

Transgression Completion for Chern–Simons Black Hole Thermodynamics and Quantum Criticality

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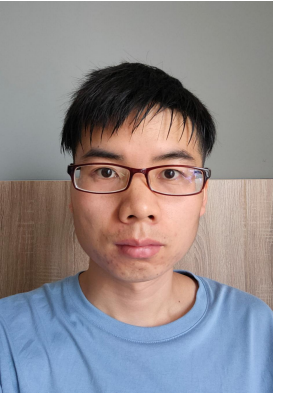
References:

Transgression Completion for Chern-Simons Black Hole Thermodynamics,

Rong-Gen Cai, Li Li, Jun-Kun Zhao, arXiv: 2410.00717

Universally diverging Gruneisen ratio of holographic quantum criticality,

Jun-Kun Zhao, Enze Lv, Wei Li, Li Li, Rept.Prog.Phys. 89 (2026) 9, 098001



Outline

- **Introduction**
- **Thermodynamics for CS black holes**
- **Application: quantum criticality**
- **Summary**

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- **Introduction**
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The black holes of nature are the most perfect macroscopic objects there are in the universe: the only elements in their construction are our concepts of space and time.



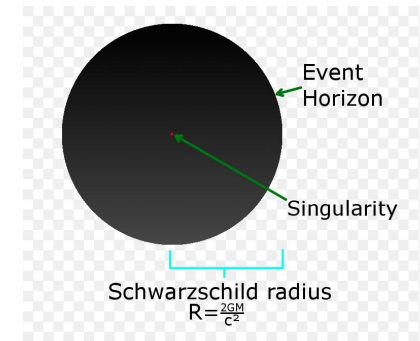
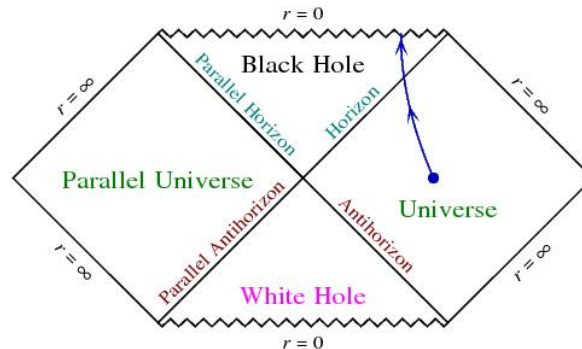
--Subrahmanyan Chandrasekhar

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--Subrahmanyan Chandrasekhar

$$ds^2 = - \left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\Omega^2$$



The Universality : black holes could be the thermodynamic limit of underlying quantum gravitational degrees of freedom.

Classical and semi-classical properties of black holes are expected to provide important clues about the nature of **quantum gravity (QG)**.

✦ Significant obstacle for **QG**:

Absence of any experimental or observational result!

✦ The only **test**

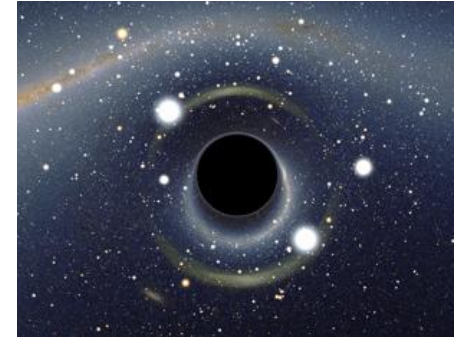
The theoretical and mathematical consistency!



The understanding of the fundamental laws of black hole mechanics could be a necessary (if not sufficient) constraint on the theory of **QG**.



