Four-loop quark-quark splitting functions in QCD

杨通智（华南师范大学）

with Thomas Gehrmann, Andreas von Manteuffel and Vasily Sotnikov

Based on arXiv:2604.09534 and arXiv:2609.37563

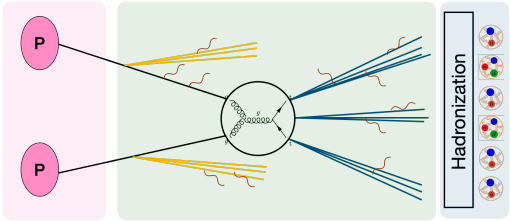
第六届量子场论及其应用研讨会

16-19 October 2026, Jinan, Shandong University





QCD factorization at hadron colliders



$$d\sigma = \underbrace{f_{P\to a}(x_1) \times f_{P\to b}(x_2)}_{\text{Parton distribution function}} \otimes \underbrace{d\hat{\sigma}}_{\text{Perturbative}} \otimes \underbrace{F}_{\text{Fragmentation}}$$

- Universal PDFs: non-perturbative inputs for hard processes, fitted from data.
- Universal evolution: DGLAP, governed by splitting functions P_{ij}

$$\frac{df_{i|N}}{d\ln Q} = 2\sum_k P_{ik} \otimes f_{k|N}, \quad P_{ik} = a_s P_{ik}^{(0)} + a_s^2 P_{ik}^{(1)} + a_s^3 P_{ik}^{(2)} + a_s^4 P_{ik}^{(3)} + \cdots$$

- Process-dependent hard scattering: calculated in perturbation theory.

The history of DGLAP

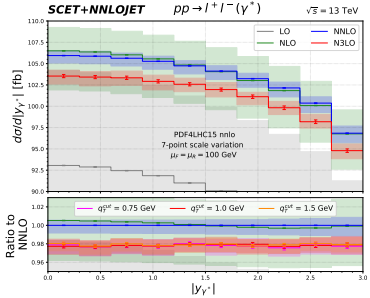
- Momentum space
  - 1972: Gribov & Lipatov derive the same equation – before QCD even existed, in a simpler theory (vector and pseudoscalar Yukawa)
  - 1977: Dokshitzer and Altarelli–Parisi independently extend it to QCD
  - Probability interpretation, implication for factorization
- Moment space
  - Wilson OPE approach: $\gamma(n)$ govern the evolution of local twist-two operators
  - A number for each n, hard to interpret physically.

Two routes converge: related by a Mellin transform,

$$\gamma_{ij}(n) = -\int_0^1 \mathrm{d}x\, x^{n-1} P_{ij}(x).$$

Gribov–Lipatov (1972); Dokshitzer (1977); Altarelli–Parisi (1977); Wilson (1969).

Why four-loop splitting functions?



N^{3}LO Drell–Yan with NNLO PDFs

Drell–Yan: Chen, Gehrmann, Glover, Huss, Yang, Zhu (2022).
Conventions: unpolarized, space-like $\overline{\rm MS}$; n_f massless flavors; $\mu_R = \mu_F = \mu$.

Precision predictions need consistent PDF evolution

- N^{3}LO hard predictions are already available.
- Consistent N^{3}LO PDF evolution needs four-loop $P_{ij}^{(3)}$.
- Exact kernels remove the uncertainty of moment-based reconstruction.

Theory motivation: $\mathcal{N}=4$ and reciprocity

$\mathcal{N}=4$ SYM benchmark

- Planar twist-two anomalous dimensions to seven loops.
- Principle of Maximal transcendentality: highest-weight QCD terms determine the $\mathcal{N}=4$ result.
- Four-loop non-planar $\mathcal{N}=4$ results are expressed by harmonic sums.

Generalized reciprocity

- Space-like PDF evolution (S) and time-like fragmentation (T).

  Non-singlet relation:

$$\gamma_{\rm ns}^{S}(n)=\gamma_{\rm ns}^{T}\big(n+2\gamma_{\rm ns}^{S}(n)\big)$$

- Singlet: proposed for matrix eigenvalues; verified through three loops.

Exact QCD results enable new tests of reciprocity and analytic structure.

SYM: Marboe, Velizhanin (2016); Kotikov, Lipatov, Onishchenko, Velizhanin (2004); Kniehl, Velizhanin (2021).
Reciprocity: Dokshitzer, Marchesini, Salam (2006); Basso, Korchemsky (2006); Mitov, Moch, Vogt (2006); Chen, Yang, Zhu, Zhu (2020).

