Complete Three-Loop Flavor-Nonsinglet Nonforward Kernels in QCD

Yao Ji

The Chinese University of Hong Kong, Shenzhen

The 6th workshop on QFT and its applications, Jinan, 2026

Based on work with Long-Bin Chen, Bo-Xuan Shi and Yu-Ming Wang

to appear

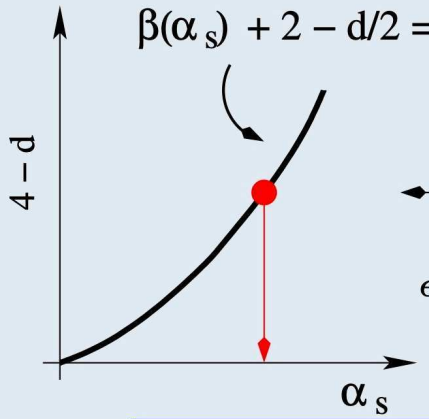


Application of conformal techniques in QCD

- **Conformal constraints from QCD at a critical point in $d = 4 - 2\epsilon$**

  For each perturbative coupling, tune ϵ so that the d-dimensional beta function vanishes,

  $\Longrightarrow$ **scale invariance** $\Longrightarrow$ **conformal invariance** $\Longrightarrow$ $SL(2,R)$ **for collinear dynamics**



conformal theory

$$\epsilon^*(a_s) = -\beta_0 a_s + \dots, \qquad a_s = \frac{\alpha_s}{4\pi}$$

$$\mathcal{A}_{\rm QCD} = \mathcal{A}^{\rm conf}_{\rm QCD} + \mathcal{O}\left(\frac{\beta(\alpha_s)}{\alpha_s}\right)$$

$\epsilon_* < 0$ at small positive coupling.

$$\beta_d^*(a_s) = -2a_s[\epsilon^* + \bar{\beta}(a_s)] = 0, \qquad \epsilon_* = -\bar{\beta}(a_s), \quad a_s = \frac{\alpha_s}{4\pi}.$$

- **The $\overline{\rm MS}$ evolution coefficients carry over to four dimensions**

  Reconstruct the ϵ-independent 4d $\overline{\rm MS}$-scheme kernel by trading ϵ_* and $\bar{\beta}$.

[V. Braun and A. Manashov (2013)]

Light-ray operators and nonforward evolution

- **LP QCD factorization require matrix elements of twist-2 operators**

$$\mathcal{O}_{V,A}(z_1,z_2;\mu)=[\bar{q}(z_1n)\not{n}\{1,\gamma_5\}[z_1n,z_2n]q(z_2n)]_R,\qquad n^2=0.$$

- **Renormalization brings scale dependence and calls for resummation via RGE**

$$\mu\frac{d}{d\mu}\mathcal{O}(z_1,z_2;\mu)+[\mathbb{H}(a_s)\mathcal{O}](z_1,z_2;\mu)=0,\quad [\mathbb{H}f](z_1,z_2)=\int_0^1 d\alpha d\beta\, w(\alpha,\beta)f(z_{12}^\alpha,z_{21}^\beta),$$
$$z_{12}^\alpha=(1-\alpha)z_1+\alpha z_2,$$

w includes support, endpoint distributions and constant terms.

- **Forward matrix elements lose total derivatives**

$$\mathcal{O}(z_1,z_2)\sim\sum(z_1-z_2)^m\underbrace{\mathcal{O}_{\mu_1,\cdots,\mu_m}(0)n^{\mu_1}\cdots n^{\mu_n}}_{\mathcal{O}_n^{(m)}\Longrightarrow\gamma_{\rm DGLAP}(m)}+\underbrace{(z_1-z_2)^m(z_1+z_2)^k(n\cdot\partial)^k\mathcal{O}_n^{(m)}(0)}_{\neq 0\text{ for nonforward distributions}}$$

[V. Braun, G. Korchemsky, D. Müller (2003)]

