A Representation Transformation of Parametric Feynman Integrals

based on: Phys. Lett. B 862 (2025) 139340

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Oct. 17, 2026, Jinan

Representations of Feynman integrals

Polynomial transformation

Representation transformation

The duality

Summary

Representations of Feynman integrals

Momentum representation

$$J = \int \mathrm{d}^d l_1 \mathrm{d}^d l_2 \cdots \mathrm{d}^d l_L \, \frac{1}{D_1^{\lambda_1+1} D_2^{\lambda_2+1} \cdots D_n^{\lambda_n+1}} \tag{1}$$

$D_i = Q_i^2 - m_i^2$, with Q_i linear combinations of loop momenta and external momenta.

Schwinger parametrization

$$\frac{1}{D_i^{\lambda_i+1}} = \frac{e^{-\frac{\lambda_i+1}{2} i\pi}}{\Gamma(\lambda_i+1)} \int_0^\infty dx_i \; e^{i x_i D_i} x_i^{\lambda_i}, \qquad \mathrm{Im}\{D_i\} > 0 \tag{2}$$

$$\begin{aligned} J \to & \int \mathrm{d}x_1 \mathrm{d}x_2 \cdots \mathrm{d}x_n \, x_1^{\lambda_1} x_2^{\lambda_2} \cdots x_n^{\lambda_n} \\ & \times \int \mathrm{d}^d l_1 \mathrm{d}^d l_2 \cdots \mathrm{d}^d l_L \, \exp\left(i \sum_{i=1}^n x_i D_i \right) \end{aligned} \tag{3}$$

Feynman-parameter representation

$$
\begin{aligned}
J \to I =& \frac{\Gamma(-\lambda_f)}{\prod_{i=1}^{n}\Gamma(\lambda_i+1)} \int \mathrm{d}\Pi^{(n)}\, U^{\lambda_u} F^{\lambda_f} \prod_{i=1}^{n} x_i^{\lambda_i} \\
=& \frac{\Gamma(-\lambda_u-\lambda_f)}{\Gamma(-\lambda_u)\prod_{i=1}^{n}\Gamma(\lambda_i+1)} \int \mathrm{d}x_{n+1}\mathrm{d}\Pi^{(n)} \\
& (Ux_{n+1}+F)^{\lambda_u+\lambda_f} x_{n+1}^{-\lambda_u-1} \prod_{i=1}^{n} x_i^{\lambda_i} \\
\equiv& \frac{\Gamma(-\lambda_0)}{\prod_{i=1}^{n+1}\Gamma(\lambda_i+1)} \int \mathrm{d}\Pi^{(n+1)}\, \mathcal{F}^{\lambda_0} \prod_{i=1}^{n+1} x_i^{\lambda_i}
\end{aligned}
\tag{4}
$$

$$
\mathrm{d}\Pi^{(n)} \equiv \mathrm{d}x_1\mathrm{d}x_2\cdots\mathrm{d}x_n\delta\left(1-\mathcal{E}^{(1)}(x)\right),
$$

$$
\lambda_0 = -\frac{d}{2}.
$$

Lee-Pomeransky representation Lee,Pomeransky(2013)

$$
\begin{aligned}
\mathcal{E}^{(1)}(x) &= x_{n+1} \\
G &= \mathcal{F}|_{x_{n+1}=1}
\end{aligned}
\tag{5}
$$

Applications of the parametric representation

- Landau analysis

$$x_i \frac{\partial \mathcal{F}}{\partial x_i} = 0 \tag{6}$$

- Numerical integration
- Direct symbolic integration
- Integral reduction
- Differential-equation method

Predating multi-loop era Regge(1968)

Advantages of the parametric representation

- No auxiliary propagators
- Integrands are Lorentz invariant
- Most symmetric among all representations
- Allowing direct integration

The Baikov representation

$$J \to \int \prod_{i=1}^{n} \mathrm{d}v_n \, P^{-\lambda_0-\frac{1}{2}(L+E+1)} \prod_{i=1}^{n} v_i^{-\lambda_i-1} \tag{7}$$

$$\begin{aligned} v_i =& D_i \,, \\ P =& \mathrm{Det}\left(q_i \cdot q_j\right) \,. \end{aligned}$$

Cuts in the Baikov representation Abreu,Britto,Duhr,Gardi(2017)

$$\frac{1}{v_i} \to \delta(v_i) \to \oint_{v_i=0} \to \mathrm{Res}_{v_i=0} \tag{8}$$

The parametric representation

$$\int \mathrm{d}\Pi^{(n+1)}\, \mathcal{F}^{-\frac{d}{2}} \prod_{i=1}^{n+1} x_i^{\lambda_i} \tag{9}$$

The Baikov representation

$$\int \prod_{i=1}^{n} \mathrm{d}v_n\, P^{\frac{d}{2}-\frac{1}{2}(L+E+1)} \prod_{i=1}^{n} v_i^{-\lambda_i-1} \tag{10}$$

Duality?