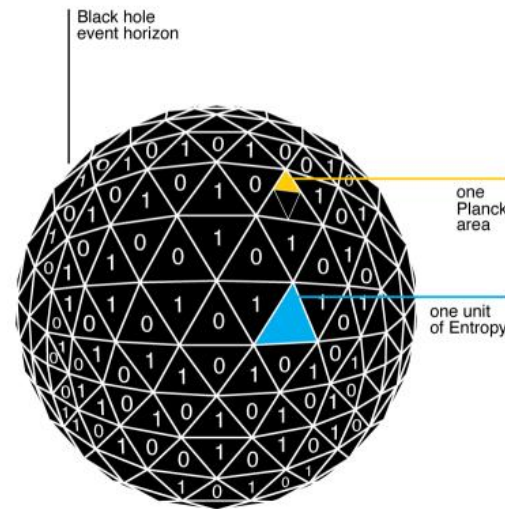
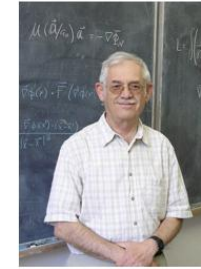
➤ Four laws of black hole mechanics



Law	Thermodynamics	Stationary black holes
Zeroth	T constant on a body in thermal equilibrium	κ constant on the horizon of a black hole
First	$dE = T dS - p dV + \mu dN$	$dM = \frac{\kappa}{8\pi G} dA + \Omega_H dJ + \Phi dq$
Second	$dS \geq 0$	$dA \geq 0$
Third	$T = 0$ cannot be reached	$\kappa = 0$ cannot be reached

J.M.Bardeen, B.Carter and S.W.Hawking, Commun.Math.Phys.31, 161 (1973)

Quantum effect: black hole is a thermodynamic system



$$S_{BH} = \frac{k_B c^3}{4 \hbar G} A$$

Area Law: The black hole entropy is proportional to the **area of its event horizon** divided by a quarter of Planck area

$$l_P^2 = \frac{\hbar G}{c^3} \sim 2.61223 \times 10^{-70} \text{ m}^2$$

- ◆ Black hole thermodynamics: a very deep and fundamental relationship between **gravitation**, **thermodynamics** and **quantum theory**!
- ◆ To unify GR and quantum mechanism is one of the greatest challenges of modern Physics !

Quantum Gravity ?

Quantum gravity defined by path integral [Gibbons and Hawking, 1977]

$$Z = \int \mathcal{D}g e^{iS_L[g]} \xrightarrow[t_E = -iS_L]{t_E = it} Z = \int \mathcal{D}g e^{-S_E[g]} \quad t_E \sim t_E + 1/T$$

semi-classical approximation

$$Z \simeq e^{-S_E}$$

Quantum statistical relation (QSR)

[Gibbons, Perry, Pope, 2005]

$$W = -T \ln Z = T S_E$$

Quantum gravity defined by path integral [Gibbons and Hawking, 1977]

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Quantum statistical relation (QSR)

[Gibbons, Perry, Pope, 2005]

$$W = -T \ln Z = T S_E$$

A **standard framework** for discussing the thermodynamics of black holes in textbooks.

Schwarzschild BH:

$$W = \frac{M}{2G} = \frac{1}{16\pi T} \quad S = -\partial_T W = \frac{1}{16\pi T^2} = \frac{A}{4G}$$

$$E = W + TS = \frac{1}{8\pi T} = M$$

Significant applications: Hawking-Page transition, PV critically, applied holography...

QSR yields thermodynamic potential consistent with the laws of thermodynamics. Remarkable and **by no means obvious**, particularly for gravitational systems involving matter fields.

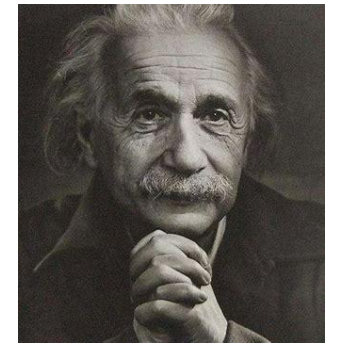


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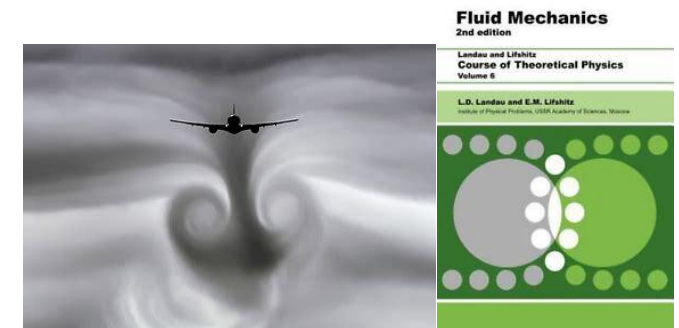


Two universal theories: Thermodynamics and Hydrodynamics

“[Thermodynamics is] the only physical theory of universal content concerning which I am convinced that, within the framework of the applicability of its basic concepts, it will never be overthrown.”

