Intersection Matrix of Feynman Integral

Xing Wang (王 星)

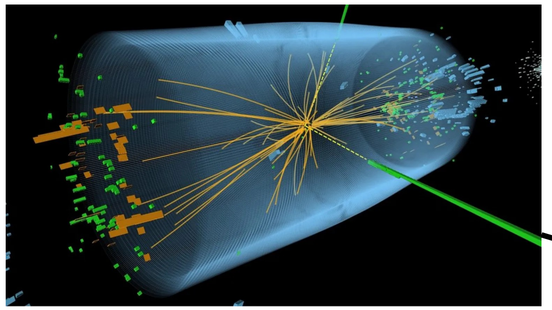
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Based on 2608.03646

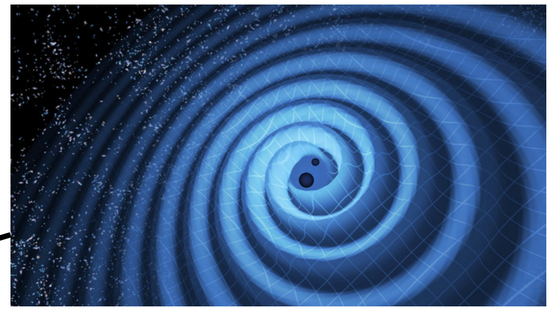
with Iris Bree, Federico Gasparotto, Sebastian Pögel, Stefan Weinzierl and Xiaofeng Xu (徐小峰)

第六届量子场论及其应用研讨会@济南章丘

Motivation in the Era of AI



precision prediction with QFTs



$$O\left(\alpha, \{x\}\right) = \sum_b r_b \, I_b$$

I_b: **Master Integrals**

- LLM+Agents can make the **pipeline** very efficient at two loops or even beyond.
- Bottleneck: **Efficiency** of reduction to MIs and evaluation of MIs analyticall/numerically

new ideas in IBP and MIs evaluation

Differential Equation

Kinematics $(x_1, x_2, \cdots, x_{N_B})$ vary$\longrightarrow$ **differential equations** [Kotikov '91; Remiddi '97; Gehrmann, Remiddi '00].

The **primary** method for analytic and (semi-)numerical calculation of FIs.

$$\mathrm{d}_B = \sum_{i=1}^{N_B} \mathrm{d}x_i \frac{\partial}{\partial x_i} \quad \rightsquigarrow \quad \mathrm{d}_B \begin{pmatrix} I_1 \\ I_2 \\ I_3 \\ \vdots \\ I_{N_F} \end{pmatrix} = \boldsymbol{A}_{N_F \times N_F}(\varepsilon, x) \begin{pmatrix} I_1 \\ I_2 \\ I_3 \\ \vdots \\ I_{N_F} \end{pmatrix}$$

Solving this linearly coupled PDE system is non-trivial!

ε-factorisation (canonicalization)

ε-factorisation:

Via **rotations** of basis (i.e., $\vec{I} \to \overrightarrow{K}$) and **variable change**, ***ε-dependence factorises in the connection matrix, with suitable boundary conditions.*** [Henn '13]

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$$\mathrm{d}_B \begin{pmatrix} K_1 \\ K_2 \\ K_3 \\ \vdots \\ K_{N_F} \end{pmatrix} = \varepsilon \hat{A}_{N_F \times N_F}(\{x\}) \begin{pmatrix} K_1 \\ K_2 \\ K_3 \\ \vdots \\ K_{N_F} \end{pmatrix} \Longrightarrow$$

$$\vec{K} = \sum \varepsilon^n \vec{K}^{(n)} \longrightarrow \vec{K}^{(n+1)} = \int \hat{A}_{N_F \times N_F} \vec{K}^{(n)} + \text{boundary}$$

$$\hookrightarrow \quad \vec{K} = \mathbf{P} \exp\left(\varepsilon \int \hat{A}_{N_F \times N_F} \right) \vec{K}_{\text{boundary}}$$

MIs can be written as Chen's iterated integrals [Chen '77].