- $(n\cdot\partial)^k\mathcal{O}_n^{(m)}$ **are descendants of** $\mathcal{O}_n^{(m)}$**,** $\gamma_n^{(m)}$ **and** $\gamma_n^{(m,k)}$ **are related via CFT**

Conformal symmetry at the critical point

- **The collinear conformal generators form a $SL(2,R)$ algebra**

$$S^{(0)}_{-;i} = -\partial_{z_i}\,, \qquad S^{(0)}_{0;i} = z_i\partial_{z_1} + 1\,, \qquad S^{(0)}_{+;i} = z_i^2\partial_{z_i} + 2z_i\,,$$
$$S^{(0)}_{\pm,0} = S^{(0)}_{\pm,0;1} + S^{(0)}_{\pm,0;2}\,, \qquad [S^{(0)}_0, S^{(0)}_\pm] = \pm S^{(0)}_\pm\,, \qquad [S^{(0)}_+, S^{(0)}_-] = 2S^{(0)}_0\,,$$

- **$\mathbb{H}$ commutes with a_s-corrected $\widetilde{S}_{\pm,0}$ at the WF critical point $\beta(a_s,\epsilon^*)=0$**

$$[\widetilde{S}_{\pm,0}, \mathbb{H}] = 0\,, \qquad \widetilde{S}_- = S^{(0)}_-\,, \qquad \widetilde{S}_0 = S^{(0)}_0 + \bar{\beta} + \tfrac{1}{2}\mathbb{H},$$
$$\widetilde{S}_+ = S^{(0)}_+ + (z_1+z_2)(\bar{\beta} + \tfrac{1}{2}\mathbb{H}) + z_{12}\Delta, \qquad z_{12} = z_1 - z_2.$$

Δ is the scheme-specific conformal anomaly. Here $\bar{\beta} = \beta_0 a_s + \beta_1 a_s^2 + \cdots = -\epsilon_*$

Setting $\Delta = 0$ corresponds to conformal-scheme

- **$\overline{\text{MS}}$ kernel coefficients are independent of ϵ^*** [V. Braun, A. Manashov (2013)]

Reconstruct $\mathbb{H}$ at the critical point and retain the same expression in $d=4$

Conformal scheme and the role of the anomaly

- **Similarity transforms remove the anomaly term: $\mathbf{H}$ and $\mathbf{S}$ are Δ free**

$$\mathbf{H} = e^{X}\mathbb{H}e^{-X}, \qquad \mathbf{S}_i = e^{X}\widetilde{S}_i e^{-X},$$
$$\mathbf{S}_+ = S_+^{(0)} + (z_1+z_2)(\bar{\beta} + \tfrac{1}{2}\mathbf{H}), \qquad [\mathbf{S}_i, \mathbf{H}] = 0.$$

$\mathbf{H}$ and $\mathbf{S}_{\pm,0}$ are a_s-dependent CS quantities

- **The anomaly fixes the translation from CS to $\overline{\text{MS}}$**

$$X = a_s X^{(1)} + a_s^2 X^{(2)} + \cdots, \qquad [S_+^{(0)}, X^{(1)}] = z_{12}\Delta^{(1)}, \cdots$$

Higher-loop anomaly determines $X^{(n)}$ recursively. We set the canonically invariant freedom in X to zero as a gauge choice: i.e., $X^{(n)}_{\text{inv}} = 0$ with $[S_+^{(0)}, X^{(n)}_{\text{inv}}] = 0$

- **Two-loop anomaly + three-loop DGLAP imply three-loop nonforward kernel**

Both $\mathbf{H}$ and $\mathbb{H}$ have the same eigenvalues $\gamma(N)$ on z_{12}^{N-1} by similarity transformation.