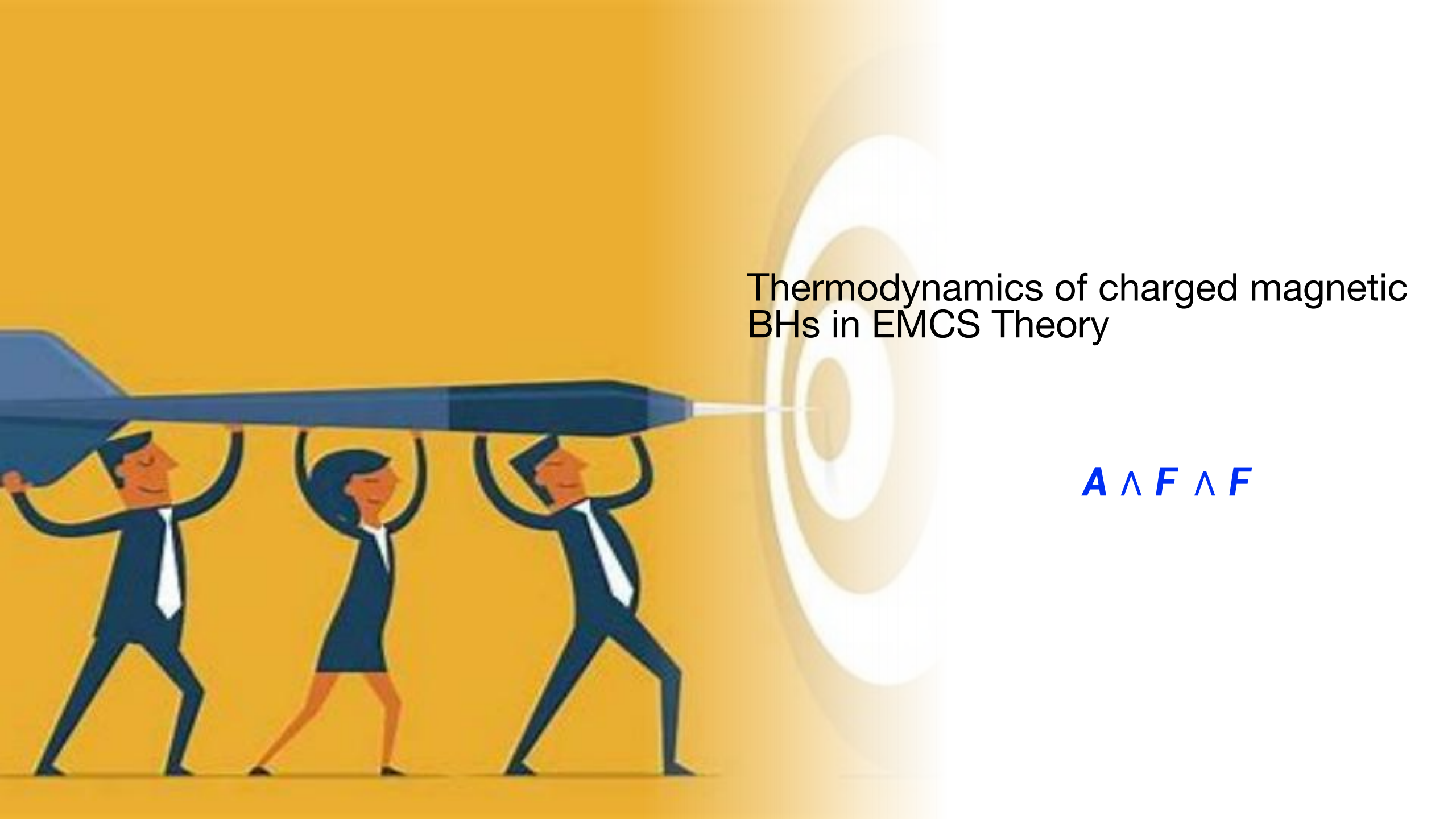


Hydrodynamics is an effective long-distance description for a given system at non-zero temperature. It provides a celebrated and appealing framework for capturing the collective dynamics of interacting systems (microscopic details only enter through equation of state and transport coefficients).