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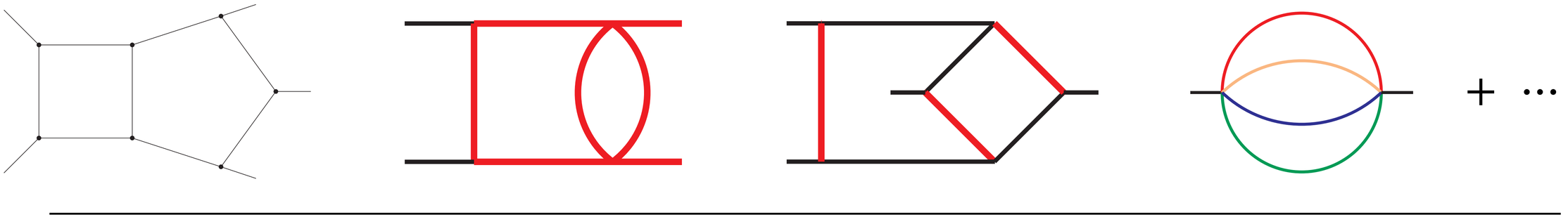
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Once the ε-factorised (canonical, 典则的) form is derived, FIs (MIs) are viewed as solved.

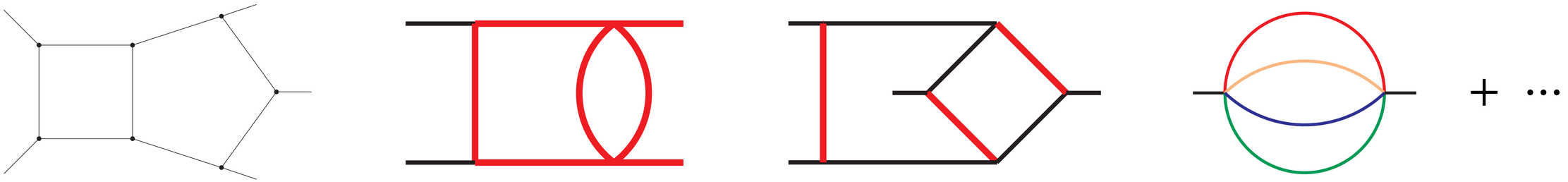
Lots of progress recently, e.g., Baune, Bönisch, Broedel, ε-collaboration, Dlapa, Duhr, Frellesvig, Görges, Henn, Klement, Jiang, Maggio, Nega, Porkert, Sauer, Sohnle, Stawinski, Tancredi, Wager, Wilhelm, Yan, Yang, Zhang, Zhu + many more…

A Universal Framework of Canonicalization



[ε-collaboration, ‘24 - ‘26]

A Universal Framework of Canonicalization

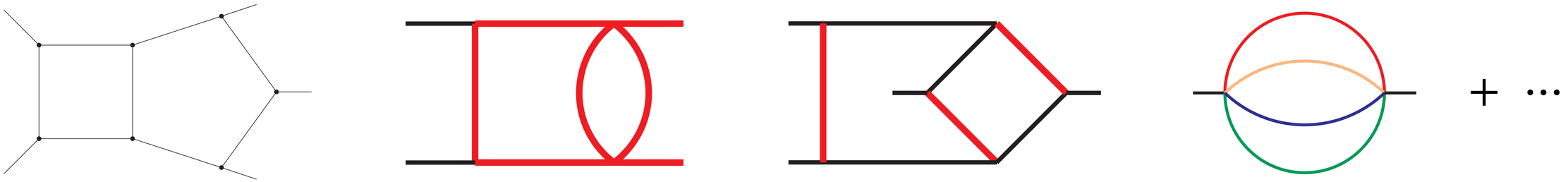


Step I: Any basis $\vec{I}$, filtrations leads to a good basis $\vec{I} = \boldsymbol{R}_1 \cdot \vec{J}$.

$$\mathrm{d}_B \vec{J} = \underbrace{\left[\frac{1}{\varepsilon^{N_V}} \boldsymbol{A}^{(-N_V)}(x) + \cdots + \boldsymbol{A}^{(0)}(x) + \varepsilon \boldsymbol{A}^{(1)}(x)\right]}_{\boldsymbol{A}(\varepsilon, x)} \vec{J}$$