[V. Braun, A. Manashov, S. Moch, M. Strohmaier (2017)]

Canonical invariance and the forward spectrum

- **A second transformation brings the CS operators to canonical form**

$$T(\mathbf{H}) = \sum_{k=0}^{\infty} \frac{L^k}{k!} (\bar{\beta} + \tfrac{1}{2}\mathbf{H})^k, \qquad L = \ln(z_{12} + i0),$$

$$T\mathbf{S}_i = S_i^{(0)} T, \qquad T\mathbf{H} = \mathscr{H}_{\rm inv} T, \qquad [S_i^{(0)}, \mathscr{H}_{\rm inv}] = 0.$$

$\mathbf{H}$ and $\mathscr{H}_{\rm inv}$ are again related via similarity transformation, so as $\mathbf{S}_i$ and $S_i^{(0)}$

- **The spectrum of the canonical kernel is exactly the Lipatov-Gribov reciprocity**

$$\mathbb{H} z_{12}^{N-1} = \gamma(N) z_{12}^{N-1}, \quad \mathscr{H}_{\rm inv} z_{12}^{N-1} = \gamma_{\rm inv}(N) z_{12}^{N-1}, \quad \gamma(N) = \gamma_{\rm inv}\left(N + \bar{\beta} + \tfrac{1}{2}\gamma(N)\right),$$

$$[\mathscr{H}_{\rm inv} f](z_1, z_2) = \int_0^1 d\alpha\, d\beta\, w(\tau) f(z_{12}^{\alpha}, z_{21}^{\beta}), \qquad \tau = \frac{\alpha\beta}{(1-\alpha)(1-\beta)}.$$

Integer $N \geq 1$ labels the local primaries.

Canonical invariance reduces regular weights to the conformal ratio: single variable

- **The DGLAP spectrum fixes $\mathscr{H}_{\rm inv}$ order by order** [YJ, A. Manashov, S. Moch (20



The three-loop reconstruction

- **Known input: exact canonical kernel and lower-order anomaly**

  Three-loop DGLAP spectrum [S. Moch, Vermaseren, Vogt (2004)] $\Longrightarrow$ $\mathscr{H}_{\rm inv}^{(1,2,3)}$ [YJ, A. Manashov, S. Moch (2023)]
  Anomaly $\Delta^{(1,2)}$ **and** $X^{(1,2)}$ **for** $\mathrm{CS}\mapsto\overline{\mathrm{MS}}$ **in** [V. Braun, A. Manashov, S. Moch, M. Strohamier (2017)]

- **First recover the CS kernel with an ordered recursion**

$$\mathbf{H}=\mathscr{H}_{\rm inv}+\sum_{k\geq 1}\frac{T_k}{k!}(\bar\beta+\tfrac{1}{2}\mathbf{H})^k,$$
$$T_0=\mathscr{H}_{\rm inv},\qquad T_k=[T_{k-1},L].$$

- **Then restore the anomaly-dependent** $\overline{\mathrm{MS}}$ **mixing**

$$\mathbb{H}^{(3)}=\mathbf{H}^{(3)}+[\mathbf{H}^{(2)},X^{(1)}]+[\mathbf{H}^{(1)},X^{(2)}]+\tfrac{1}{2}[[\mathbf{H}^{(1)},X^{(1)}],X^{(1)}].$$

  - **We evaluate the full ordered convolutions analytically**
  - **Previously** γ_{mn} **in Gegenbauer basis is known only for** $0\leq m,n\leq 7$

[V. Braun, A. Manashov, S. Moch, M. Strohmaier (2017)]



Explicit momentum-space ERBL and GPD kernels

- **Fourier transform yields familiar x-space kernels useful in phenomenology**

$$V(x,y) = \int_0^1 d\alpha\, d\beta\, w(\alpha,\beta)\delta(x-(1-\alpha)y-\beta(1-y))\,, \quad \textbf{ERBL kernel}$$

$$\mu\frac{d\phi(x,\mu)}{d\mu} = -\int_0^1 dy\, V(x,y)\phi(y,\mu), \qquad 0\le x,y\le 1,$$

$$\mathcal{K}(x,y;\xi) = \int_0^1 d\alpha\, d\beta\, w(\alpha,\beta)\delta(x-(1-\alpha-\beta)y-\xi(\alpha-\beta))\,, \quad \textbf{GPD kernel}$$

