Strings in grand unified theories

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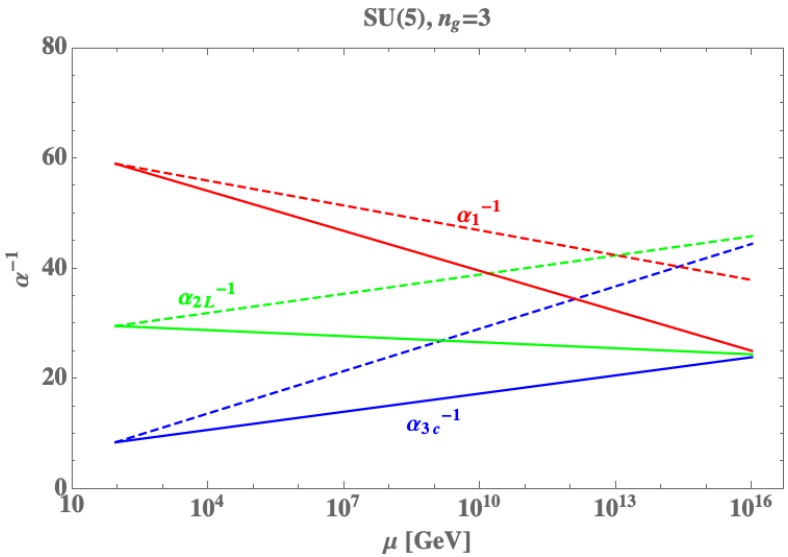
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Historical reviews

- GUTs were proposed in terms of the simple Lie algebras of $\mathfrak{su}(5)$ with $3 \times \left[\overline{\mathbf{5_F}} \oplus \mathbf{10_F}\right]$ by [‘74, Georgi-Glashow] (GG), or $\mathfrak{so}(10)$ with $3 \times \mathbf{16_F}$ by [‘75, Fritzsch-Minkowski]. The main ingredients: (i) local Lie algebras $\mathfrak{g}_{\rm SM} \equiv \mathfrak{su}(3)_c \oplus \mathfrak{su}(2)_W \oplus \mathfrak{u}(1)_Y \subset \mathfrak{g}_{\rm GUT}$, AF of the QCD [‘73, Gross, Wilczek, Politzer], (ii) the two-generational chiral fermions (with charm quark theorized in ‘70 by Glashow-Iliopoulos-Maiani, and discovered in late ‘74).
- The SUSY extension to the $\mathfrak{su}(5)$ can unify three SM gauge couplings at $\mu \sim 10^{16}\,\mathrm{GeV}$ [‘81, Dimopoulos-Georgi], with sparticle masses $\sim \mathcal{O}(1)$ TeV.



Historical reviews

- The chiral fermions in the $\mathfrak{su}(5)$ are decomposed as
$\overline{\mathbf{5_F}} = \underbrace{(\overline{\mathbf{3}}\,,\mathbf{1}\,,+\frac{1}{3})_{\mathbf{F}}}_{{d_R}^c} \oplus \underbrace{(\mathbf{1}\,,\overline{\mathbf{2}}\,,-\frac{1}{2})_{\mathbf{F}}}_{\ell_L}$ and
$\mathbf{10_F} = \underbrace{(\mathbf{3}\,,\mathbf{2}\,,+\frac{1}{6})_{\mathbf{F}}}_{q_L} \oplus \underbrace{(\overline{\mathbf{3}}\,,\mathbf{1}\,,-\frac{2}{3})_{\mathbf{F}}}_{{u_R}^c} \oplus \underbrace{(\mathbf{1}\,,\mathbf{1}\,,+1)_{\mathbf{F}}}_{{e_R}^c}.$
- Besides of the well-acknowledged challenges within the *minimal* GUTs, there are two longstanding problems within the SM that have never been solved w/w.o. the SUSY extension: (i) the SM flavor puzzle, (ii) the strong CP problem.
- The formulation of the QM solved several fundamental puzzles in the late 19^{th} century: (i) blackbody radiation [1900, Max Planck], (ii) photoelectric effects [1905, Albert Einstein], (iii) hydrogen spectrum [1913, Niels Bohr], by hypothesizing the quantized energies/angular momenta of particles.

The flavor puzzle: Yukawa couplings

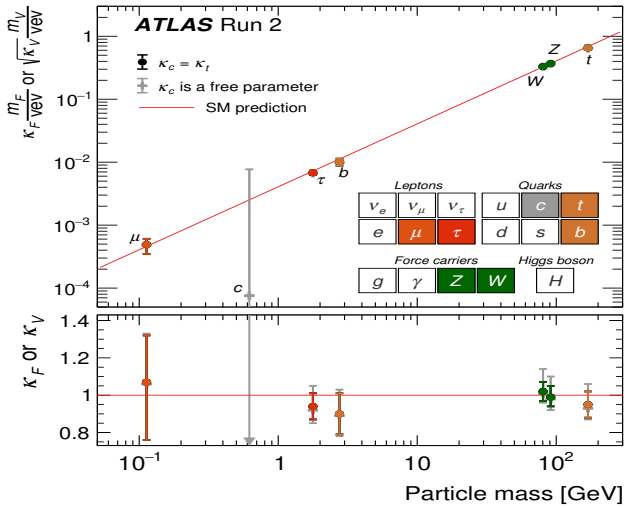


Figure: The LHC measurements of the SM Higgs boson, [2207.00092].

- A natural expectation to the experiments: the hierarchical/non-universal Yukawa couplings of *one single* SM Higgs boson $y_f = \sqrt{2} m_f / v_{\rm EW}$ for all SM quarks/leptons.
- The flavor puzzle for the quarks/leptons: How can one single SM Higgs boson distinguish different generations given their identical irreps?

Three generations w.o. generation

- The main conjecture [‘79, Georgi, ‘80, Nanopolous]: no three repetitive generations in the UV (GUT), but they emerge in the IR (SM). For a unified Lie algebra of $\mathfrak{su}(N)$, with the anti-symmetric chiral fermions of

$$\{f_L\}_{\mathfrak{su}(N)} = \sum_k n_k \, [N\,, k]_{\mathbf{F}} \;, \;\; n_k \in \mathbb{Z}\,. \tag{1}$$

- Georgi's "third law" of GUT [‘79]: no repetition of a particular irrep of $\mathfrak{su}(N)$, i.e., $n_k = 0$ or $n_k = 1$ in Eq. (1). Georgi's minimal solution is

$$\{f_L\}_{\mathfrak{su}(11)} \;=\; [11\,, 4]_{\mathbf{F}} \oplus [11\,, 8]_{\mathbf{F}} \oplus [11\,, 9]_{\mathbf{F}} \oplus [11\,, 10]_{\mathbf{F}} \;, \tag{2}$$

  with $\dim_{\mathbf{F}} = 561$.
- My understanding: no $3 \times \left[\overline{\mathbf{5_F}} \oplus \mathbf{10_F}\right]$ rather than no $3 \times \overline{\mathbf{5_F}} \oplus 3 \times \mathbf{10_F}$?

The $\mathfrak{su}(8)$ theory

- The minimal setup: an $\mathfrak{su}(8)$ theory with rank-2 and rank-3 chiral IRAFFSs of

$$\{f_L\}_{\mathfrak{su}(8)}^{n_g=3} = \left[\overline{\mathbf{8_F}}^{\omega} \oplus \mathbf{28_F}\right] \bigoplus \left[\overline{\mathbf{8_F}}^{\dot{\omega}} \oplus \mathbf{56_F}\right], \ \dim_{\mathbf{F}} = 156\,,$$
$$\Omega = (\omega\,, \dot{\omega})\,, \ \omega = (3\,, \mathrm{IV}\,, \mathrm{V}\,, \mathrm{VI})\,, \ \dot{\omega} = (\dot{\mathrm{I}}\,, \dot{2}\,, \dot{\mathrm{V}}\mathrm{II}\,, \dot{\mathrm{V}}\mathrm{III}\,, \dot{\mathrm{I}}\mathrm{X})\,. \tag{3}$$

- Georgi's decompositions:

$$\begin{aligned}
\overline{\mathbf{8_F}}^{\Omega} &= 3 \times \mathbf{1_F}^{\Omega} \oplus \overline{\mathbf{5_F}}^{\Omega}\,, \\
\mathbf{28_F} &= 3 \times \mathbf{1_F} \oplus 3 \times \mathbf{5_F} \oplus \mathbf{10_F}\,, \\
\mathbf{56_F} &= \mathbf{1_F} \oplus 3 \times \mathbf{5_F} \oplus 3 \times \mathbf{10_F} \oplus \overline{\mathbf{10_F}}\,.
\end{aligned} \tag{4}$$

Six $(\mathbf{5_F}\,, \overline{\mathbf{5_F}})$ pairs, one $(\mathbf{10_F}\,, \overline{\mathbf{10_F}})$ pair from the $\mathbf{56_F}$, and $3 \times \left[\overline{\mathbf{5_F}} \oplus \mathbf{10_F}\right]_{\mathrm{SM}}$.