Non-singlet and pure-singlet contributions

- Flavor non-singlet

$$q_{ij}^{\pm} = (q_i \pm \bar{q}_i) - (q_j \pm \bar{q}_j)$$

$$\frac{\mathrm{d}\, q_{ij}^{\pm}}{\mathrm{d}\ln \mu^2} = P_{\rm ns}^{\pm} \otimes q_{ij}^{\pm}$$

$$q^V = \sum_i (q_i - \bar{q}_i), \quad P_{\rm ns}^V = P_{\rm ns}^- + P_{\rm ns}^s$$

- No mixing with the gluon PDF.

- Flavor singlet

$$\Sigma = \sum_i (q_i + \bar{q}_i)$$

$$\frac{\mathrm{d}}{\mathrm{d}\ln \mu^2} \begin{pmatrix} \Sigma \\ g \end{pmatrix} = \begin{pmatrix} P_{qq} & P_{qg} \\ P_{gq} & P_{gg} \end{pmatrix} \otimes \begin{pmatrix} \Sigma \\ g \end{pmatrix}$$

$$\boxed{P_{qq} = P_{\rm ns}^+ + P_{\rm ps}}$$

- $P_{\rm ps}$ starts at two loops.

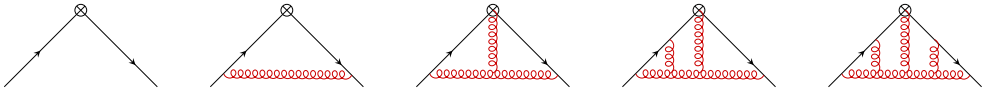
At four loops, the two results now complete $P_{qq}^{(3)}$.

Gehrmann, von Manteuffel, Sotnikov, Yang, arXiv:2604.09534; Gehrmann, von Manteuffel, Sotnikov, Yang, arXiv:2609.37563.

Splitting functions from off-shell matrix elements

$$O_q(n) = \frac{i^{n-1}}{2}\,\bar{\psi}\not{\Delta}(\Delta\cdot D)^{n-1}\psi, \qquad \Delta^2 = 0$$

Singlet: sum over flavors. Flavor non-singlet: insert a traceless flavor matrix.



$$A_{ij}(n,p) = \langle j(p)|\,O_i(n)\,|j(p)\rangle, \qquad p^2 < 0$$

- Off-shell momentum supplies a scale for the two-point integrals.
- UV poles determine the operator renormalization constants and $\gamma(n)$.
- In the singlet sector, the subtraction must include gauge-variant (GV) operators.

Renormalization: what changes in the singlet sector?

- Non-singlet: multiplicative renormalization

$$O_{\rm ns}^{\rm R} = Z_{\rm ns}\, O_{\rm ns}^{\rm B}$$

- Singlet: physical and gauge-variant operators mix

$$O_q^{\rm R} = Z_{qq} O_q^{\rm B} + Z_{qg} O_g^{\rm B} + Z_{qA}\, O_{ABC}^{\rm B} + [ZO]_q^{\rm GV}$$

$$O_{ABC} = O_A + O_B + O_C$$

O_A: gluon fields

O_B: quark and gluon fields

O_C: ghost and gluon fields

$$[ZO]_q^{\rm GV} = \sum_{\alpha \text{ beyond } ABC} Z_{q\alpha}\, O_\alpha^{\rm B} = \mathcal{O}(a_s^3)$$

- Additional GV counterterms beyond O_{ABC}; their vertices are determined as a combined $[ZO]$, without separating each Z and O.

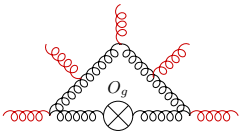
Gehrmann, von Manteuffel, Yang, JHEP 04 (2023) 041; arXiv:2409.10303.

GV Feynman rules from UV finiteness

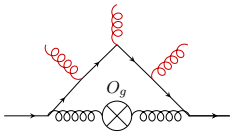
- Determine the counterterm vertices directly from off-shell OMEs
- For example, with two external ghosts and m gluons:

$$Z_{gA}^{(1)} \langle c|O_C|c+mg\rangle_{1\mathrm{PI}}^{(0)} = -\,\mathrm{Pole}_\epsilon \langle c|O_g|c+mg\rangle_{1\mathrm{PI}}^{(1),\mathrm{B}}$$

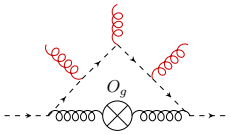
$$Z_{gA} = a_s Z_{gA}^{(1)} + \cdots, \qquad Z_{gA}^{(1)} = -\frac{C_A}{\epsilon\, n(n-1)}$$



gluon external states



quark external states



ghost external states

Pole cancellation fixes all-n rules order by order in loops and legs.