Thermodynamics of charged magnetic BHs in EMCS Theory

$$A \wedge F \wedge F$$



Motivations

- The CS term appears naturally in supergravity and shows numerous intriguing phenomena.
- BHs in this theory and their rich properties have attracted substantial interests, including black hole in Gödel universe, rotating black holes, dipole black rings.
- EMCS theory describes holographically quantum field theory with chiral anomaly in the strongly interacting regime.
- It holographically gives magnetic-field-driven quantum phase transition and the chiral magnetic effect
- ...

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Einstein-Maxwell-Chern-Simons theory [D'Hoker and Kraus, 2012]

Boundary QPT with chiral anomalous: $\partial_\mu J^\mu \sim k \mathbf{E} \cdot \mathbf{B}$

$$S = \frac{1}{16\pi G} \int d^5x \sqrt{-g} \left(R + \frac{12}{L^2} - \frac{1}{4} F_{ab} F^{ab} + \frac{k}{24} \epsilon^{abcde} A_a F_{bc} F_{de} \right)$$

$$k = k_{susy} = \frac{2}{\sqrt{3}}$$

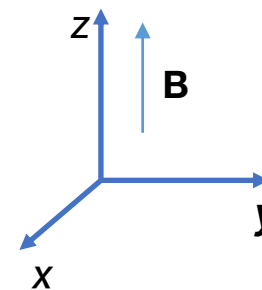
a consistent truncation of Type IIB supergravity or M-theory
[Buchel and Liu 2007; Gauntlett, Colgain and Varela 2007]

Charged magnetic black holes

$$ds^2 = \frac{1}{r^2} \left[- (f - h^2 p^2) dt^2 + 2ph^2 dt dz + g(dx^2 + dy^2) + h^2 dz^2 + \frac{dr^2}{f} \right]$$

$$A = A_t dt - \frac{B}{2} y dx + \frac{B}{2} x dy - A_z dz ,$$

magnetic field



What do we know for a system with finite charge and magnetic field?

- Standard thermodynamic first law [Landau, Lifshitz and Pitaevskii, 1995]