- $\mathbb{H}^{(3)}$ **needs many convolutions** $\sim [\mathbb{Z}_1\cdots\mathbb{Z}_n\mathcal{O}](z_1,z_2)$, $\mathbb{Z}_i = \mathscr{H}^{(k)}_{\rm inv}, X^{(m)},\cdots$
- **Convolutions done in both position space and momentum space, cross checked**

$$[\mathbb{Z}_i f](\underline{z}) = \int_0^1 d\alpha d\beta z_i(\alpha,\beta) f(z_{12}^\alpha, z_{21}^\beta)\,,\ [\widetilde{\mathbb{Z}}_i\widetilde{f}](x) = \int_0^1 \widetilde{z}_i(x,y)\widetilde{f}(y)\,,\ [\mathbb{Z}_{12}f](\underline{z}) = [\mathbb{Z}_1\mathbb{Z}_2 f](\underline{z})$$

$$[\mathbb{Z}_{12}f](\underline{z}) = \int d\alpha d\beta \widetilde{z}_{12}(\alpha,\beta) f(z_{12}^\alpha, z_{21}^\beta) \implies [\widetilde{\mathbb{Z}}_{12}\widetilde{f}](x) = \int_0^1 dy_1 dy_2 \underbrace{\widetilde{z}_1(x,y_1)\widetilde{z}_2(y_1,y_2)}_{\widetilde{z}_{12}(x,y_2)}\widetilde{f}(y_2)$$

- **Three-loop kernel require Goncharov polylogarithms of weight** ≤ 4

Symbol alphabet: $\{x,\ 1-x,\ y,\ 1-y,\ x-y,\ 1-x-y\}$.

Distribution structure and consistency checks

- **The plus prescription preserves the normalization of the LCDA**

$$\int_0^1 dx\, V^{(3)}(x,y) = 0, \quad \Longrightarrow \quad V^{(3)}(x,y) = [P^{(3)}(x,y)]_+^x$$

$P^{(3)}$ is the density before subtraction, and g is a smooth test function.