Gehrmann, von Manteuffel, Yang, JHEP 04 (2023) 041; arXiv:2409.10303. Diagrams illustrate the extraction of five-leg counterterm vertices.

The GV ingredients needed for four-loop P_{qq}

$$\langle q|O_q^{\rm R}|q\rangle = Z_q\Big[Z_{qq}\langle q|O_q^{\rm B}|q\rangle + Z_{qg}\langle q|O_g^{\rm B}|q\rangle + Z_{qA}\langle q|O_{ABC}^{\rm B}|q\rangle\Big]$$

Coupling and gauge-parameter renormalization are implicit.

Required matrix elements

- $\langle q|O_q|q\rangle$: four loops.
- $\langle q|O_g|q\rangle$: through three loops.
- $\langle q|O_{ABC}|q\rangle$: through two loops.

Why the GV sector is manageable

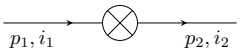
- $Z_{qA} = \mathcal{O}(a_s^2)$, needed through a_s^3.
- $\langle q|O_{ABC}|q\rangle = \mathcal{O}(a_s)$.
- $\langle q|[ZO]_q^{\rm GV}|q\rangle$ vanishes through four loops.

The new ingredient is the four-loop quark OME; the mixing terms are known.

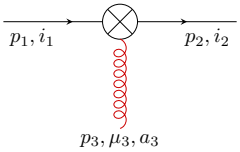
Gehrmann, von Manteuffel, Sotnikov, Yang, arXiv:2609.37563; Gehrmann, von Manteuffel, Yang, JHEP 04 (2023) 041.

Feynman rules for the operator vertices

- Functional variation of O_q produces vertices with an arbitrary number of gluons



$$\longrightarrow \quad \frac{1}{2} \not{\Delta}\, \delta_{i_1 i_2} \left(\Delta \cdot p_1\right)^{n-1}$$



$$\longrightarrow \quad \frac{g_s}{2} \Delta^{\mu_3}\, T^{a_3}_{i_2 i_1} \not{\Delta} \sum_{j_1=0}^{n-2} \left(-\Delta \cdot p_2\right)^{j_1} \left(\Delta \cdot p_1\right)^{n-2-j_1}$$

2 gluons, 3 gluons, $\cdots$

- The red factors are non-standard: they carry the spin n and are not ordinary vertices

Tracing parameter: from spin powers to propagators

- Set $A = \Delta\cdot p_1$, $B = -\Delta\cdot p_2$ for the one-gluon vertex.

$$\sum_{n=1}^{\infty} t^n (\Delta\cdot k)^{n-1} = \frac{t}{1 - t\Delta\cdot k},$$
$$\sum_{n=2}^{\infty} t^n \sum_{j=0}^{n-2} A^{n-2-j} B^j = \frac{t^2}{(1-tA)(1-tB)}.$$

- Coefficients of t^n give the Mellin-moment sequence

$$\frac{1}{1-t} = \sum_{n=0}^{\infty} t^n \cdot 1 \quad \longrightarrow \quad 1$$

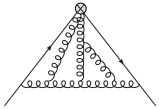
$$\ln(1-t) \longrightarrow -\frac{1}{n}, \qquad \frac{\ln(1-t)}{1-t} \longrightarrow -S_1(n) \quad (n \geq 1).$$

Ablinger, Blümlein, Hasselhuhn, Klein, Schneider, Wißbrock (2012).