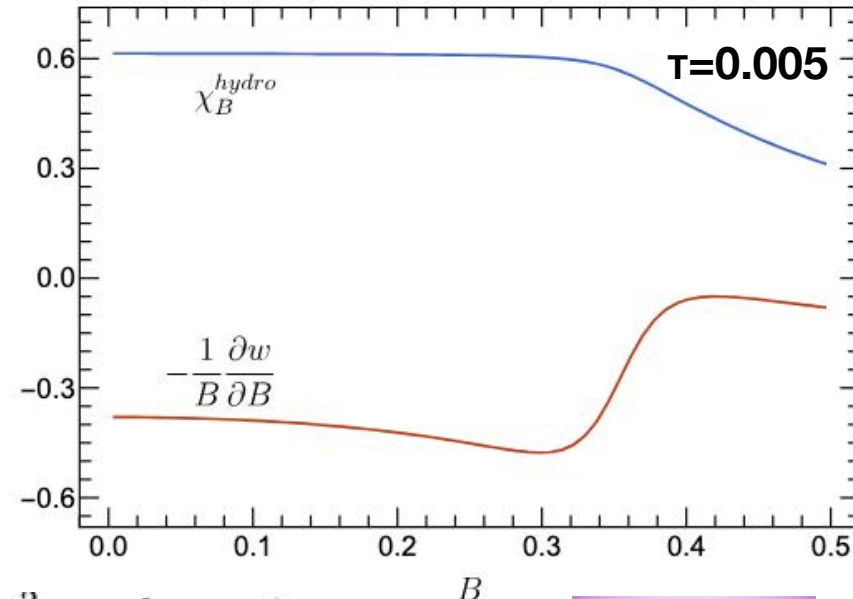
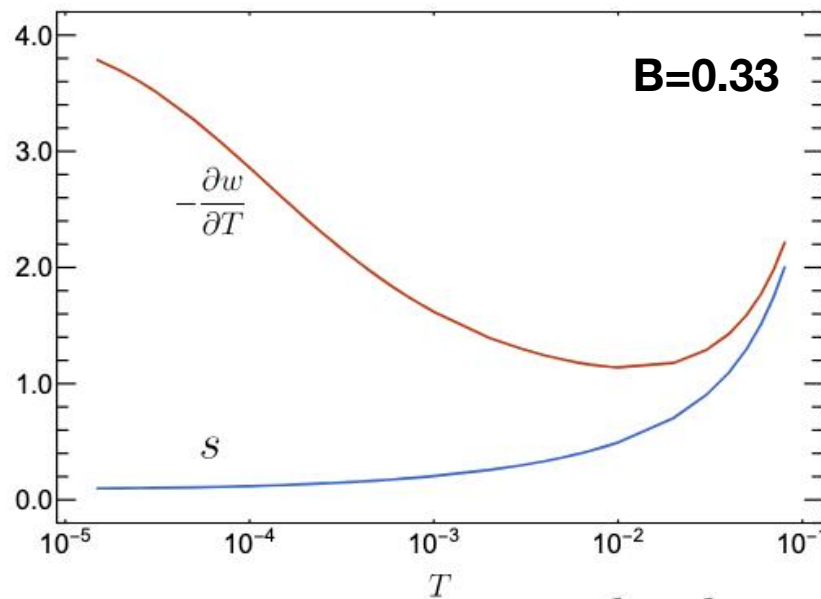
$$\begin{aligned}\delta w &= -s\delta T - \rho\delta\mu - M_B\delta B, \\ \delta\epsilon &= T\delta s + \mu\delta\rho - M_B\delta B, \end{aligned} \quad \mathbf{w} = \frac{W}{V}, \quad \epsilon = w + Ts + \mu\rho$$

- Magnetic susceptibility from hydrodynamics [Kovtun, 2016]

$$\chi_B B^2 = \langle T^{zz} \rangle - \langle T^{xx} \rangle = \frac{B^2}{4} + 8h_4.$$

Numerical tests of the first law of thermodynamics for dyonic black holes obtained by QSR

$$-\left(\frac{\partial w}{\partial T}\right)_{B,\mu} \neq s, \quad -\frac{1}{B} \left(\frac{\partial w}{\partial B}\right)_{T,\mu} \neq \chi_B^{hydro}$$



$$k = k_{\text{susy}} = \frac{2}{\sqrt{3}} \text{ and } \mu = 1.$$

Significant deviation from first law and hydrodynamics !



How to address this issue?



How to address this issue?



☆ variation of Euclidean on-shell action

☆ Iyer-Wald formalism

☆ variation of Euclidean on-shell action

In Euclidean signature, BH horizon is smooth when Euclidean time $\tau = it$ is periodic, with a period given by

$$\Delta\tau = 4\pi/(f'e^{-\chi/2})|_{r=r_h}$$

$$\delta I_E = \Delta\tau V \left\{ e^{-\frac{\chi_0}{2}} \left[-\rho \delta\mu - (e^{\chi_0} \epsilon - \mu\rho) \frac{\delta\chi_0}{2} \right] - k\delta Q_{cs} - e^{-\chi_0/2} M_B \delta B \right\}$$

The non-local term Q_{cs} arises from the CS interaction ($k \mathbf{A} \wedge \mathbf{F} \wedge \mathbf{F}$)

$$Q_{cs} = \frac{B}{6} \int_0^{r_h} (A'_z A_t - A_z A'_t) dr$$

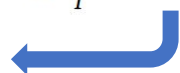
and M_B is defined as

$$M_B = - \left(\int_0^{r_h} \left[\frac{B}{r} (e^{-\chi/2} h - 1) + \frac{k}{2} (A'_z A_t - A_z A'_t) \right] dr + B \ln r_h \right)$$