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Parametric IBP WC(2020,2021)

$$\left(\mathcal{D}_0 \frac{\partial \mathcal{F}}{\partial \hat{x}_i} - \hat{z}_i\right) \hat{x}_{n+1} I = 0 \tag{11}$$

$$\hat{z}_i I = \int \prod_{i=1}^{n} \mathrm{d}v_i \, v_i P^{-\lambda_0 - \frac{1}{2}(L+E+1)} \prod_{j=1}^{n} v_j^{-\lambda_j - 1} \tag{12}$$

Polynomial transformation WC(2025)

$$u_i \equiv \frac{1}{\mathcal{F}} \frac{\partial \mathcal{F}}{\partial x_i}, \tag{13a}$$

$$\mathcal{G}(u) \equiv \left. \frac{1}{\mathcal{F}} \right|_{x=x(u)}. \tag{13b}$$

The homogeneity of $\mathcal{F}$ gives

$$x_i = \frac{1}{\mathcal{G}} \frac{\partial \mathcal{G}}{\partial u_i}. \tag{14}$$

Multiplicative Legendre transform Etingof,Kazhdan,Polishchuk(2002)

The Baikov polynomial

$$P = -\left. \mathcal{G} \right|_{x_{n+1}=-1} \tag{15}$$

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The w distribution WC(2020)

$$w_\lambda(x) \equiv e^{\frac{\lambda+1}{2}i\pi} \int_{-\infty}^{\infty} \mathrm{d}v \ \frac{1}{(v+i0^+)^{\lambda+1}} e^{-ivx} \ . \tag{16}$$

$$w_{-n}(x) = 2\pi\delta^{(n)}(x)\,, n \in N,$$
$$w_\lambda(x) = 2\pi\theta(x)\frac{x^\lambda}{\Gamma(\lambda+1)},\ \lambda \geq 0\ .$$

Regularized w distribution

$$\tilde{w}_\lambda(x) \equiv \Gamma(\lambda+1) w_\lambda(x) \tag{17}$$

Plus distribution

$$\tilde{w}_{-1}^{(n)}(x) = 2n\pi\theta(x)\left[\frac{\ln^{n-1} x}{x}\right]_+ \tag{18}$$

$$\frac{1}{\Gamma(\lambda+1)}\int_0^\infty \mathrm{d}x\, x^\lambda = \frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{d}x\, w_\lambda(x) \tag{19}$$

$$\begin{aligned}
&\int \mathrm{d}\Pi_x^{(n+1)} \mathcal{F}^{\lambda_0} \prod_{i=1}^{n+1} x_i^{\lambda_i} \\
\to &\int_{-\infty}^{\infty} \mathrm{d}\Pi_v^{(n+1)} \prod_{i=1}^{n+1} v_i^{-\lambda_i-1} \int_{-\infty}^{\infty} \prod_{i=1}^{n+1} \mathrm{d}x_i\ \mathcal{F}^{\lambda_0} \exp\left(-i\sum_{i=1}^{n+1} v_i x_i\right) \\
\to &\int_{-\infty}^{\infty} \mathrm{d}\Pi_v^{(n+1)} \mathcal{G}(v)^{\lambda_0'} \prod_{i=1}^{n+1} v_i^{-\lambda_i'-1} \\
\to &\int \prod_{i=1}^{n} \mathrm{d}v_n\, P^{-\lambda_0-\frac{1}{2}(L+E+1)} \prod_{i=1}^{n} v_i^{-\lambda_i-1}
\end{aligned} \tag{20}$$

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The duality

$\mathcal{G} = \frac{1}{\mathcal{F}}$	$\mathcal{F} = \frac{1}{\mathcal{G}}$
$u = \frac{1}{\mathcal{F}} \frac{\partial \mathcal{F}}{\partial x}$	$x = \frac{1}{\mathcal{G}} \frac{\partial \mathcal{G}}{\partial u}$
$\frac{1}{u^{\lambda+1}} = e^{-\frac{\lambda+1}{2} i\pi} \int_{-\infty}^{\infty} \frac{\mathrm{d}x}{2\pi} \; w_\lambda(x) e^{iux}$	$w_\lambda(x) = e^{\frac{\lambda+1}{2} i\pi} \int_{-\infty}^{\infty} \mathrm{d}u \; \frac{1}{u^{\lambda+1}} e^{-iux}$

Landau equations

$$x = 0 \qquad \leftrightarrow \qquad \frac{\partial \mathcal{G}}{\partial u_i} = 0\,; \tag{21}$$

$$\frac{\partial \mathcal{F}}{\partial x_i} = 0 \qquad \leftrightarrow \qquad u_i = 0\,. \tag{22}$$

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A transformation on homogeneous polynomials is proposed in this work, which is further applied to parametric Feynman integrals. The two representations related through this transformation are dual to each other. It naturally leads to dualities of Landau equations and linear integral relations between the two representations. For integrals with momentum-space correspondences, the dual representation is equivalent to the Baikov representation.

Thanks for your attention.