The $\mathfrak{su}(8)$ theory: global symmetries

- The global symmetries of the $\mathfrak{su}(8)$ theory:

$$\begin{aligned}
&\widetilde{\mathcal{G}}_{\rm global}\left[\mathfrak{su}(8)\right] = \left[\widetilde{\rm SU}(4)_{\omega}\otimes\widetilde{\rm U}(1)_{T_2}\otimes\widetilde{\rm U}(1)_{{\rm PQ}_2}\right]\\
&\bigotimes\left[\widetilde{\rm SU}(5)_{\dot{\omega}}\otimes\widetilde{\rm U}(1)_{T_3}\otimes\widetilde{\rm U}(1)_{{\rm PQ}_3}\right],\\
&\left[\mathfrak{su}(8)\right]^2\cdot\widetilde{\rm U}(1)_{T_2\,,3}=0\,,\quad \left[\mathfrak{su}(8)\right]^2\cdot\widetilde{\rm U}(1)_{{\rm PQ}_2\,,3}\neq 0\,.
\end{aligned}\tag{5}$$

- The Higgs fields and the Yukawa couplings:

$$\begin{aligned}
-\mathcal{L}_Y \;=\;& Y_{\mathcal{B}}\overline{\mathbf{8_F}}^{\omega}\mathbf{28_F}\overline{\mathbf{8_H}}_{\,,\omega}+Y_{\mathcal{T}}\mathbf{28_F}\mathbf{28_F}\mathbf{70_H}\\
&+\;Y_{\mathcal{D}}\overline{\mathbf{8_F}}^{\dot{\omega}}\mathbf{56_F}\overline{\mathbf{28_H}}_{\,,\dot{\omega}}+\frac{c_4}{M_{\rm pl}}\mathbf{56_F}\mathbf{56_F}\overline{\mathbf{28_H}}^{\dagger}_{\,,\dot{\omega}}\mathbf{63_H}+H.c.\,.
\end{aligned}\tag{6}$$

  NB: $\mathbf{56_F}\mathbf{56_F}\mathbf{28_H}=0$ ['08, S. Barr], $d=5$ operator suppressed by $1/M_{\rm pl}$ is possible, with $M_{\rm pl}=(8\pi G_N)^{1/2}=2.4\times10^{18}\,{\rm GeV}$.

- *Gravity breaks global symmetries.*

Symmetry breaking pattern in the $\mathfrak{su}(8)$ theory

- One possible (maximally) symmetry breaking pattern [‘74, L.F.Li] of $\mathfrak{su}(8) \to \mathfrak{g}_{441} \to \mathfrak{g}_{341} \to \mathfrak{g}_{331} \to \mathfrak{g}_{\rm SM} \to \mathfrak{su}(3)_c \oplus \mathfrak{u}(1)_{\rm EM}$.

Higgs	$\mathfrak{g}_{441} \to \mathfrak{g}_{341}$	$\mathfrak{g}_{341} \to \mathfrak{g}_{331}$	$\mathfrak{g}_{331} \to \mathfrak{g}_{\rm SM}$	$\mathfrak{g}_{\rm SM} \to \mathfrak{su}(3)_c \oplus \mathfrak{u}(1)_{\rm EM}$
$\overline{\mathbf{8_H}}_{,\omega}$	✓$(\omega = \mathrm{IV})$	✓$(\omega = \mathrm{V})$	✓$(\omega = 3\,, \mathrm{VI})$	✓
$\overline{\mathbf{28_H}}_{,\dot\omega}$	✗	✓$(\dot\omega = \dot 1\,,$ $\dot{\mathrm{V}}\mathrm{II})$	✓$(\dot\omega = \dot 2\,, \dot{\mathrm{V}}\mathrm{III}\,,$ $\dot{\mathrm{I}}\mathrm{X})$	✓
$\mathbf{70_H}$	✗	✗	✗	✓

Conjecture: the $(\mathbf{1}\,, \overline{\mathbf{2}}\,, +\frac{1}{2})'''_{\mathbf{H}} \subset \mathbf{70_H}$ is the only SM Higgs doublet.

Main results of the $\mathfrak{su}(8)$ theory

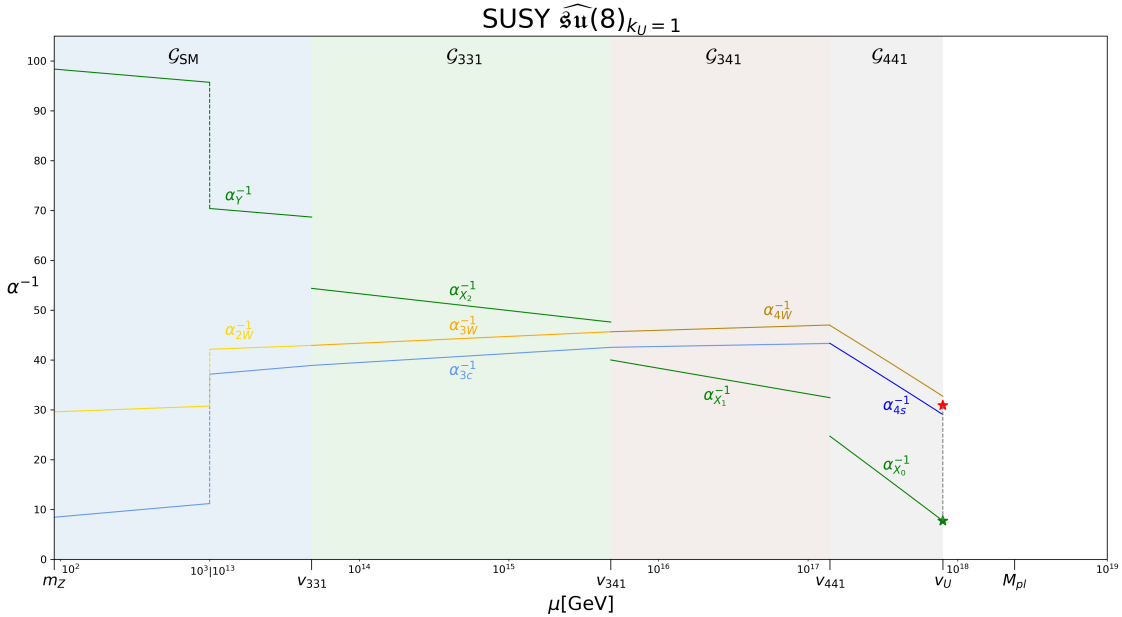
$\mathfrak{su}(8)$	$\mathfrak{g}_{441}$	$\mathfrak{g}_{341}$	$\mathfrak{g}_{331}$	$\mathfrak{g}_{\rm SM}$
$\mathbf{8}_{\mathbf{F}}^{-11}$	$(\mathbf{4},\mathbf{1},+\frac{1}{4})_{\mathbf{F}}^{\Omega}$	$(\mathbf{3},\mathbf{1},+\frac{1}{3})_{\mathbf{F}}^{\Omega}$	$(\mathbf{3},\mathbf{1},+\frac{1}{3})_{\mathbf{F}}^{\Omega}$	$(\mathbf{3},\mathbf{1},+\frac{1}{3})_{\mathbf{F}}^{\Omega}$: $\mathcal{D}_R^{\Omega c}$
		$(\mathbf{1},\mathbf{1},0)_{\mathbf{F}}^{\Omega}$	$(\mathbf{1},\mathbf{1},0)_{\mathbf{F}}^{\Omega}$	$(\mathbf{1},\mathbf{1},0)_{\mathbf{F}}^{\Omega}$: $\mathcal{N}_L^{\Omega}$
	$(\mathbf{1},\bar{\mathbf{4}},-\frac{1}{4})_{\mathbf{F}}^{\Omega}$	$(\mathbf{1},\bar{\mathbf{3}},-\frac{1}{3})_{\mathbf{F}}^{\Omega}$	$(\mathbf{1},\bar{\mathbf{3}},-\frac{1}{3})_{\mathbf{F}}^{\Omega}$	$(\mathbf{1},\bar{\mathbf{2}},-\frac{1}{2})_{\mathbf{F}}^{\Omega}$: $\mathcal{L}_L^{\Omega} = (\mathcal{E}_L^{\Omega}, -\mathcal{N}_L^{\Omega})^T$
				$(\mathbf{1},\mathbf{1},0)_{\mathbf{F}}^{\Omega'}$: $\mathcal{N}_L^{\Omega'}$
			$(\mathbf{1},\mathbf{1},0)_{\mathbf{F}}^{\Omega''}$	$(\mathbf{1},\mathbf{1},0)_{\mathbf{F}}^{\Omega''}$: $\mathcal{N}_L^{\Omega''}$