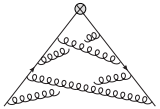
Four-loop diagrams in the two flavor sectors

- Non-singlet $P_{\rm ns}^{\pm}$

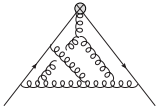
Insertion on an open quark line



$C_F C_A^3$



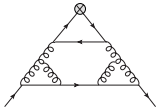
C_F^4



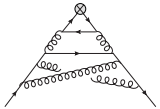
$d_F^{abcd} d_A^{abcd}/N_c$

- Pure singlet $P_{\rm ps}$

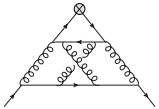
Insertion on a closed quark loop



$n_f C_F C_A^2$



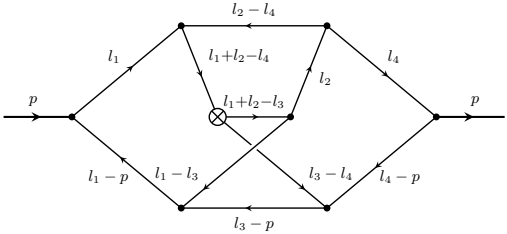
$n_f C_F^3$



$n_f (d_F^{abcd})^2/N_c$

Selected diagrams from the two papers; crossed vertices denote operator insertions.

IBP reduction at four loops



Example: 11 quadratic + 2 linear propagators

Scale of the non-singlet calculation

- About 16,000 diagrams.
- 549 top-level sectors.
- About 3 million raw integrals.
- About 6,000 master integrals.

- Finite-field sampling, optimized IBP equations and spanning sectors.
- Pure singlet: comparable complexity, slightly fewer master integrals.

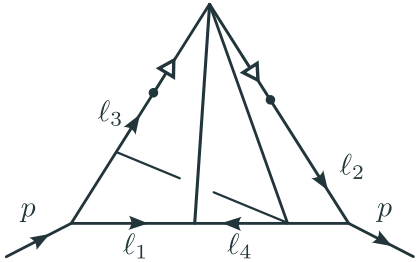
IBP: Chetyrkin, Tkachov (1981); Laporta (2000). Finite fields: von Manteuffel, Schabinger (2014); Peraro (2015).
Equation selection: Driesse, Jakobsen, Mogul, Plefka, Sauer, Usovitsch (2024); Guan, Liu, Ma, Wu (2024); Bern, Herrmann, Roiban, Ruf, Smirnov, Smirnov, Zeng (2024); von Hippel, Wilhelm (2025); Song, Yang, Cao, Luo, Zhu (2025); Lange, Usovitsch, Wu (2025).
Spanning cuts/sectors: Larsen, Zhang (2015); Guan, Liu, Ma, Wu (2024).

Elliptic sectors in the master integrals

- Solve the DE:

$$\frac{\mathrm{d}\vec{I}(t,\epsilon)}{\mathrm{d}t} = A(t,\epsilon)\vec{I}(t,\epsilon)$$

- Some non-planar sectors have elliptic geometry.
- Example: four masters on the maximal cut.
- Solving every master in closed form would be costly.



$$K(t) = \int_0^1 \frac{\mathrm{d}\tau}{\sqrt{(1-\tau^2)(1-t^2\tau^2)}}$$

Use series solutions and reconstruct the final anomalous dimension.

Gehrmann, von Manteuffel, Sotnikov, Yang, arXiv:2604.09534.

Where do elliptic contributions cancel?

- In the full bare OME, or only in the final anomalous dimension?
- Is this a four-loop property, or does it persist to all loop orders?

A three-loop hint from the non-singlet $q \to q$ OME

$$A_{qq}^{(3),\mathrm{B}}(n,\epsilon) = \sum_{r=-3}^{0} \epsilon^r A_r(n) + \epsilon A_1(n) + \cdots$$

- Through ϵ^0: ordinary harmonic sums suffice.
- At ϵ^1: generalized harmonic sums occur.
- ϵ^0 enters four-loop renormalization; ϵ^1 can enter at five loops via $1/\epsilon^2$ counterterms.

The four-loop result does not establish cancellation to all orders.

High Mellin moments from differential equations

$$I(t,\epsilon) = \sum_{r=r_{\min}}^{r_{\max}} \epsilon^r \sum_{n=0}^{\infty} c_{r,n}\, t^n$$

1. Substitute the series into the differential equations.
2. Obtain recursion relations for $c_{r,n}$.
3. Fix the regular solutions at $t = 0$ with four-loop self-energy integrals.
4. Generate enough moments for analytic reconstruction.

Non-singlet

- About 4,000 moments; 28 self-energy masters for the boundary data.

Pure singlet

- Same series strategy, followed by the full singlet counterterm subtraction.

Gehrmann, von Manteuffel, Sotnikov, Yang, arXiv:2604.09534; Gehrmann, von Manteuffel, Sotnikov, Yang, arXiv:2609.37563.