[ε-collaboration, '24 - '26]

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Step II: $\vec{J} = \boldsymbol{R}_2 \cdot \vec{K}$ to get canonical basis $\vec{K}$: $\mathrm{d}_B \vec{K} = \varepsilon \hat{\boldsymbol{A}}\, \vec{K}$

$$\underbrace{\boldsymbol{R}_2^{(-N_V)} \boldsymbol{R}_2^{(-N_V+1)} \cdots \boldsymbol{R}_2^{(0)}}_{\boldsymbol{R}_2} \vec{K} = \vec{J}$$

[ε-collaboration, '24 - '26]

A Universal Framework of Canonicalization

- **Step I:** Both $\boldsymbol{R}_1$ and $\boldsymbol{A}(\varepsilon, x)$ are rational in ε and x;
- **Step II:** $\boldsymbol{R}_2 = \boldsymbol{R}_2^{(-N_V)} \boldsymbol{R}_2^{(-N_V+1)} \cdots \boldsymbol{R}_2^{(0)}$ are made of transcendantal functions.
- They are constrained by special PDEs to remove unwanted ε terms in $\boldsymbol{A}(\varepsilon, x)$.

$$\boldsymbol{R}_2{}^{-1} \cdot \left[\boldsymbol{A}(\varepsilon, x) - \mathrm{d}_B \boldsymbol{R}_2\right] \stackrel{!}{=} \varepsilon \hat{\boldsymbol{A}}(x)$$

$$\text{e.g., } \boldsymbol{R}_2 = \underbrace{\begin{bmatrix} R_{11}^{(0)} & 0 \\ \frac{1}{\epsilon} R_{21}^{(-1)} & R_{22}^{(0)} \end{bmatrix}}_{\boldsymbol{R}_2^{(-1)}} \underbrace{\begin{bmatrix} 1 & 0 \\ R_{21}^{(0)} & 1 \end{bmatrix}}_{\boldsymbol{R}_2^{(0)}}$$

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We want to solve as less PDEs as possible!

A Lesson from Equal-mass Bananas

$$\Theta = q\frac{\mathrm{d}}{\mathrm{d}q}$$

$$
\begin{aligned}
l = 0 &: \quad 1 \\
l = 1 &: \quad \Theta \\
l = 2 &: \quad \Theta \cdot \Theta \\
l = 3 &: \quad \Theta \cdot \Theta \cdot \Theta \\
l = 4 &: \quad \Theta \cdot \Theta \cdot \frac{1}{Y_2} \cdot \Theta \cdot \Theta \\
l = 5 &: \quad \Theta \cdot \Theta \cdot \frac{1}{Y_2} \cdot \Theta \cdot \frac{1}{Y_2} \cdot \Theta \cdot \Theta \\
l = 6 &: \quad \Theta \cdot \Theta \cdot \frac{1}{Y_2} \cdot \Theta \cdot \frac{1}{Y_3} \cdot \Theta \cdot \frac{1}{Y_2} \cdot \Theta \cdot \Theta \\
l = 7 &: \quad \Theta \cdot \Theta \cdot \frac{1}{Y_2} \cdot \Theta \cdot \frac{1}{Y_3} \cdot \Theta \cdot \frac{1}{Y_3} \cdot \Theta \cdot \frac{1}{Y_2} \cdot \Theta \cdot \Theta \\
l = 8 &: \quad \Theta \cdot \Theta \cdot \frac{1}{Y_2} \cdot \Theta \cdot \frac{1}{Y_3} \cdot \Theta \cdot \frac{1}{Y_4} \cdot \Theta \cdot \frac{1}{Y_3} \cdot \Theta \cdot \frac{1}{Y_2} \cdot \Theta \cdot \Theta \\
&\vdots \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \vdots
\end{aligned}
$$

[Pögel, Wang and Weinzierl, '24]