$\mathfrak{su}(8)$	$\mathfrak{g}_{441}$	$\mathfrak{g}_{341}$	$\mathfrak{g}_{331}$	$\mathfrak{g}_{\rm SM}$
$\mathbf{28}_{\mathbf{F}}$	$(\mathbf{6},\mathbf{1},-\frac{1}{2})_{\mathbf{F}}$	$(\mathbf{3},\mathbf{1},-\frac{1}{3})_{\mathbf{F}}$	$(\mathbf{3},\mathbf{1},-\frac{1}{3})_{\mathbf{F}}$	$(\mathbf{3},\mathbf{1},-\frac{1}{3})_{\mathbf{F}}$: $\mathfrak{D}_L$
		$(\bar{\mathbf{3}},\mathbf{1},-\frac{2}{3})_{\mathbf{F}}$	$(\bar{\mathbf{3}},\mathbf{1},-\frac{2}{3})_{\mathbf{F}}$	$(\bar{\mathbf{3}},\mathbf{1},-\frac{2}{3})_{\mathbf{F}}$: $t_R{}^c$
	$(\mathbf{1},\mathbf{6},+\frac{1}{2})_{\mathbf{F}}$	$(\mathbf{1},\mathbf{6},+\frac{1}{2})_{\mathbf{F}}$	$(\mathbf{1},\mathbf{3},+\frac{1}{3})_{\mathbf{F}}$	$(\mathbf{1},\mathbf{2},+\frac{1}{2})_{\mathbf{F}}$: $(e_R{}^c, \mathfrak{n}_R{}^c)^T$
				$(\mathbf{1},\mathbf{1},0)_{\mathbf{F}}$: $\mathfrak{n}_R$
			$(\mathbf{1},\bar{\mathbf{3}},+\frac{2}{3})_{\mathbf{F}}$	$(\mathbf{1},\bar{\mathbf{2}},+\frac{1}{2})'_{\mathbf{F}}$: $(\mathfrak{n}'_R{}^c, -e'_R{}^c)^T$
				$(\mathbf{1},\mathbf{1},+1)_{\mathbf{F}}$: $\tau_R{}^c$
	$(\mathbf{4},\mathbf{4},0)_{\mathbf{F}}$	$(\mathbf{3},\mathbf{4},-\frac{1}{12})_{\mathbf{F}}$	$(\mathbf{3},\mathbf{3},0)_{\mathbf{F}}$	$(\mathbf{3},\mathbf{2},+\frac{1}{6})_{\mathbf{F}}$: $(t_L, b_L)^T$
				$(\mathbf{3},\mathbf{1},-\frac{1}{3})'_{\mathbf{F}}$: $\mathfrak{D}'_L$
			$(\mathbf{3},\mathbf{1},-\frac{1}{3})''_{\mathbf{F}}$	$(\mathbf{3},\mathbf{1},-\frac{1}{3})''_{\mathbf{F}}$: $\mathfrak{D}''_L$
		$(\mathbf{1},\mathbf{4},+\frac{1}{4})_{\mathbf{F}}$	$(\mathbf{1},\mathbf{3},+\frac{1}{3})''_{\mathbf{F}}$	$(\mathbf{1},\mathbf{2},+\frac{1}{2})''_{\mathbf{F}}$: $(e''_R{}^c, \mathfrak{n}''_R{}^c)^T$
				$(\mathbf{1},\mathbf{1},0)'_{\mathbf{F}}$: $\mathfrak{n}'_R{}^c$
			$(\mathbf{1},\mathbf{1},0)''_{\mathbf{F}}$	$(\mathbf{1},\mathbf{1},0)''_{\mathbf{F}}$: $\mathfrak{n}''_R{}^c$

$\mathfrak{su}(8)$	$\mathfrak{g}_{441}$	$\mathfrak{g}_{341}$	$\mathfrak{g}_{331}$	$\mathfrak{g}_{\rm SM}$
$\mathbf{56}_{\mathbf{F}}$	$(\mathbf{1},\bar{\mathbf{4}},+\frac{3}{4})_{\mathbf{F}}$	$(\mathbf{1},\bar{\mathbf{4}},+\frac{3}{4})_{\mathbf{F}}$	$(\mathbf{1},\bar{\mathbf{3}},+\frac{2}{3})'_{\mathbf{F}}$	$(\mathbf{1},\bar{\mathbf{2}},+\frac{1}{2})'''_{\mathbf{F}}$: $(\mathfrak{n}'''_R{}^c, -e'''_R{}^c)^T$
				$(\mathbf{1},\mathbf{1},+1)'_{\mathbf{F}}$: $\mu_R{}^c$
			$(\mathbf{1},\mathbf{1},+1)''_{\mathbf{F}}$	$(\mathbf{1},\mathbf{1},+1)''_{\mathbf{F}}$: $\mathfrak{E}_R{}^c$
	$(\bar{\mathbf{4}},\mathbf{1},-\frac{3}{4})_{\mathbf{F}}$	$(\bar{\mathbf{3}},\mathbf{1},-\frac{2}{3})'_{\mathbf{F}}$	$(\bar{\mathbf{3}},\mathbf{1},-\frac{2}{3})'_{\mathbf{F}}$	$(\bar{\mathbf{3}},\mathbf{1},-\frac{2}{3})'_{\mathbf{F}}$: $u_R{}^c$
		$(\mathbf{1},\mathbf{1},-1)_{\mathbf{F}}$	$(\mathbf{1},\mathbf{1},-1)_{\mathbf{F}}$	$(\mathbf{1},\mathbf{1},-1)_{\mathbf{F}}$: $\mathfrak{E}_L$
	$(\mathbf{4},\mathbf{6},+\frac{1}{4})_{\mathbf{F}}$	$(\mathbf{3},\mathbf{6},+\frac{1}{6})_{\mathbf{F}}$	$(\mathbf{3},\mathbf{3},0)'_{\mathbf{F}}$	$(\mathbf{3},\mathbf{2},+\frac{1}{6})'_{\mathbf{F}}$: $(c_L, s_L)^T$
				$(\mathbf{3},\mathbf{1},-\frac{1}{3})''_{\mathbf{F}}$: $\mathfrak{D}''_L$
			$(\mathbf{3},\bar{\mathbf{3}},+\frac{1}{3})_{\mathbf{F}}$	$(\mathbf{3},\bar{\mathbf{2}},+\frac{1}{6})''_{\mathbf{F}}$: $(\mathfrak{d}_L, -\mathfrak{u}_L)^T$
				$(\mathbf{3},\mathbf{1},+\frac{2}{3})_{\mathbf{F}}$: $\mathfrak{U}_L$
		$(\mathbf{1},\mathbf{6},+\frac{1}{2})'_{\mathbf{F}}$	$(\mathbf{1},\mathbf{3},+\frac{1}{3})'_{\mathbf{F}}$	$(\mathbf{1},\mathbf{2},+\frac{1}{2})'''_{\mathbf{F}}$: $(\epsilon'''_R{}^c, \mathfrak{n}'''_R{}^c)^T$
				$(\mathbf{1},\mathbf{1},0)''_{\mathbf{F}}$: $\mathfrak{n}''_R{}^c$
			$(\mathbf{1},\bar{\mathbf{3}},+\frac{2}{3})''_{\mathbf{F}}$	$(\mathbf{1},\bar{\mathbf{2}},+\frac{1}{2})''''_{\mathbf{F}}$: $(\mathfrak{n}''''_R{}^c, -\epsilon''''_R{}^c)^T$
				$(\mathbf{1},\mathbf{1},+1)''_{\mathbf{F}}$: $e_R{}^c$
	$(\mathbf{6},\mathbf{4},-\frac{1}{4})_{\mathbf{F}}$	$(\mathbf{3},\mathbf{4},-\frac{1}{12})'_{\mathbf{F}}$	$(\mathbf{3},\mathbf{3},0)''_{\mathbf{F}}$	$(\mathbf{3},\mathbf{2},+\frac{1}{6})''_{\mathbf{F}}$: $(u_L, d_L)^T$
				$(\mathbf{3},\mathbf{1},-\frac{1}{3})'''_{\mathbf{F}}$: $\mathfrak{D}'''_L$
			$(\mathbf{3},\mathbf{1},-\frac{1}{3})''''_{\mathbf{F}}$	$(\mathbf{3},\mathbf{1},-\frac{1}{3})''''_{\mathbf{F}}$: $\mathfrak{D}''''_L$
		$(\bar{\mathbf{3}},\mathbf{4},-\frac{5}{12})_{\mathbf{F}}$	$(\bar{\mathbf{3}},\mathbf{3},-\frac{1}{3})_{\mathbf{F}}$	$(\bar{\mathbf{3}},\mathbf{2},-\frac{1}{6})_{\mathbf{F}}$: $(\mathfrak{d}_R{}^c, \mathfrak{u}_R{}^c)^T$
				$(\bar{\mathbf{3}},\mathbf{1},-\frac{2}{3})'_{\mathbf{F}}$: $\mathfrak{U}_R{}^c$
			$(\bar{\mathbf{3}},\mathbf{1},-\frac{2}{3})''_{\mathbf{F}}$	$(\bar{\mathbf{3}},\mathbf{1},-\frac{2}{3})'''_{\mathbf{F}}$: $c_R{}^c$