From the UV pole to an all-n result

- UV renormalization:

$$\frac{\mathrm{d}Z}{\mathrm{d}\ln\mu^2} = -\gamma Z, \qquad Z^{(4)} = \frac{\gamma^{(3)}}{4\epsilon} + \frac{Z_{4,2}}{\epsilon^2} + \frac{Z_{4,3}}{\epsilon^3} + \frac{Z_{4,4}}{\epsilon^4}$$

  Higher poles follow from lower-order anomalous dimensions and the beta function.

- Reconstruct the all-n anomalous dimension:

$$\gamma^{(3)}(n) = \sum_{a,k,\vec{m}} c_{a,k,\vec{m}} \frac{S_{\vec{m}}(n)}{(n+a)^k}$$

  $c_{a,k,\vec{m}}$ are constants independent of n, fixed by calculated moments.

$$\begin{aligned} \gamma_{\rm ns}^{\pm}: &\quad a \in \{0, 1\}, \\ \gamma_{\rm ps} \text{ (singlet ansatz)}: &\quad a \in \{0, \pm 1, \pm 2\}, \\ \gamma_{\rm ns}^{V} \text{ (including the sea term)}: &\quad a \in \{0, \pm 1, \pm 2\}. \end{aligned}$$

- For pure singlet, subtract physical and GV mixing, then take

$$\gamma_{\rm ps}^{(3)} = \gamma_{qq}^{(3)} - \gamma_{\rm ns}^{+,(3)}.$$

Analytic results for arbitrary n and x

Non-singlet: $\gamma_{\rm ns}^{\pm,(3)}$, $\gamma_{\rm ns}^{V,(3)}$

- Harmonic sums up to weight 7.
- In x space: harmonic polylogarithms up to weight 6.
- Constants up to weight 7, including ζ_7.

Pure singlet: $\gamma_{\rm ps}^{(3)}$

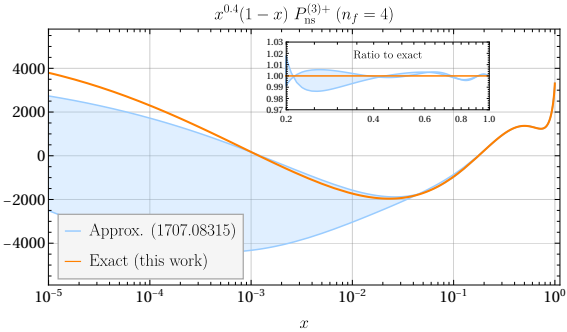
- Harmonic sums up to weight 6.
- In x space: harmonic polylogarithms and zeta products up to weight 6.
- All seven color structures determined.

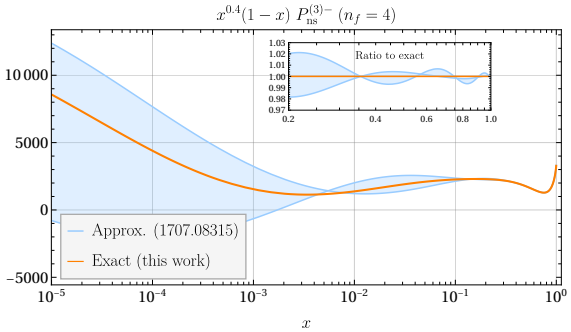
$$P^{(3)}(x) = -\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \mathrm{d}n\, x^{-n}\gamma^{(3)}(n)$$

Complete analytic kernels, with the full momentum-fraction dependence.

Gehrmann, von Manteuffel, Sotnikov, Yang, arXiv:2604.09534; Gehrmann, von Manteuffel, Sotnikov, Yang, arXiv:2609.37563. Inverse Mellin transformation with `HarmonicSums`.

Non-singlet $P_{\rm ns}^{+}$ and $P_{\rm ns}^{-}$: exact vs approximate

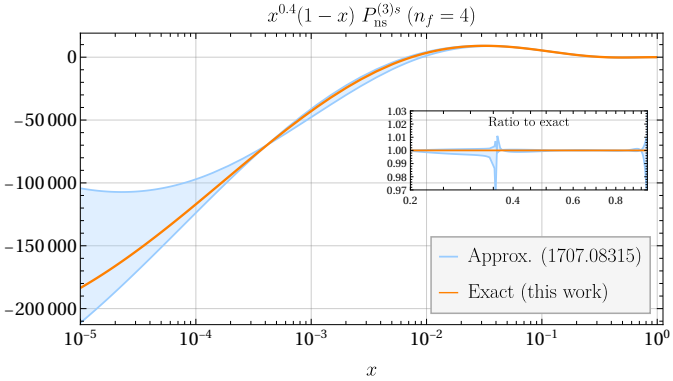




- $n_f = 4$; displayed: $x^{0.4}(1-x)P_{\rm ns}^{\pm,(3)}(x)$ for $x < 1$.
- Good agreement overall; $P_{\rm ns}^{+,(3)}$ slightly exceeds the old band at very small x.

