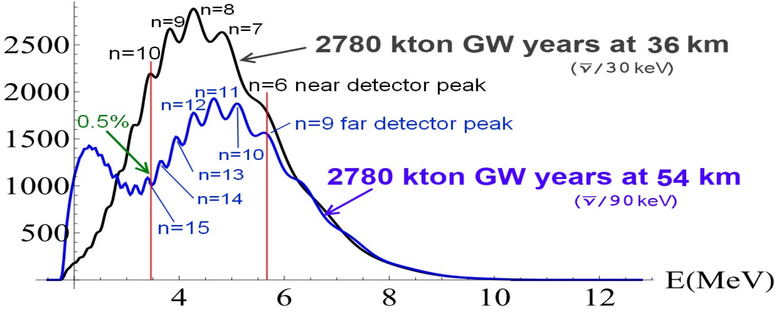
This requires that the statistical and the systematic errors in the energy be less than half of the state of the art.

The systematic error is the unknown part of the detector energy response, only the nonlinear part affects the relative energy differences and so the hierarchy.

The main challenge to the hierarchy determination at a reactor experiment is the unknown nonlinear energy response.

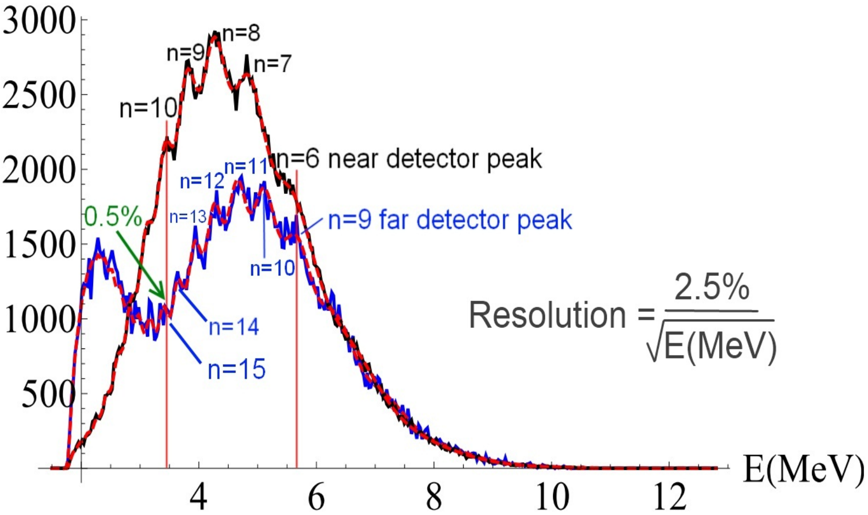
Ref: Ciuffoli et al., arXiv:1211.6818

This problem can be circumvented in principle if one uses two identical detectors at distinct baselines: use only *relative* energy measurements - independent of the correlated systematic error



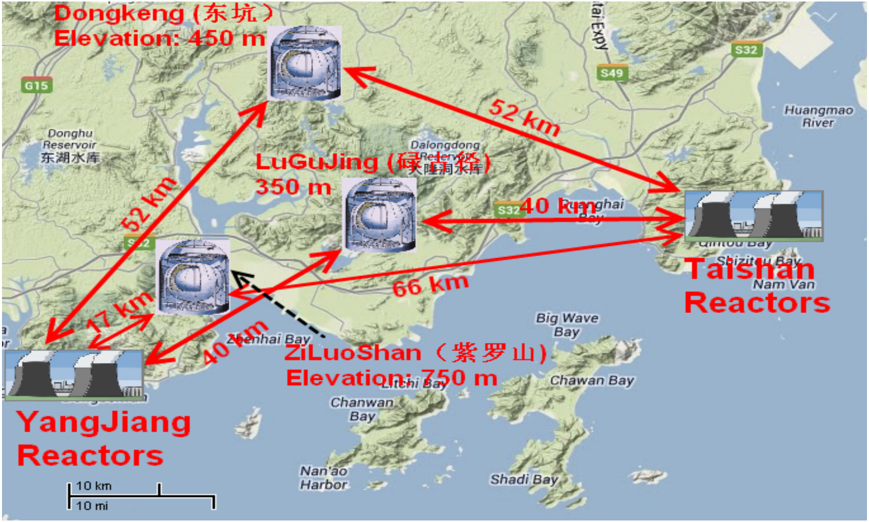
Here the 10th peak at 36 km is at a *higher* energy than the 15th peak at 54 km, so the hierarchy is inverted.