- Results: the precise flavor identifications lead to the observed quark/lepton mass hierarchies and the mixing pattern, with one single SM Higgs boson in the spectrum.
  `https://arxiv.org/abs/2307.07921` `https://arxiv.org/abs/2402.10471`
  `https://arxiv.org/abs/2406.09970` `https://arxiv.org/abs/2411.12979`
  Talk: `https://www.koushare.com/live/details/40893`

The RGE of the $\mathcal{N}=1$ $\widehat{\mathfrak{su}}(8)_{k_U=1}$ theory



- In the $\widehat{\mathfrak{su}}(8)_{k_U=1}$ theory [2411.12979], we find the conformal embedding of $\widehat{\mathfrak{su}}(4)_{k_s=1} \oplus \widehat{\mathfrak{su}}(4)_{k_W=1} \oplus \widehat{\mathfrak{u}}(1)_{k_1=1/4}$. The gauge coupling unification is achieved through the relation of

$$g_{4s}^2 = g_{4W}^2 = \frac{1}{4} g_{X_0}^2 = \frac{8\pi G_N}{\alpha'} \,. \tag{7}$$

What's Next?

- The BSM scales are too high for any upcoming terrestrial experiments.
- Usually, the GUT contains 't Hooft-Polyakov monopole with $\mathrm{SU}(5) \to \mathrm{SU}(3)_c \otimes \mathrm{SU}(2)_W \otimes \mathrm{U}(1)_Y$, and the domain walls with discrete $\mathbb{Z}_2$ in the Higgs sector.
  Q: Do we have strings in the $\mathrm{SU}(8)$ GUT?
- Usually, strings arise in the breaking of global/local $\mathrm{U}(1) \to \emptyset$, Landau-Ginzburg and the Abelian-Higgs model. What type of strings are possible in the non-Abelian theories?
- Next, I will consider the $\mathrm{SU}(6) \to \mathcal{G}_{331}$ as a toy model, with $2 \times \overline{\mathbf{6_F}} \oplus \mathbf{15_F}$, and two $\Phi_{\overline{\mathbf{3}},\omega} \equiv (\mathbf{1}, \overline{\mathbf{3}}, +\frac{1}{3})_{\mathbf{H},\omega} \subset \overline{\mathbf{6_H}}_{,\omega}$. To study the classical stability of non-topological/topological strings, developed by Hindmarsh, Vachaspati, Dvali, and etc. Based on: `https://arxiv.org/abs/2604.06530` and `https://arxiv.org/abs/2609.27791`.

The embedded Z string in the SM

- The semilocal string: to consider a $\mathrm{U}(1) \otimes \widetilde{\mathrm{SU}}(2)$ model with $\Phi \equiv (\phi_1\,, \phi_2)^T \in \mathbb{C}$, [1991-1992, Hindmarsh, Vachaspati].
- Promote to the local $\mathrm{SU}(2)_W \otimes \mathrm{U}(1)_Y$, with the string profiles of

$$\Phi_{\mathrm{ANO}} = \frac{v}{\sqrt{2}} \bar{f}(r) e^{in\varphi} \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \vec{Z} = -n \frac{\bar{z}(r)}{r} \vec{e}_\varphi\,, \tag{8}$$

  and the integer-quantized magnetic flux of

$$\Phi_Z^{\mathrm{non-topo}} = \int d^2x\, Z_{12} = -\frac{4\pi n}{g_Z}\,, \quad g_Z^2 \equiv g_{2W}^2 + g_Y^2\,. \tag{9}$$

  $n \in \mathbb{Z}$ is known as the winding number.
- For more generic local $\mathrm{SU}(N) \otimes \mathrm{U}(1)$ theory, the integer-quantized magnetic flux of

$$\Phi_{Z'}^{\mathrm{non-topo}} = \int d^2x\, \mathcal{Z}'_{12} = -\frac{2\pi N}{g_{Z'}} n\,. \tag{10}$$

The classical stability of the Z string in the SM [cont.]

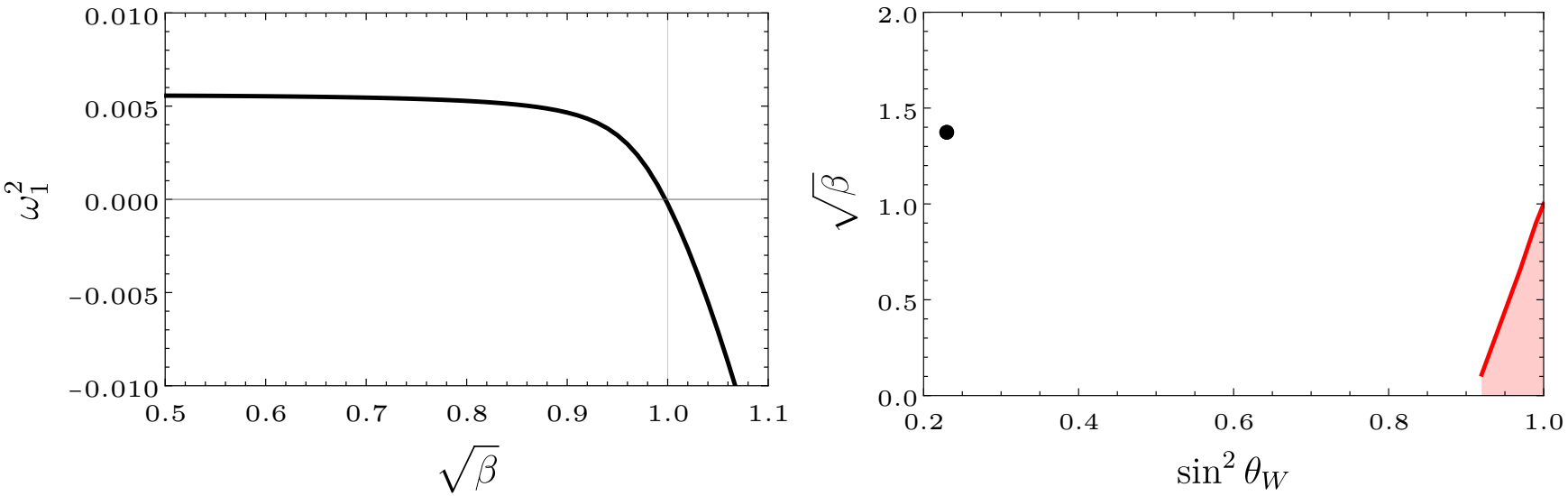


Figure: The stability region of the EW Z string: the semilocal limit of $\vartheta_W \to \frac{\pi}{2}$ and $\beta \equiv \frac{m_H^2}{m_Z^2} < 1$. Plot credit: Eto, Hamada, Jinno, Nitta, and Yamada, 2402.19471.