Exact: Gehrmann, von Manteuffel, Sotnikov, Yang, arXiv:2604.09534. Approximations: Moch, Ruijl, Ueda, Vermaseren, Vogt (2017).

The valence sea contribution $P^{s}_{\rm ns}$



$$P^{V}_{\rm ns} = P^{-}_{\rm ns} + P^{s}_{\rm ns}$$

- Completes the valence evolution kernel.
- New n_f contributions in $\gamma^{s,(3)}_{\rm ns}$.
- Determines the complete small-x expansion.

All non-singlet channels are now known analytically at four loops.

Exact: Gehrmann, von Manteuffel, Sotnikov, Yang, arXiv:2604.09534. Approximation: Moch et al. (2017); $n_f = 4$.

The non-singlet threshold limit

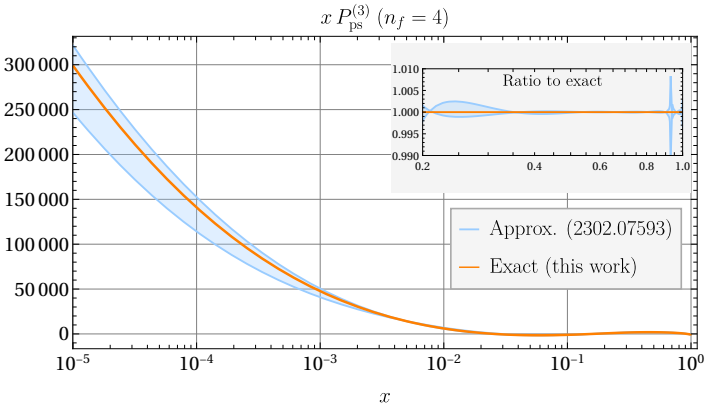
As $x \to 1$:

$$P_{\rm ns}^{\pm,(3)}(x) = A_4 \left[\frac{1}{1-x}\right]_+ + B_4\,\delta(1-x) + C_4 \ln(1-x) + D_4 - A_4 + \text{power-suppressed terms}$$

- A_4: confirms the known four-loop cusp anomalous dimension.
- B_4: analytic virtual anomalous dimension; agrees with earlier numerical results.
- B_4 makes the four-loop rapidity anomalous dimension analytic: a universal ingredient for N^{4}LL resummation.

Gehrmann, von Manteuffel, Sotnikov, Yang, arXiv:2604.09534. Cusp: Henn, Korchemsky, Mistlberger; von Manteuffel, Panzer, Schabinger. Virtual: Das, Moch, Vogt. Rapidity: Moult, Zhu, Zhu; Duhr, Mistlberger, Vita.

New result: the complete pure-singlet kernel



- All seven color structures

$$n_f\{C_F C_A^2,\ C_F^2 C_A,\ C_F^3\},$$
$$n_f \frac{(d_F^{abcd})^2}{N_c},$$
$$n_f^2\{C_F C_A,\ C_F^2\}, \qquad n_f^3 C_F$$

- Agreement within the approximation band for $10^{-5} \le x < 1$.

Exact: arXiv:2609.37563. Approximation: Falcioni, Herzog, Moch, Vogt, PLB 842 (2023) 137944.

The apparent pole at $n = 2$ and the literature's $\delta(n-2)$

- The reconstructed ζ_3 coefficient in the literature contains

$$\delta(n-2) \equiv \delta_{n,2}$$

  A non-analytic selector on the even moments

- The all-n expression explains the exceptional moment

$$\frac{\frac{3}{4} + S_{-2}(n)}{n-2} = \frac{7}{4} - \frac{3}{2}\zeta_3 + \mathcal{O}(n-2)$$

  - Continue $S_{-2}(n)$ from even moments and take the removable limit.
  - The rational part generates extra ζ_3 at $n = 2$.
  - The full anomalous dimension is regular at $n = 2$.