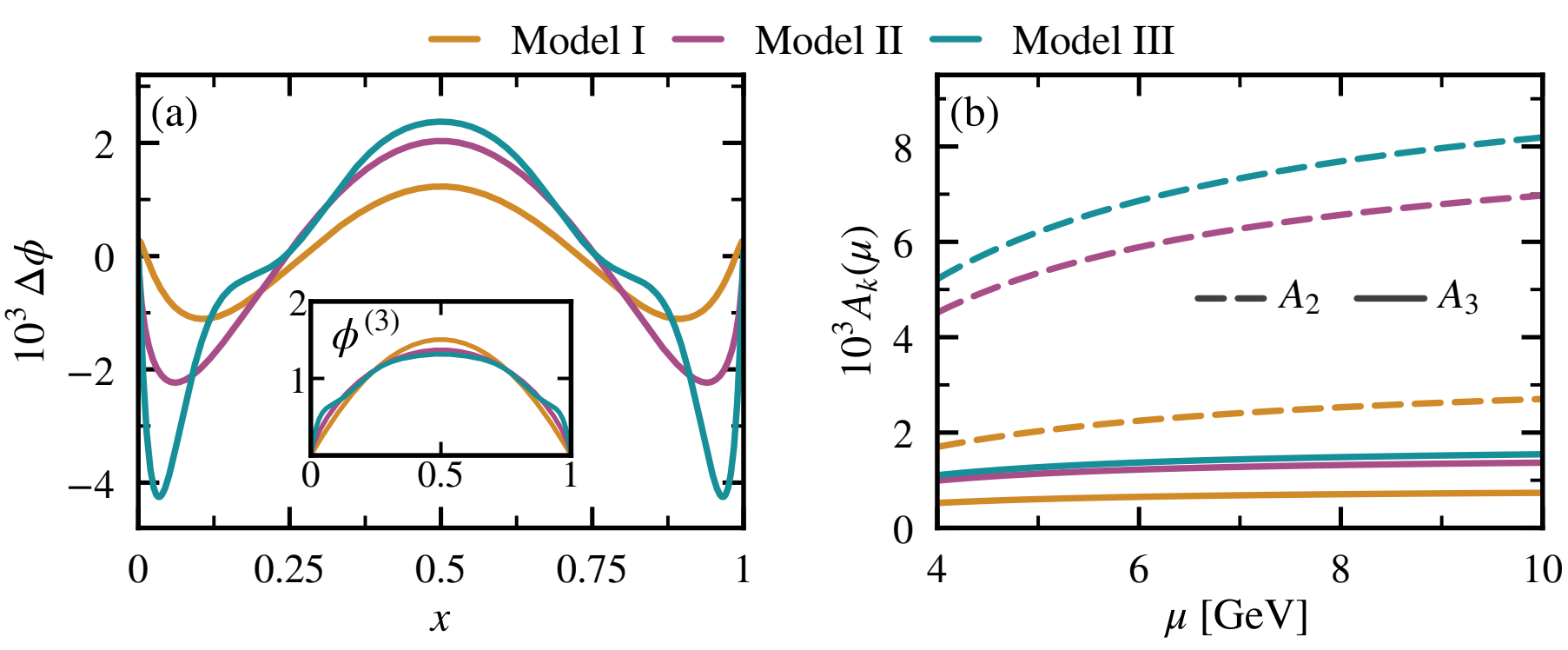
- **The diagonal singularity has the expected quark cusp coefficient**

$$\lim_{x\to y} |x-y| P^{(3)}(x,y) = -\Gamma^{(3)}_{\text{cusp}}, \qquad 0 < y < 1.$$

- **Checks we have performed:**
  - **Gegenbauer projection reproduces the published matrix for $0 \le m, n \le 7$**
  - **The GPD forward limit reproduces the three-loop nonsinglet DGLAP kernel**
  - **Position- and momentum-space convolutions agree, with numerical checks**
  - **The final $[\mathbb{H}^{(3)} f]$ and the initial full convolutions agree for various test functions**
  - **Three-loop DVCS vector coefficient: cancellation of all collinear and infrared poles**