The Z' string in the $\mathcal{G}_{331} \to \mathcal{G}_{\rm SM}$

- The massive off-diagonal $\mathcal{G}_{331}$ gauge bosons of $(W'^{\pm}_{\mu}, N_{\mu}, \bar{N}_{\mu})$

$$W'^{\pm}_{\mu} \equiv \frac{1}{\sqrt{2}}(W^4_{\mu} \mp i W^5_{\mu}), \quad N_{\mu}/\bar{N}_{\mu} \equiv \frac{1}{\sqrt{2}}(W^6_{\mu} \mp i W^7_{\mu}), \tag{11}$$

  and the diagonal $\mathcal{G}_{331}$ gauge boson

$$\begin{aligned} Z'_{\mu} &\equiv c_{\vartheta_S} W^8_{\mu} - s_{\vartheta_S} X_{\mu}, \\ B_{\mu} &\equiv s_{\vartheta_S} W^8_{\mu} + c_{\vartheta_S} X_{\mu}. \end{aligned} \tag{12}$$

- The masses of

$$\begin{aligned} & m^2_{W'^{\pm}} = m^2_{N,\bar{N}} = \frac{1}{12} g^2_{Z'} c^2_{\vartheta_S} v^2_{331}, \quad m^2_{Z'} = \frac{1}{9} g^2_{Z'} v^2_{331} \\ & g^2_{Z'} \equiv 3 g^2_{3W} + g^2_X . \end{aligned} \tag{13}$$

The Z' string in the $\mathcal{G}_{331} \to \mathcal{G}_{\rm SM}$

- The non-topological $(1\,,1)$ Z' string: two Higgs profile functions, both with unit windings of

$$\vec{Z}'_{\rm ANO} = -\frac{\bar{\zeta}(r)}{r}\vec{e}_{\varphi}\,, \quad \Phi_{\overline{\mathbf{3}}\,,\omega} = \frac{V_\omega}{\sqrt{2}}\bar{f}_\omega(r)e^{i\varphi}\begin{pmatrix}0\\0\\1\end{pmatrix}\,, \quad \omega = (1\,,2)\,, \tag{14}$$

  and the integer-quantized magnetic flux

$$\Phi_{Z'}^{\rm non-topo} = \int d^2x\,\partial_{[1}Z'_{{\rm ANO}\,2]} = \frac{6\pi}{g_{Z'}}\,, \tag{15}$$

  with the BCs: $\bar{\zeta}(\xi=0)=0$ and $\bar{\zeta}(\xi=\infty)=-\frac{3}{g_{Z'}}$.

- The generic Higgs potential with two Higgs triplets

$$\begin{aligned} V(\Phi_\omega) = &\sum_\omega \frac{1}{2}\lambda_\omega\left(|\Phi_{\overline{\mathbf{3}}\,,\omega}|^2 - \frac{1}{2}V_\omega^2\right)^2 + \frac{1}{2}\lambda_3\left(\sum_\omega|\Phi_{\overline{\mathbf{3}}\,,\omega}|^2 - \frac{1}{2}\bar{V}^2\right)^2 \\ &+ \lambda_4\left(|\Phi_{\overline{\mathbf{3}}\,,1}|^2|\Phi_{\overline{\mathbf{3}}\,,2}|^2 - |\Phi^\dagger_{\overline{\mathbf{3}}\,,1}\Phi_{\overline{\mathbf{3}}\,,2}|^2\right) + \lambda_5\left|\Phi^\dagger_{\overline{\mathbf{3}}\,,1}\Phi_{\overline{\mathbf{3}}\,,2} - \frac{1}{2}V_1V_2\right|^2. \end{aligned} \tag{16}$$

The non-topological Z' string in the $\mathcal{G}_{331} \to \mathcal{G}_{\rm SM}$

- The string tension with the dimensionless coordinates of $\xi \equiv \frac{m_{Z'}}{\sqrt{2}} r$

$$
\begin{aligned}
\frac{\mu_{\rm ANO}}{2\pi v_{331}^2} &= \int \xi d\xi\, \rho(\xi)\,, \quad \rho(\xi) = \rho_{\bar{\zeta}}(\xi) + \rho_{\bar{f}_\omega}(\xi) + \rho_V(\xi)\,, \\
\rho_{\bar{\zeta}}(\xi) &= \frac{g_{Z'}^2}{36}\left(\frac{1}{\xi}\frac{d\bar{\zeta}(\xi)}{d\xi}\right)^2\,, \\
\rho_{\bar{f}_\omega}(\xi) &= \frac{1}{2}\sum_{\omega=1\,,2}\frac{V_\omega^2}{v_{331}^2}\left[\left(\frac{d\bar{f}_\omega(\xi)}{d\xi}\right)^2 + \left(1+\frac{g_{Z'}}{3}\bar{\zeta}(\xi)\right)^2\left(\frac{\bar{f}_\omega(\xi)}{\xi}\right)^2\right], \\
\rho_V(\xi) &= \frac{1}{4}\beta_1 c_{\tilde{\beta}}^4\left(\bar{f}_1^2(\xi)-1\right)^2 + \frac{1}{4}\beta_2 s_{\tilde{\beta}}^4\left(\bar{f}_2^2(\xi)-1\right)^2 \\
&\quad + \frac{1}{4}\beta_3\left(c_{\tilde{\beta}}^2\bar{f}_1^2(\xi) + s_{\tilde{\beta}}^2\bar{f}_2^2(\xi) - 1\right)^2 + \frac{1}{2}\beta_5 s_{\tilde{\beta}}^2 c_{\tilde{\beta}}^2\left(\bar{f}_1(\xi)\bar{f}_2(\xi)-1\right)^2\,, \\
&\quad \text{where} \quad \beta_i \equiv \frac{9\lambda_i}{g_{Z'}^2}\,.
\end{aligned}
\tag{17}
$$

The non-topological Z' string in the $\mathcal{G}_{331} \to \mathcal{G}_{\rm SM}$

- The classical field equations

$$\frac{d^2\bar{f}_1(\xi)}{d\xi^2}+\frac{1}{\xi}\frac{d\bar{f}_1(\xi)}{d\xi}-\left(1+\frac{g_{Z'}}{3}\bar{\zeta}(\xi)\right)^2\frac{\bar{f}_1(\xi)}{\xi^2}=\beta_1 c_{\tilde{\beta}}^2\left(\bar{f}_1^2(\xi)-1\right)\bar{f}_1(\xi)$$
$$+\beta_3\left(c_{\tilde{\beta}}^2\bar{f}_1^2(\xi)+s_{\tilde{\beta}}^2\bar{f}_2^2(\xi)-1\right)\bar{f}_1(\xi)\,, \tag{18a}$$

$$\frac{d^2\bar{f}_2(\xi)}{d\xi^2}+\frac{1}{\xi}\frac{d\bar{f}_2(\xi)}{d\xi}-\left(1+\frac{g_{Z'}}{3}\bar{\zeta}(\xi)\right)^2\frac{\bar{f}_2(\xi)}{\xi^2}=\beta_2 s_{\tilde{\beta}}^2\left(\bar{f}_2^2(\xi)-1\right)\bar{f}_2(\xi)$$
$$+\beta_3\left(c_{\tilde{\beta}}^2\bar{f}_1^2(\xi)+s_{\tilde{\beta}}^2\bar{f}_2^2(\xi)-1\right)\bar{f}_2(\xi)\,, \tag{18b}$$

$$\frac{d^2\bar{\zeta}(\xi)}{d\xi^2}-\frac{1}{\xi}\frac{d\bar{\zeta}(\xi)}{d\xi}=\frac{6}{g_{Z'}}\left(1+\frac{g_{Z'}}{3}\bar{\zeta}(\xi)\right)\left(c_{\tilde{\beta}}^2\bar{f}_1^2(\xi)+s_{\tilde{\beta}}^2\bar{f}_2^2(\xi)\right)\,, \tag{18c}$$

with BCs of: $\bar{f}_\omega(\xi=0)=\bar{\zeta}(\xi=0)=0$, $\bar{f}_\omega(\xi=\infty)=1$, and $\bar{\zeta}(\xi=\infty)=-\frac{3}{g_{Z'}}$.

The non-topological Z' string in the $\mathcal{G}_{331} \to \mathcal{G}_{\rm SM}$

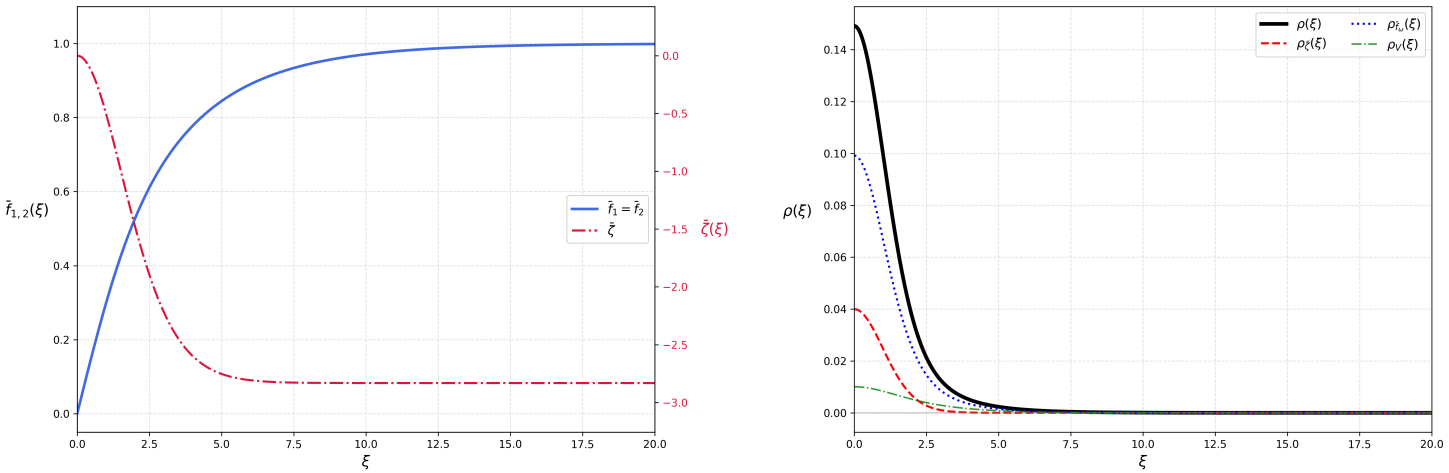


Figure: The parameters are: $\lambda_1 = \lambda_2 = \lambda_5 = 0.15$, $\lambda_3 = -0.145$, $t_{\tilde{\beta}} = 1.0$, and $g_{Z'} = 1.06$.