Falcioni, Herzog, Moch, Pelloni, Vogt, arXiv:2512.10783, Eq. (3), PLB 875 (2026) 140278; Gehrmann, von Manteuffel, Sotnikov, Yang, arXiv:2609.37563.

Pure singlet: new small-x information

- Small-x expansion:

$$\frac{P_{\rm ps}^{(3)}(x)}{n_f} = \frac{1}{x}\left[K_2 \ln^2 x + m_{-1,1}^{(3)} \ln x + m_{-1,0}^{(3)}\right] + \sum_{k=0}^{6} b_k \ln^k x + \cdots$$

$$K_2 = C_A^2 C_F \left(\frac{10496}{243} + \frac{256}{3}\zeta_3\right)$$

- The leading $\ln^2 x/x$ term confirms the high-energy prediction.
- New: complete coefficients of $\ln x/x$ and $1/x$.
- The ancillary results give the expansion through next-to-leading power.

The subleading $1/x$ terms are now fixed exactly.

Gehrmann, von Manteuffel, Sotnikov, Yang, arXiv:2609.37563, small-x expansion. Leading-power prediction: Catani and Hautmann (1994).

A common small-x puzzle: quartic color factors

- Non-singlet: a quartic invariant at $\zeta_2 \ln^4 x$ as $x \to 0$

$$P_{\rm ns}^{+,(3)}(x) \supset -48\,\zeta_2\,\frac{d_F^{abcd} d_A^{abcd}}{N_c}\ln^4 x$$

- Pure singlet: the $\zeta_2 \ln^4 x$ coefficient at next-to-leading power

$$\left.\frac{P_{\rm ps}^{(3)}}{n_f}\right|_{\zeta_2 \ln^4 x} = -\frac{136}{3}C_A C_F^2 + \frac{34}{9}C_A^2 C_F + \frac{524}{9}C_F^3 + 64\frac{(d_F^{abcd})^2}{N_c}$$

- Both results differ from double-logarithmic resummation beyond leading color.
- Agreement is recovered in the leading-color limit.

The structural origin of the subleading-color mismatch remains open.

Exact: both works presented here. Resummation: Davies, Kom, Moch, Vogt, JHEP 08 (2022) 135.

Independent checks of the analytic results

Non-singlet

- Previously known n_f terms.
- Available fixed moments through $n = 16$.
- Four-loop cusp anomalous dimension.
- Numerical virtual anomalous dimension.

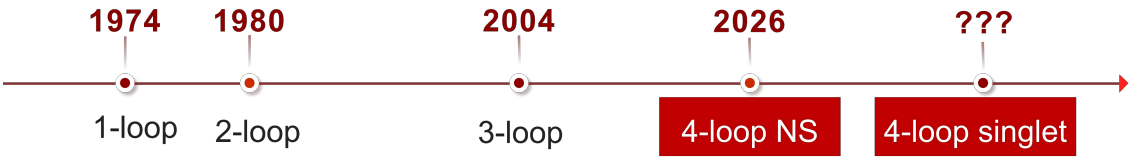
Pure singlet

- Exact n_f^3 and n_f^2 terms.
- Available even moments through $n = 22$.
- Known all-n zeta terms and weight-five contribution.
- Finite $n = 2$ limit after cancellation of apparent poles.

Agreement with fixed-order partial results

Moch et al. (2017); Davies et al. (2017); Gehrmann et al. (2024); Kniehl et al. (2025); Falcioni et al. (2023, 2026).

Summary: the quark-quark entry is complete



- Non-singlet $P_{\rm ns}^{\pm,V}$: complete analytic results and threshold information.
- Pure singlet $P_{\rm ps}$: complete analytic result with physical and GV mixing.
- Exact P_{qq} removes reconstruction uncertainty; the full singlet matrix remains to be completed.
- Open questions: subleading-color small-x resummation and the function space beyond four loops.

Thank you for your attention!

Generalized harmonic sums and new t-space letters

- File notation: $\mathrm{GS}[\vec{m}, \vec{z}, n] \equiv S_{\vec{m}}(\vec{z}; n)$.

$$S_{m_1,\ldots,m_r}(z_1, \ldots, z_r; n) = \sum_{n \geq k_1 \geq \cdots \geq k_r \geq 1} \frac{z_1^{k_1} \cdots z_r^{k_r}}{k_1^{m_1} \cdots k_r^{m_r}}.$$

  Here $m_i > 0$; $z_i = \pm 1$ give ordinary (possibly alternating) harmonic sums.
  The ϵ^1 terms include $S_{1,6}(\frac{1}{2}, 2; n)$ and $2^{-n} S_6(2; n)$.