A Lesson from Equal-mass Bananas

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\begin{aligned}
l=0:&\quad 1\\
l=1:&\quad \Theta\\
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l=4:&\quad \Theta\cdot\Theta\cdot\frac{1}{Y_2}\cdot\Theta\cdot\Theta\\
l=5:&\quad \Theta\cdot\Theta\cdot\frac{1}{Y_2}\cdot\Theta\cdot\frac{1}{Y_2}\cdot\Theta\cdot\Theta\\
l=6:&\quad \Theta\cdot\Theta\cdot\frac{1}{Y_2}\cdot\Theta\cdot\frac{1}{Y_3}\cdot\Theta\cdot\frac{1}{Y_2}\cdot\Theta\cdot\Theta\\
l=7:&\quad \Theta\cdot\Theta\cdot\frac{1}{Y_2}\cdot\Theta\cdot\frac{1}{Y_3}\cdot\Theta\cdot\frac{1}{Y_3}\cdot\Theta\cdot\frac{1}{Y_2}\cdot\Theta\cdot\Theta\\
l=8:&\quad \Theta\cdot\Theta\cdot\frac{1}{Y_2}\cdot\Theta\cdot\frac{1}{Y_3}\cdot\Theta\cdot\frac{1}{Y_4}\cdot\Theta\cdot\frac{1}{Y_3}\cdot\Theta\cdot\frac{1}{Y_2}\cdot\Theta\cdot\Theta\\
&\quad\vdots
\end{aligned}
$$

[Pögel, Wang and Weinzierl, '24]

$$
\varepsilon\hat{\boldsymbol{A}} = \varepsilon
\begin{pmatrix}
A_{11} & 1 & 0 & \cdots & 0 & 0 & 0\\
A_{21} & A_{22} & Y_2 & \cdots & 0 & 0 & 0\\
A_{31} & A_{32} & A_{33} & \ddots & 0 & 0 & 0\\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots\\
A_{(l-2)1} & A_{(l-2)2} & A_{(l-2)3} & \cdots & A_{33} & Y_2 & 0\\
A_{(l-1)1} & A_{(l-1)2} & A_{(l-2)2} & \cdots & A_{32} & A_{22} & 1\\
* & A_{(l-1)1} & A_{(l-2)1} & \cdots & A_{31} & A_{21} & A_{11}
\end{pmatrix}
$$

self-duality: $A_{ij} = A_{(l+1-j)(l+1-i)}$

Self-Duality



Constant Intersection Matrix in the Menthined Universal Framework

Algebraic Constraints are simpler than PDEs.

Use them as many as possible.

Pairings and Intersections

$$I_i = \int_\Gamma u(\varepsilon)\,\Psi_i \qquad \Longrightarrow \qquad I_i \;\sim\; \langle \Psi_i | \;\in H^n$$