$\rho(\xi \to \infty) \to 0$, guarantees that the non-topological string tension is finite.

The non-topological Z' string in the $\mathcal{G}_{331} \to \mathcal{G}_{\rm SM}$

Remarks on the non-topological Z' string profile with two Higgs triplets:

- At $\xi \to \infty$, the asymptotic behaviors for the Higgs profiles read

$$\bar{f}_1(\xi) \sim 1 - \frac{1}{\sqrt{\xi}} \left(C_1 \exp\left(-\sqrt{\beta_1 + 2\beta_3 + \beta_5}\xi \right) + C_2 \exp\left(-\sqrt{\beta_1}\xi \right) \right) ,$$
$$\bar{f}_2(\xi) \sim 1 - \frac{1}{\sqrt{\xi}} \left(C_1 \exp\left(-\sqrt{\beta_1 + 2\beta_3 + \beta_5}\xi \right) - C_2 \exp\left(-\sqrt{\beta_1}\xi \right) \right) , \quad (19)$$

  with $t_{\tilde{\beta}} = 1.0$, $\beta_1 = \beta_2$.

- A more stringent Higgs potential parameter constraints: $\beta_3 > -\frac{1}{2}(\beta_1 + \beta_5)$ or $\beta_3 > -\frac{1}{2}(\beta_2 + \beta_5)$ for the classical non-topological string solution, compared to the stability constraints.

The non-topological Z' string in the $\mathcal{G}_{331} \to \mathcal{G}_{\rm SM}$

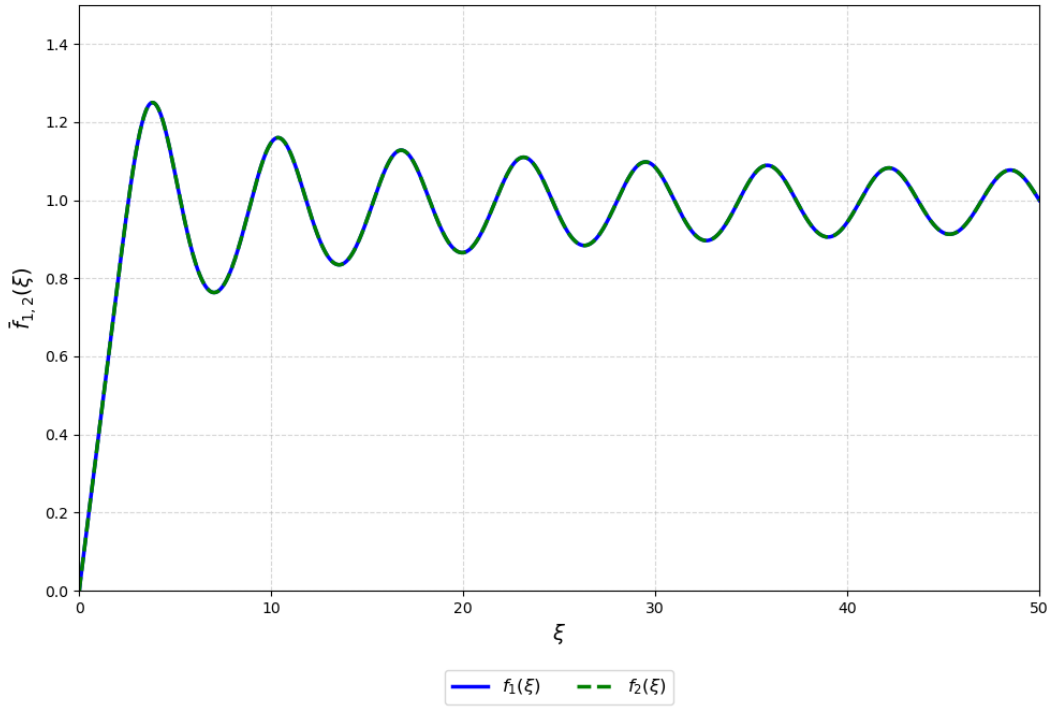


Figure: Unwanted parameters of: $\lambda_1 = \lambda_2 = 0.15$, $\lambda_3 = -0.14$, $\lambda_4 = \lambda_5 = 0$, $t_{\tilde{\beta}} = 1.0$, and $g_{Z'} = 1.06$. Credit to: GUO Mian.

The classical stability of the non-topological Z' string

The classical stability analyses:

- To impose the perturbations to the Z' string background

$$\Phi_\omega = \begin{pmatrix} \delta\pi^{-}_{W',\omega} \\ \delta\pi^{0}_{\bar{N},\omega} \\ \phi_{\mathrm{ANO},\omega} \end{pmatrix}, \quad (\delta W'^{\pm}_m, \delta N_m, \delta\bar{N}_m). \tag{20}$$

  The perturbations to the $\mathrm{U}(1)_Y$-neutral components are unnecessary.
- The perturbed Z' string tension expansion to the quadratic terms

$$\mu = \mu_{\mathrm{ANO}} + \delta\mu_{(2)}, \quad \delta\mu_{(2)} = \delta\mu_{W'} + \delta\mu_\pi + \delta\mu_c. \tag{21}$$

- The gauge fixing terms

$$\begin{aligned} \mathcal{L}_{\mathrm{fix}} &= F(\delta W'^{+}_m)\cdot F(\delta W'^{-}_m), \\ F(\delta W'^{\pm}_m) &= \frac{m_{Z'}}{\sqrt{2}}\left(\vec{\nabla}_\xi\cdot\delta\vec{W}'^{\pm} \mp \frac{ig_{Z'}c^2_{\vartheta_S}}{2}\frac{\bar{\zeta}(\xi)}{\xi}(s_\varphi\delta W'^{+}_{\xi_1} - c_\varphi\delta W'^{+}_{\xi_2})\right. \\ &\quad \left. \pm ic_{\vartheta_S}\sqrt{\frac{3}{2}}\sum_\omega\frac{V_\omega}{v_{331}}\bar{f}_\omega(\xi)e^{\pm i\varphi}\delta\pi^{\pm}_{W',\omega}\right). \end{aligned} \tag{22}$$

The classical stability of the non-topological Z' string [cont.]

- Both the $(\delta\pi^{\pm}_{W'\,,\omega}\,,\delta W'^{\pm}_{m})$ and the $(\delta\pi^{0}_{N\,,\omega}\,,\delta\pi^{0}_{\bar{N}\,,\omega}\,,\delta N_{m}\,,\delta\bar{N}_{m})$ lead to the identical results for the (in)stabilities.
- An eigenvalue problem for the coupled Helmholtz equations in the basis of $\delta\Theta\equiv\left(s_{\ell\,,1}\,,s_{\ell\,,2}\,,w_{\uparrow\,,\ell},w_{\downarrow\,,\ell}\right)^{T}$

$$\hat{\mathcal{O}}=\begin{pmatrix}\mathcal{D}_{11} & \mathcal{D}_{12} & \mathcal{B}_{\uparrow} & \mathcal{B}_{\downarrow}\\ \mathcal{D}_{12} & \mathcal{D}_{22} & \mathcal{B}_{\uparrow} & \mathcal{B}_{\downarrow}\\ \mathcal{B}_{\uparrow} & \mathcal{B}_{\uparrow} & \mathcal{D}_{\uparrow} & 0\\ \mathcal{B}_{\downarrow} & \mathcal{B}_{\downarrow} & 0 & \mathcal{D}_{\downarrow}\end{pmatrix}\,, \tag{23}$$

with $\delta\pi^{+}_{W'\,,\omega}=\sum_{\ell}s_{\ell\,,\omega}e^{-i\ell\varphi}$, $\delta W'^{+}_{\uparrow}=\sum_{\ell}-iw_{\uparrow\,,\ell}e^{-i\ell\varphi}$, and etc.

The classical stability of the non-topological Z' string [cont.]