- The generating series exposes the new letters

$$\sum_{n=0}^{\infty} t^n S_1(z; n) = -\frac{\ln(1 - zt)}{1 - t}, \qquad \frac{\mathrm{d}}{\mathrm{d}t}[-\ln(1 - zt)] = \frac{1}{z^{-1} - t}.$$

$$z = \tfrac{1}{2}: \quad \frac{1}{2 - t}, \qquad z = 2: \quad \frac{1}{\frac{1}{2} - t}.$$

Will the extra letters cancel at five loops? We do not know.

Examples checked in `NS_3LoopToEpsNSpace.m`. The displayed generating identity is illustrative.

Backup: the general singlet mixing framework

$$\begin{pmatrix} O_q \\ O_g \\ O_{ABC} \end{pmatrix}^{\rm R} = \begin{pmatrix} Z_{qq} & Z_{qg} & Z_{qA} \\ Z_{gq} & Z_{gg} & Z_{gA} \\ 0 & 0 & Z_{AA} \end{pmatrix} \begin{pmatrix} O_q \\ O_g \\ O_{ABC} \end{pmatrix}^{\rm B} + \begin{pmatrix} [ZO]_q^{\rm GV} \\ [ZO]_g^{\rm GV} \\ [ZO]_A^{\rm GV} \end{pmatrix}$$

$$Z_{gA} = \mathcal{O}(a_s), \qquad Z_{qA} = \mathcal{O}(a_s^2), \qquad [ZO]_g^{\rm GV} = \mathcal{O}(a_s^2), \quad [ZO]_q^{\rm GV} = \mathcal{O}(a_s^3)$$

- Multi-leg, off-shell OMEs determine the counterterm Feynman rules.
- Two-point OMEs with these counterterm insertions determine the physical anomalous dimensions.
- For four-loop γ_{qq}, $\langle q|[ZO]_q^{\rm GV}|q\rangle = 0$; the $Z_{qA}\,O_{ABC}$ contribution remains essential.

Gehrmann, von Manteuffel, Yang, JHEP 04 (2023) 041; arXiv:2409.10303; Gehrmann, von Manteuffel, Sotnikov, Yang, arXiv:2609.37563.

Numerical representations for PDF evolution

- Exact analytic information plus fast evaluation

$$P_{\rm ps}^{(3)}(x) = \underbrace{P_{\rm ps,\,small\text{-}x}^{(3)}(x)}_{\text{exact through next-to-leading power}} + \underbrace{R(x)}_{\text{numerically parametrized}}$$

- Fit the remainder with polynomials in x, $\ln x$ and $\ln(1-x)$.
- Pure-singlet parametrization: deviations up to 10^{-5} absolute and 10^{-8} relative.
- Ancillary files contain the exact kernels, expansions and parametrizations.
- Precise numerical representations also accompany the non-singlet result.

Gehrmann, von Manteuffel, Sotnikov, Yang, arXiv:2609.37563, numerical representation and ancillary files; Gehrmann, von Manteuffel, Sotnikov, Yang, arXiv:2604.09534.

Outlook: OMEs for the remaining singlet channels

Loops \ legs	2	3	4	5	6
0	A.D.	$[ZO]_g^{\mathrm{GV},(3)}$	$[ZO]_g^{\mathrm{GV},(2)}$	O_{ABC}	O_q, O_g
1	$[ZO]_g^{\mathrm{GV},(3)}$	$[ZO]_g^{\mathrm{GV},(2)}$	O_{ABC}	O_g	
2	$[ZO]_g^{\mathrm{GV},(2)}$	O_{ABC}	O_g		
3	O_{ABC}	O_g			
4	O_q, O_g				

Rows count loops; columns count external legs in the OME hierarchy. A.D. = anomalous dimensions; $[ZO]_g^{\mathrm{GV},(L)} = L$-loop GV counterterm.

- Done: one-loop five-point OMEs and two-loop four-point GV counterterms; four-loop $q \to q$ OMEs.
- In progress: $\gamma_{qg}^{(3)}$, $\gamma_{gq}^{(3)}$ and $\gamma_{gg}^{(3)}$.

Gehrmann, von Manteuffel, Yang, JHEP 04 (2023) 041; arXiv:2409.10303.