Three-loop corrections

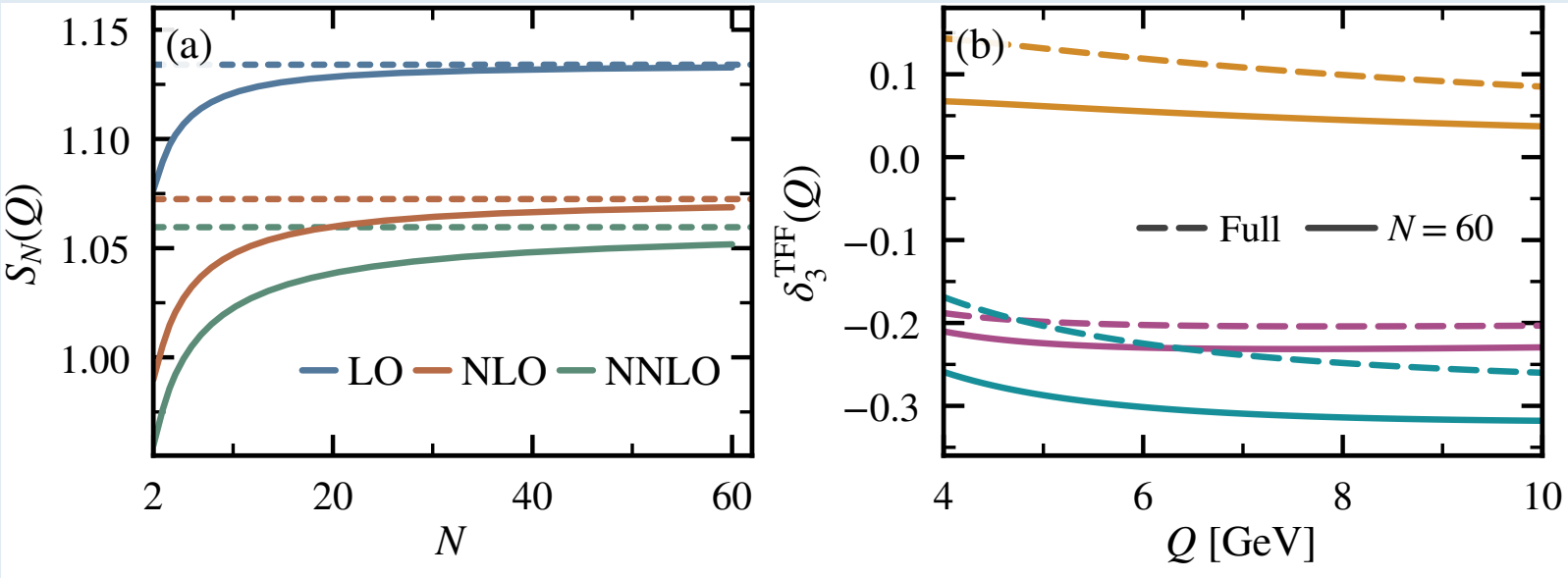


$$\Delta\phi^{[3]} = \phi^{[3]} - \phi^{[2]}, \qquad A_k(\mu) = \int_0^1 dx\, |\phi^{[k]}(x,\mu) - \phi^{[k-1]}(x,\mu)|$$

- **The three-loop deformation changes sign and integrates to zero**
  The 3L evolved DA depends on the nonperturbative input throughout $0 < x < 1$.
  at $\mu_0 = 2$ GeV , $\mu = 10$ GeV, $A_3/A_2 = 0.19$–0.27 for Models I–III
- **The new corrections consistently show good perturbative convergence.**

Gegenbauer truncation can bring nonnegligible effects

- **Use two-loop $\gamma^*\gamma \to \pi^0$ to test for Gegenbauer truncation error**



$$S(Q) = \frac{Q^2 F_{\gamma\pi}^{\rm LP}(Q)}{\sqrt{2} f_\pi}, \qquad \delta_N^{(h)} = \frac{S_x^{(h)} - S_N^{(h)}}{S_x^{(h)}},$$

$$\text{Panel (b): } 100\,\delta_3^{\rm TFF}(Q), \qquad \delta_3^{\rm TFF} = \frac{S^{[3]} - S^{[2]}}{S_x^{[2]}}$$

LO / NLO / NNLO truncation errors: $\{0.11, 0.35, 0.74\}\%$.

- **NNLO: 0.74% truncation error versus $\simeq 0.20\%$ evolution increment**

Conclusions and outlook

- Complete analytic three-loop flavor-nonsinglet nonforward kernels
  - Explicit position-space and momentum-space ERBL and GPD kernels
- Conformal symmetry makes the reconstruction possible
  - Forward spectrum fixes the canonical invariant kernel
  - The two-loop conformal anomaly supplies the CS-to-$\overline{\mathrm{MS}}$ translation
- Three-loop corrections support controlled precision phenomenology
  - LCDA-shape increments remain smaller than the two-loop increments
  - $F_{\gamma\gamma^*\to\pi^0}$ isolated increment below 0.34% at $10\ \mathrm{GeV}$ for the tested inputs
  - Direct x-space evolution removes finite-moment truncation errors that can exceed the three-loop effect
- The full kernel allows predictions at NNLL accuracy for many exclusive processes

  also needed for three-loop CF infrared subtraction and higher-loop kernel constructions