Sources of the non-topological Z' string instability:

1. The instabilities due to the Higgs profile

$$\mathcal{D}_{11} = -\vec{\nabla}_\xi^2 + \frac{3c_{\vartheta_S}^2}{2} c_{\tilde{\beta}}^2 \bar{f}_1^2(\xi) \underbrace{+\beta_1 c_{\tilde{\beta}}^2 (\bar{f}_1^2(\xi) - 1)}_{<0(\xi\to 0)\ \text{with}\ \beta_1>0} \underbrace{+\beta_3 \left(c_{\tilde{\beta}}^2 (\bar{f}_1^2(\xi) - 1) + s_{\tilde{\beta}}^2 (\bar{f}_2^2(\xi) - 1) \right)}_{>0(\xi\to 0)\ \text{with}\ \beta_3<0} + \underbrace{\beta_4 s_{\tilde{\beta}}^2 \bar{f}_2^2(\xi)}_{>0\ ,\xi\to 0}, \tag{24}$$

2. The spin-magnetic couplings of $\mathcal{D}_{\uparrow/\downarrow} = -\vec{\nabla}_\xi^2 + \frac{3c_{\vartheta_S}^2}{2} \sum_\omega \bar{f}_\omega^2 \frac{V_\omega^2}{v_{331}^2} \pm g_{Z'} c_{\vartheta_S}^2 \frac{1}{\xi} \frac{\partial \bar{\zeta}}{\partial \xi}$, inevitable due to the cubic gauge self-couplings in the non-Abelian theories.
3. In the semilocal limit of $\vartheta_S \to \frac{\pi}{2}$, there can still be off-diagonal elements of $\mathcal{D}_{12}$ due to the mixings between $\delta\pi_{W',1}^{\pm}$ and $\delta\pi_{W',2}^{\mp}$ from the perturbed Higgs potential as well as the gauge fixing terms.

The classical stability of the non-topological Z' string [cont.]

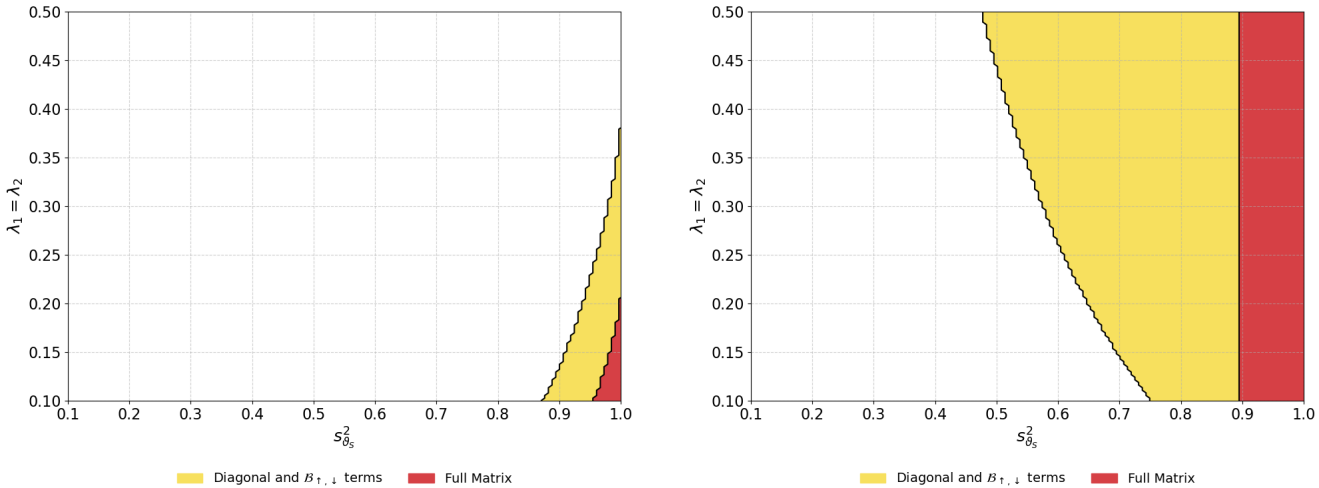


Figure: The stability regions with a fixed $\lambda_3 = -0.05$ (left panel) and a varying $\lambda_3 = -\frac{1}{2}(\lambda_1 + \lambda_5) + 0.005$ (right panel), with $g_{Z'} = 1.06$. Left panel: $\lambda_4 = \lambda_5 = 0.1$, $t_{\tilde{\beta}} = 2$ in the left panel, Right panel: $\lambda_1 = \lambda_2 = \lambda_5$, $\lambda_4 = 0$, $t_{\tilde{\beta}} = 1$. Subject to the constraint of $\lambda_3 > -\frac{1}{2}(\lambda_1 + \lambda_5)$.

Result: the stable regions point to the semilocal limit of $\vartheta_S \to \frac{\pi}{2}$, unrealistic for the GUT.

The topological Z' string in the $\mathcal{G}_{331} \to \mathcal{G}_{\rm SM}$

- The global $\widetilde{\rm U}(1)$ symmetry between two Higgs fields in the $\mathcal{G}_{331}$: $(\Phi_{\overline{\mathbf{3}},1}\,,\Phi_{\overline{\mathbf{3}},2}) \to (e^{-i\alpha}\Phi_1\,,e^{i\alpha}\Phi_2)$ in Eq. (16) w.o. the λ_5 term. Similar to the situation in the 2HDM [Dvali, Senjanovic, 9305278].
- The topological $(n_1\,,n_2)$ Z' string profiles (with $n_{1,2} \in \mathbb{Z}$ and $n_1 \neq n_2$)

$$\Phi_{\overline{\mathbf{3}},1} = \frac{V_1}{\sqrt{2}}\bar{f}_1(r)e^{in_1\varphi}\begin{pmatrix}0\\0\\1\end{pmatrix}\,,\quad \Phi_{\overline{\mathbf{3}},2} = \frac{V_2}{\sqrt{2}}\bar{f}_2(r)e^{in_2\varphi}\begin{pmatrix}0\\0\\1\end{pmatrix}\,, \tag{25}$$

  and the minimal structures being $(n_1\,,n_2) = (\pm 1\,,0)$ or $(n_1\,,n_2) = (0\,,\pm 1)$ strings.
- The non-integer magnetic flux of the Z' string

$$\Phi_{Z'}^{(n_1\,,n_2)} = \frac{6\pi}{g_{Z'}}\left(c_{\tilde{\beta}}^2 n_1 + s_{\tilde{\beta}}^2 n_2\right)\,, \tag{26}$$

  $\Phi_{Z'}^{(1\,,0)} = \frac{1}{2}\Phi_{Z'}^{\rm non-topo}$ in Eq. (15) when $\tilde{\beta} = \frac{\pi}{4}$.

The topological Z' string in the $\mathcal{G}_{331} \to \mathcal{G}_{\rm SM}$

- The classical field equations

$$\frac{d^2\bar{f}_1(\xi)}{d\xi^2}+\frac{1}{\xi}\frac{d\bar{f}_1(\xi)}{d\xi}-\left(n_1+\frac{g_{Z'}}{3}\bar{\zeta}(\xi)\right)^2\frac{\bar{f}_1(\xi)}{\xi^2}=\beta_1 c_{\tilde{\beta}}^2\left(\bar{f}_1^2(\xi)-1\right)\bar{f}_1(\xi)$$
$$+\beta_3\left(c_{\tilde{\beta}}^2\bar{f}_1^2(\xi)+s_{\tilde{\beta}}^2\bar{f}_2^2(\xi)-1\right)\bar{f}_1(\xi)\,, \tag{27a}$$

$$\frac{d^2\bar{f}_2(\xi)}{d\xi^2}+\frac{1}{\xi}\frac{d\bar{f}_2(\xi)}{d\xi}-\left(n_2+\frac{g_{Z'}}{3}\bar{\zeta}(\xi)\right)^2\frac{\bar{f}_2(\xi)}{\xi^2}=\beta_2 s_{\tilde{\beta}}^2\left(\bar{f}_2^2(\xi)-1\right)\bar{f}_2(\xi)$$
$$+\beta_3\left(c_{\tilde{\beta}}^2\bar{f}_1^2(\xi)+s_{\tilde{\beta}}^2\bar{f}_2^2(\xi)-1\right)\bar{f}_2(\xi)\,, \tag{27b}$$

$$\frac{d^2\bar{\zeta}(\xi)}{d\xi^2}-\frac{1}{\xi}\frac{d\bar{\zeta}(\xi)}{d\xi}=\frac{6}{g_{Z'}}\left[c_{\tilde{\beta}}^2\left(n_1+\frac{g_{Z'}}{3}\bar{\zeta}(\xi)\right)\bar{f}_1^2(\xi)\right.$$
$$\left.+s_{\tilde{\beta}}^2\left(n_2+\frac{g_{Z'}}{3}\bar{\zeta}(\xi)\right)\bar{f}_2^2(\xi)\right], \tag{27c}$$

BCs for the $(n_1\,,0)$ configuration: $\bar{f}_1(\xi=0)=\bar{\zeta}(\xi=0)=0$, $\frac{d\bar{f}_2}{d\xi}(\xi=0)=0$, and $\bar{f}_\omega(\xi=\infty)=1$, $\bar{\zeta}(\xi=\infty)=-\frac{3}{g_{Z'}}n_1c_{\tilde{\beta}}^2$.