Ref: Ciuffoli et al., arXiv:1211.6818



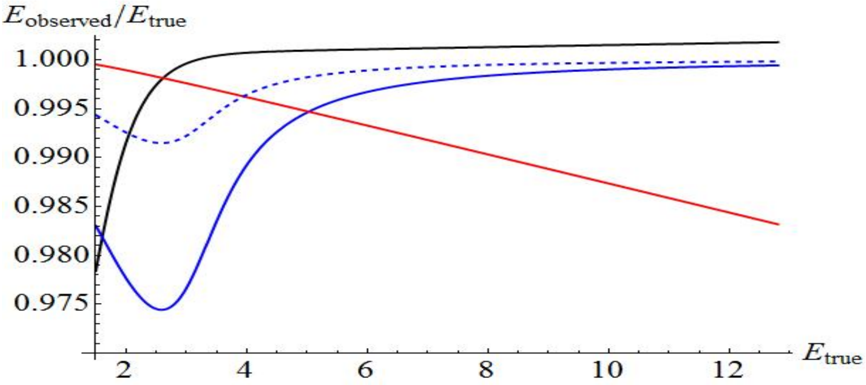
Second Detector Locations for JUNO

Ref: Ciuffoli et al., arXiv:1308.0591



Detector Nonlinear Energy Response Model

We determined the effects of several models of the detector's energy response upon $\overline{\Delta\chi^2}$



Below we will present our results for the blue dashed curve model.

Effect of Energy Response on $\overline{\Delta\chi^2}$

Site	NH	IH	NH: Nonlin	IH: Nonlin
DongKeng	14.1	-17.0	8.2	-21.5
DongKeng+LuGuJing	13.2	-16.2	7.8	-21.4
DongKeng+ZiLuoShan	13.5	-16.1	13.9	-15.3

We find that without a second detector, the nonlinearity energy response reduces $\overline{\Delta\chi^2}$ to 8.2, yielding only 2.1σ of confidence at the median experiment.

Replacing a single detector at DongKeng with a half-sized detector at DongKeng and at LuGuJing does not help, because the baselines of DongKeng and LuGuJing are too similar.

On the other hand a near detector at ZiLuoShan, fixing the total target mass, essentially erases the effects of the nonlinearity.

Other (Dis)Advantages of Multiple Detectors

Disadvantages:

2 detectors of the same target mass as 1 detector requires 25% more PMTs, also more civil engineering

In the case of JUNO, if one demands that the near detector use flux from *both* the TaiShan and YangJiang complexes, it will be at a minimum baseline of 40 km, 44 km if one wants a mountain. A similar near-far baseline reduces the advantage.

Advantages:

Breaks degeneracy between reactor flux model and θ_{12}, and also with geoneutrinos

Half mass detectors mean that light attenuation is less problematic, improving the resolution.

Determining sin(δ) with DAEδALUS

Long baseline accelerator experiments can determine both δ and the hierarchy by using the matter effect and comparing ν and $\overline{\nu}$ oscillation modes.

However the flux is low, especially in the $\overline{\nu}$ channel, and the CP signal is small, requiring expensive detectors which are difficult to construct.

The δ-dependence is greater at the second oscillation maximum, but there the flux is even lower and there is a large degeneracy between the hierarchy and δ.

One solution to this problem is to obtain $\overline{\nu}$ from decays of π at rest, this is known as the DAEδALUS project.

To determine the hierarchy one needs 2 or 3 cyclotrons creating π at various distances from the detector, such as 2, 8 and 20 km.