☆ variation of Euclidean on-shell action

$$\delta I_E = \Delta\tau V \left\{ e^{-\frac{\chi_0}{2}} \left[-\rho \delta\mu - (e^{\chi_0} \epsilon - \mu\rho) \frac{\delta\chi_0}{2} \right] - k\delta Q_{cs} - e^{-\chi_0/2} M_B \delta B \right\}$$

$$\delta w = - \left(s + e^{\frac{\chi_0}{2}} \frac{kQ_{cs}}{T} \right) \delta T - \rho \delta\mu - e^{\frac{\chi_0}{2}} k\delta Q_{cs} - M_B \delta B$$

$\delta\chi_0 = 2 \frac{\delta T}{T}$ $T = \frac{e^{\chi_0/2}}{\Delta\tau}$
 **$\Delta\tau$ fixed**

[Donos and Gauntlett, 2013]

deviation from standard form of first law is attributed to the bulk integration Q_{cs}

Perform the transformation

$$\tilde{w} = w + e^{\chi_0/2} k Q_{cs}$$

$$\begin{aligned} \tilde{w} &= \epsilon - Ts - \mu\rho, \\ \delta\tilde{w} &= -s\delta T - \rho\delta\mu - M_B\delta B, \end{aligned}$$

standard thermodynamic relation !



☆ Iyer-Wald formalism

A powerful framework for studying BH thermodynamics [Iyer, Wald, 1994]

❖ The general variation of the Lagrangian 5-form $L = L\varepsilon$ can be expressed as

$$\delta L = E(\phi)\delta\phi + d\Theta \quad \phi = (g_{ab}, A_a) \text{ dynamical fields} \quad \Theta: \text{symplectic potential form}$$

❖ Consider an infinitesimal diffeomorphism variation $\delta_\xi x = \xi^a(x)$ and define a Noether current 4-form

$$\mathbf{J} \equiv \Theta(\phi, \mathcal{L}_\xi \phi) - \xi \cdot \mathbf{L} \quad \longrightarrow \quad d\mathbf{J} = -\mathbf{E}\mathcal{L}_\xi \phi,$$

There exists a Noether charge 3-form Q satisfying $J = dQ$ once $E(\phi)$.

❖ The variation of J gives the fundamental identity

$$\omega(\phi, \delta\phi, \mathcal{L}_\xi \phi) = \delta\mathbf{J} - d(\xi \cdot \Theta) = d(\delta\mathbf{Q} - \xi \cdot \Theta) = 0, \quad \omega: \text{symplectic current}$$

which holds if $E(\phi) = 0$, $\mathcal{L}_\xi \phi = 0$ and $E(\delta\phi) = 0$

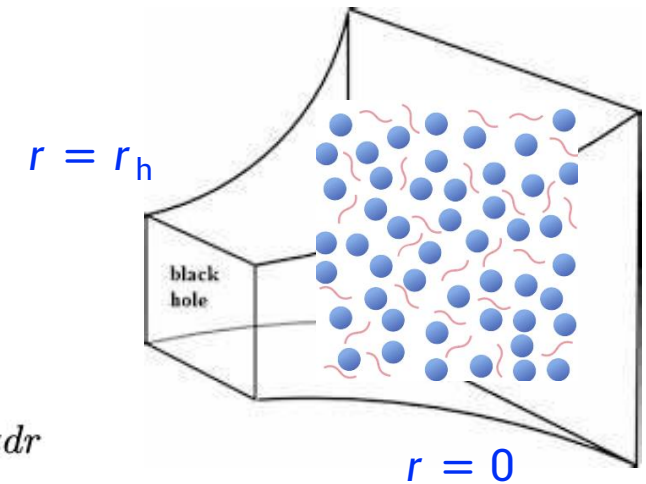
☆ Iyer-Wald formalism

Choose ξ^a to be the time-like Killing vector $\xi^a = (\partial t)^a$ which vanishes at event horizon $r = r_h$.
Let Σ be a **t=constant** hypersurface having the horizon as its interior boundary.