The topological Z' string in the $\mathcal{G}_{331} \to \mathcal{G}_{\rm SM}$

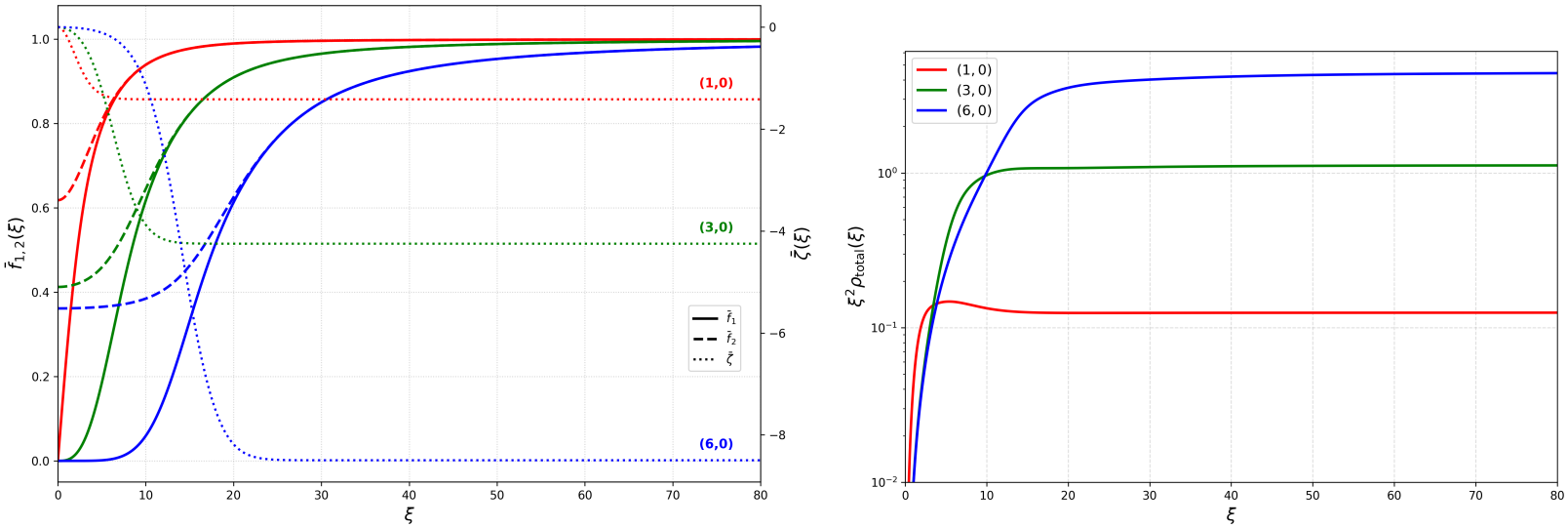


Figure: The profiles (left panel) and energy densities (right panel) for the topological $(1\,,0)$, $(3\,,0)$, and $(6\,,0)$ Z' string with $\lambda_1 = \lambda_2 = \lambda_3 = 0.15$, $\lambda_5 = 0$, $g_{Z'} = 1.06$, and $t_{\tilde{\beta}} = 1.0$.

The $\mathcal{G}_{331}$ symmetry not full restored at the string core due to the Neumann BC of the $\bar{f}_2(\xi)$, and $\mu \sim \log \Lambda$. [2609.27791]

Stable topological Z' string

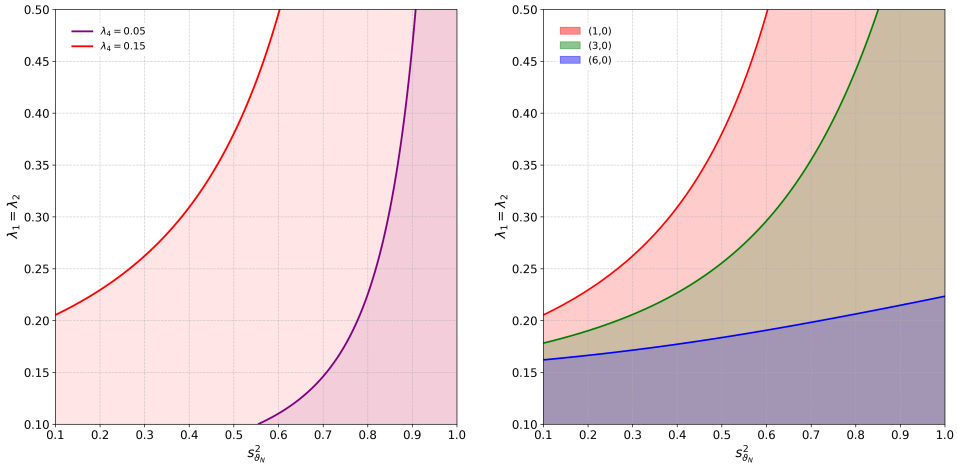


Figure: Left: the stable regions for the $(1\,,0)$ string with different λ_4. Right: the stable regions for different winding configurations with $\lambda_4 = 0.15$. Other parameters: $\lambda_3 = -0.5\lambda_1 + 0.005$, $g_{Z'} = 1.06$.

Remarks on the strings in the $\mathcal{G}_{331}$

- The non-topological and topological strings are classified by $(n_1 \neq 0\,, n_2 \neq 0)$ or $(n_1\,, 0)/(0\,, n_2)$ winding configurations.
- For non-topological string: the instability is mainly due to the spin-magnetic couplings of $\mathcal{D}_\uparrow \supset \frac{1}{\xi}\frac{\partial \bar{\zeta}}{\partial \xi}$, with the BC of $\bar{\zeta}(\xi = \infty) = -\frac{3}{g_{Z'}} n_1$ for the $(n_1\,, n_1)$ winding configuration. Variations of the Higgs potential parameters do not extend the viable ranges of $s^2_{\theta_S}$.
- For topological string: (i) the larger inputs of λ_4 enhances the parameter regions of the $s^2_{\vartheta_S}$ for a stable string, (ii) strings with the larger winding configurations tend to be unstable, due to the spin-magnetic couplings of $\mathcal{D}_\uparrow \supset \frac{1}{\xi}\frac{\partial \bar{\zeta}}{\partial \xi}$ and the BC of $\bar{\zeta}(\xi = \infty) = -\frac{3}{g_{Z'}} n_1 c^2_{\tilde{\beta}}$. Reasonable mixing angles of $0.1 \lesssim s^2_{\vartheta_S} \lesssim 1.0$ are allowed.

Summary

- We propose an $\mathfrak{su}(8)$ theory to address the SM flavor puzzle. Different generations transform differently in the UV theory and their repetitive structure only emerge in the IR, which lead to the flavor non-universality of SM quarks/leptons.
- Our construction relaxes Georgi's "third law" in 1979, and we avoid the repetitions of one IRAFFS.
- All SM quark/leptons are flavor non-universal under the extended strong/weak symmetries. All SM quark/lepton masses, as well as the CKM mixing pattern can be quantitatively recovered with $\mathcal{O}(0.1) - \mathcal{O}(1)$ direct Yukawa couplings and $\mathcal{O}(0.01)$ Higgs mixing coefficients.
- One possible maximally symmetry-breaking pattern of the $\mathfrak{su}(8) \to \mathfrak{g}_{441} \to ... \to \mathfrak{g}_{\rm SM}$ is described. Multiple Higgs fields are responsible for the intermediate symmetry-breaking stages, making them potential candidates to the topological string configuration (due to the global $\widetilde{\rm U}(1)$).

Summary

- Non-topological strings with the $(n_1\,, n_1)$ winding cannot be stable unless with the semilocal limit of $\vartheta_S \to \frac{\pi}{2}$, even if one tunes the Higgs potential parameters.
- Topological strings with the $(n_1\,, 0)$ windings due to the (relative) global $\widetilde{\mathrm{U}}(1)$'s in the Higgs potential, very likely to be stable. The explicitly $\widetilde{\mathrm{U}}(1)$-breaking terms should be included, and may lead to the domain-wall string composite [2311.15805, and in progress].
- Future topics: the classical stability of the string-wall composites with more than two Higgs fields, the stability with Josephson couplings, the fermions in the string background ['81, Jackiw-Rossi], the string superconductivity ['85, Witten].