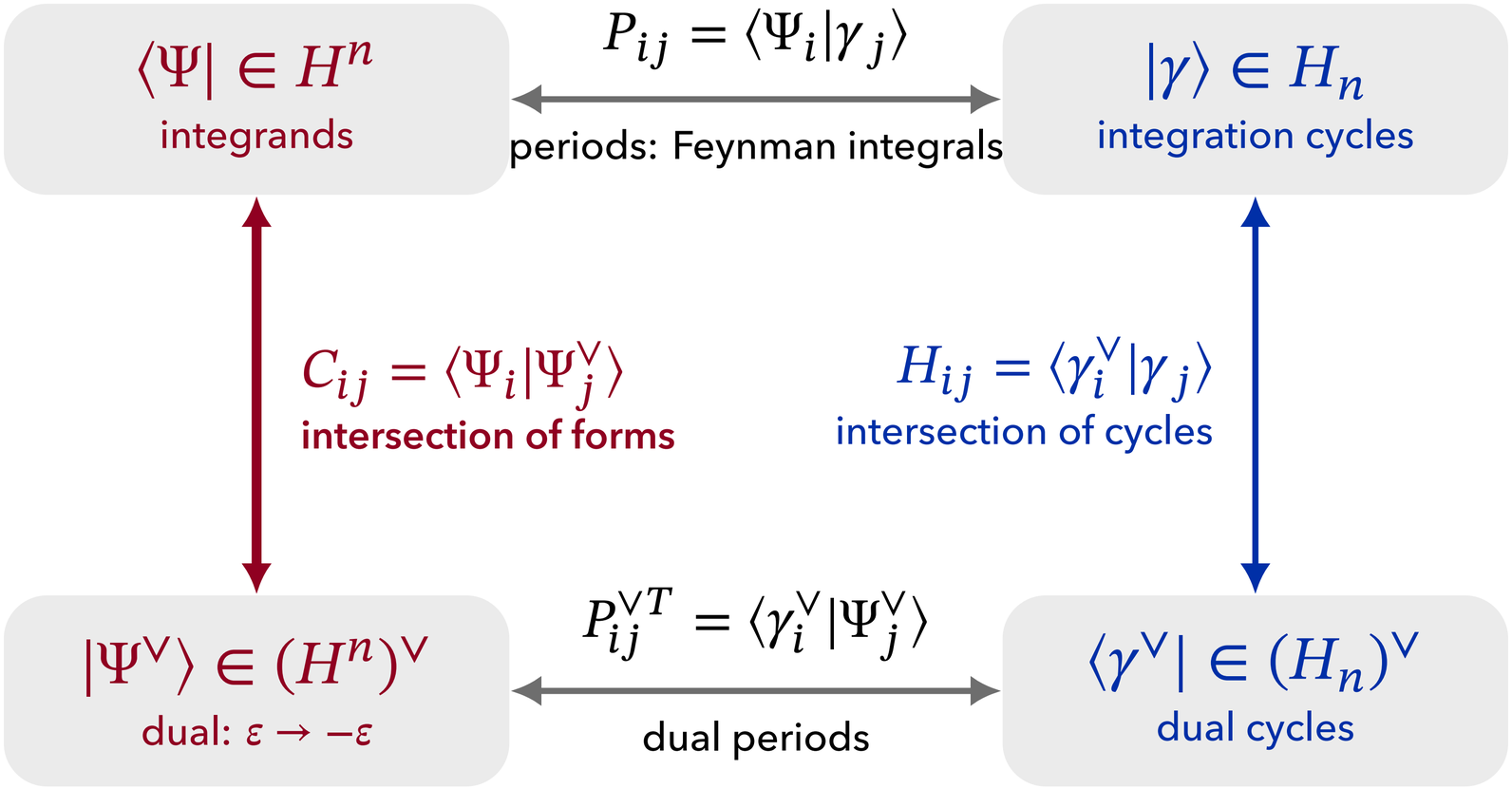
A vector space H^n, with a basis $\{\langle \Psi_i | \}$, has its dual, denoted as $H^{n\vee}$, with a basis $\{ | \Psi_i^\vee \rangle \}$.

$$\langle \Psi_i | \Psi_j^\vee \rangle = \delta_{ij}$$

$$\langle \Psi_i |_{\varepsilon \to -\varepsilon} \qquad \Longleftrightarrow \qquad | \Psi_i^\vee \rangle$$

$$X^\vee(\varepsilon) = X(-\varepsilon)$$

Pairings and Intersections



twisted Riemann bilinear relations: $\boldsymbol{P}\,\boldsymbol{H}^{-1}\,\boldsymbol{P}^{\vee T} = (2\pi i)^n\,\boldsymbol{C} = (2\pi i\varepsilon)^n\,\overline{\boldsymbol{C}}$

Intersection Matrix in Basis $\vec{J}$ and Basis $\overrightarrow{K}$

Basis J (after step 1)

$$\mathrm{d}_B \overline{\boldsymbol{C}}_J = \boldsymbol{A}_J \overline{\boldsymbol{C}}_J + \overline{\boldsymbol{C}}_J \boldsymbol{A}_J^{\vee T}$$

- **rational** in x, from $\boldsymbol{A}_J$ alone
- Laurent polynomial in ε:

$$\overline{\boldsymbol{C}}_J = \sum_{k=k_{\min}}^{0} \varepsilon^k \, \overline{\boldsymbol{C}}_J^{(k)}, \quad k_{\min} \geq -n$$