Determining sin(δ) with Multiple Detectors

Which detector should be used?

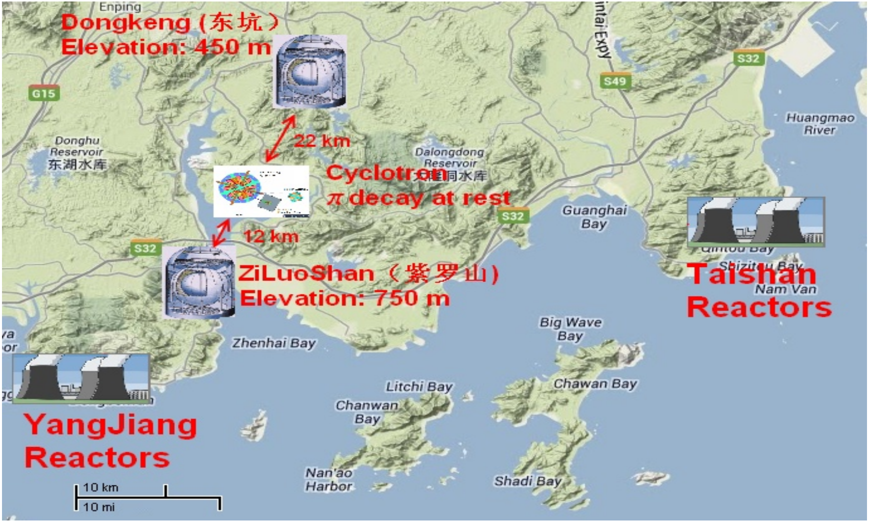
One possibility is a scintillator detector in a beam experiment which is always in ν mode, for example one might consider a 1300 km baseline and a 2° off-axis beam made using 30 GeV protons, so as to observe the second peak.

Here the $\overline{\nu}$ from the π break the hierarchy-δ degeneracy.

If JUNO has two detectors, then one can consider a *single* cyclotron generating the π at rest, which is at a different baseline from the two detectors.

This is cheaper than the usual DAEδALUS proposal because it requires only one π source, and it is potentially more precise because no error will arise from the relative intensities of the sources.

Map of DEAδALUS at JUNO



Conclusions

The medium and high energy part of the spectrum determines $\Delta M^2_{\mathrm{eff}}$ and the low energy part $|\Delta M^2_{31}|$, the hierarchy is determined by comparing them

To avoid interference, the reactors must all be within 500 m of the same distance to the detector, which can be achieved if the detector is orthogonal to a linear reactor array

The median confidence of a hierarchy determination at *any* experiment is not $\sqrt{\Delta\chi^2}$, but is about one half σ less

The unknown energy response is a big problem, but can be addressed using identical detectors at distinct baselines

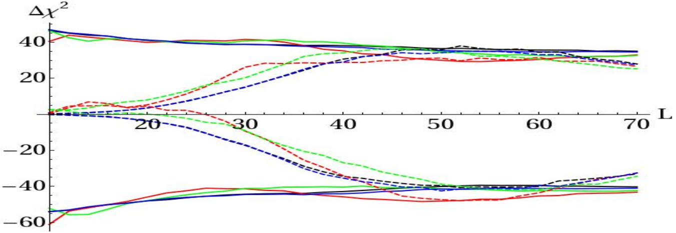
An accelerator experiment alone cannot distinguish δ from $\pi - \delta$, but perhaps in combination with a reactor experiment it can

Backup slide: χ^2 Analysis with 1 or 2 Detectors

Work in Progress with Ciuffoli, Wang, Yang, Zhang and Zhong

$\Delta\chi^2$ for 1 detector (dashed) vs 1 detector at 55 km and 1 at an arbitrary baseline (solid), each with 8640 kton GW years total

A perfect energy response, a linear shift, quadratic model and Daya Bay best fit of nonlinearity to generate the spectra and a E+constant+$1/E$ fit is used to minimize each χ^2.



Two detector experiments outperform one detector experiments with the same total target mass and distinct baselines