- parity: $\overline{\boldsymbol{C}}_J^{(k)T} = (-1)^k \, \overline{\boldsymbol{C}}_J^{(k)}$
- $\det \overline{\boldsymbol{C}}_J$ independent of ε

Basis K (after step 2)

$$\overline{\boldsymbol{C}}_K = \boldsymbol{R}_2^{-1} \, \overline{\boldsymbol{C}}_J \left(\boldsymbol{R}_2^{\vee T} \right)^{-1}$$

$$\mathrm{d}_B \overline{\boldsymbol{C}}_K = \varepsilon \left(\hat{\boldsymbol{A}} \, \overline{\boldsymbol{C}}_K - \overline{\boldsymbol{C}}_K \hat{\boldsymbol{A}}^T \right)$$

- for suitable integration constants:

$$\overline{\boldsymbol{C}}_K = \boldsymbol{N} = \boldsymbol{N}^T, \text{ constant (integers!)}$$

- ε factors out entirely:

$$\boldsymbol{C}_K = \varepsilon^{-n} \, \boldsymbol{N}$$

- const. $\boldsymbol{C}$ implies self-duality, see also,

  [Duhr, Porkert, Semper, and Stawinski, '24]

Simplest Example: Equal-mass Sunrise

Equal-mass sunrise: $\overline{\boldsymbol{C}}_J = \alpha \left(\begin{smallmatrix} 0 & 1 \\ 1 & 2\beta \end{smallmatrix}\right)$, with rational α, β and four auxiliary functions.

$$\boldsymbol{R}_2 = \begin{pmatrix} R_{11}^{(0)} & 0 \\ \varepsilon^{-1} R_{21}^{(-1)} & R_{22}^{(0)} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ R_{21}^{(0)} & 1 \end{pmatrix} \quad \Longrightarrow \quad \overline{\boldsymbol{C}}_K = \frac{1}{c} \begin{pmatrix} 0 & 1 \\ 1 & -2d \end{pmatrix}.$$

$\overline{\boldsymbol{C}}_K$ is independent of ε and $R_{21}^{(-1)}$. Its transport equation gives $\mathrm{d}_B \overline{\boldsymbol{C}}_K = 0$: two **first integrals**,

$$c = \frac{R_{11}^{(0)} R_{22}^{(0)}}{\alpha}, \qquad d = R_{21}^{(0)} - \frac{\beta\, R_{11}^{(0)}}{R_{22}^{(0)}}.$$

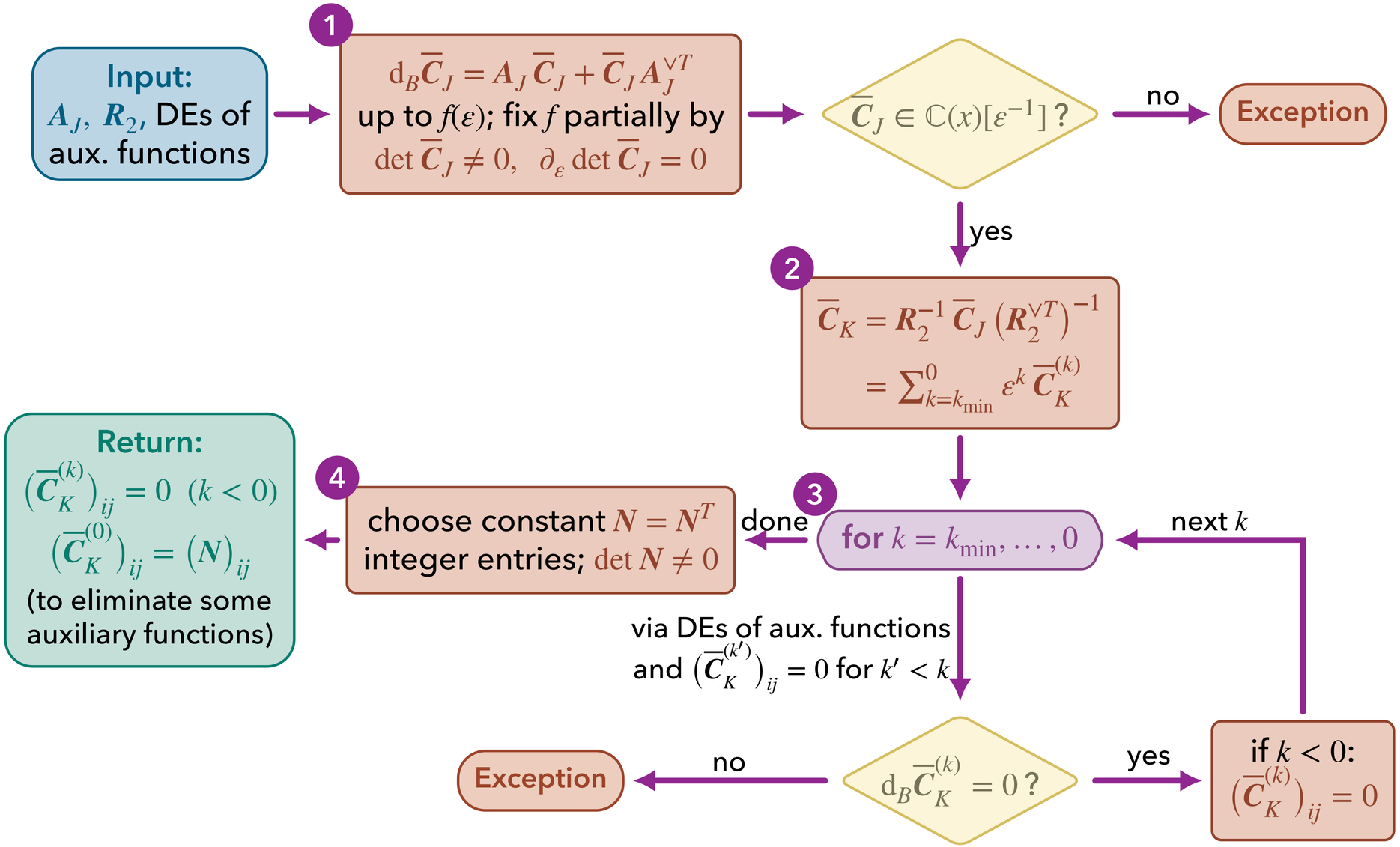
Impose $\overline{\boldsymbol{C}}_K = \boldsymbol{N} = \boldsymbol{S}_2$: $c = 1$, $d = 0$.

$$R_{22}^{(0)} = \alpha / R_{11}^{(0)}, \qquad R_{21}^{(0)} = \beta \left(R_{11}^{(0)}\right)^2 / \alpha.$$

No integration. Only the period $R_{11}^{(0)}$ remains; $R_{21}^{(-1)}$ follows by differentiation.

With $d = 0$, self-duality leaves $c \neq 0$ arbitrary. The prescribed pairing also fixes the **normalisation**.

Algorithm and Algebraic Relations on $R_{2,ij}$'s



Reducing Auxiliary Transcendantal Functions

example	in $\boldsymbol{R}_2$	algebraic	independent functions left
equal-mass sunrise	4	2	the period
non-planar triangle	7	4 (+1)	the period
three-loop equal-mass banana	10	6	$\psi = \phi^2$ and $R_{21}^{(0)}$
unequal-mass sunrise	16	11	ψ_0 and three functions
four-loop banana	20	10	ψ_0 and three functions

- Compute $\overline{\boldsymbol{C}}_J$ from the rational connection $\boldsymbol{A}_J(\varepsilon, x)$;
- Target $\overline{\boldsymbol{C}}_K = \boldsymbol{N}$, starting with the most negative power of ε;
- Replace auxiliary functions by **algebraic relations** from above.

Summary

- A unified framework towards canonicalization of general FIs;
  - Step I is a rational rotation obtained by filtraion;
  - Step II involves transcentandal functions, constrained by PDEs.
- This framework has close relations to intersection matrices.
- Enforcing constant intersection matrix can reduce #PDEs algebraiclly.