$$\partial\Sigma = S_{r=r_h} \cup S_{r=0} \cup S_{x=L_x/2} \cup S_{x=-L_x/2} \cup S_{y=L_y/2} \cup S_{y=-L_y/2} \cup S_{z=L_z/2} \cup S_{z=-L_z/2}$$

Integrate the fundamental identity over Σ

$$\begin{aligned} \delta H = 0 &= \int_{\Sigma} \omega = \int_{\Sigma} d(\delta \mathbf{Q} - \xi \cdot \boldsymbol{\Theta}) = \int_{\partial\Sigma} (\delta \mathbf{Q} - \xi \cdot \boldsymbol{\Theta}) \\ &= \int_{S_{r=r_h}} (\delta Q - \xi \cdot \boldsymbol{\Theta})_r \sqrt{\gamma(r)} dx dy dz - \int_{S_{r=\epsilon}} (\delta Q - \xi \cdot \boldsymbol{\Theta})_r \sqrt{\gamma(r)} dx dy dz \\ &\quad + \int_{S_{x=L_x/2}} (\delta Q - \xi \cdot \boldsymbol{\Theta})_x \sqrt{\gamma(x)} dy dz dr - \int_{S_{x=-L_x/2}} (\delta Q - \xi \cdot \boldsymbol{\Theta})_x \sqrt{\gamma(x)} dy dz dr \\ &\quad + \int_{S_{y=L_y/2}} (\delta Q - \xi \cdot \boldsymbol{\Theta})_y \sqrt{\gamma(y)} dx dz dr - \int_{S_{y=-L_y/2}} (\delta Q - \xi \cdot \boldsymbol{\Theta})_y \sqrt{\gamma(y)} dx dz dr \\ &= 3\delta f_4 - 8\delta h_4 - \frac{B}{2}\delta B + e^{-\frac{x_1}{2}} f_1 \left(\frac{\delta h_1}{r_h^3} - \frac{3h_1 \delta r_h}{r_h^4} \right) + \mu \delta \rho + B(\ln \epsilon) \delta B \\ &\quad + \int_{\epsilon}^{r_h} dr \left(\frac{B e^{-\chi/2} h}{r} + \frac{k}{2} (A'_z A_t - A_z A'_t) \right) \delta B, \end{aligned}$$



☆ Iyer-Wald formalism

$$0 = -\delta\epsilon + T\delta s + \mu\delta\rho + \left(\int_0^{r_h} dr \left[\frac{B}{r} \left(e^{-\chi/2} h - 1 \right) + \frac{k}{2} (A'_z A_t - A_z A'_t) \right] + B \ln r_h \right) \delta B$$



$$\delta\epsilon = T\delta s + \mu\delta\rho - M_B \delta B$$



$$T = -\frac{e^{\chi_0/2}}{4\pi} f' e^{-\chi/2} \Big|_{r=r_h}, \quad s = \frac{4\pi h}{r^3} \Big|_{r=r_h}$$

$$\epsilon = \langle T_{tt} \rangle = -3f_4 + 8h_4 + \frac{B^2}{4},$$

$$\mathcal{P}_\perp = \langle T_{xx} \rangle = \langle T_{yy} \rangle = -f_4 - \frac{B^2}{6}$$

$$\mathcal{P}_\parallel = \langle T_{zz} \rangle = -f_4 + 8h_4 + \frac{B^2}{12},$$

$$\langle T_{tz} \rangle = \langle T_{zt} \rangle = 4p_4 = -\frac{k}{2} B \mu^2,$$

$$\langle J^t \rangle = \rho, \quad \langle J^z \rangle = -2A_{z2} = kB\mu.$$

☆ Iyer-Wald formalism

$$0 = -\delta\epsilon + T\delta s + \mu\delta\rho + \left(\int_0^{r_h} dr \left[\frac{B}{r} \left(e^{-\chi/2} h - 1 \right) + \frac{k}{2} (A'_z A_t - A_z A'_t) \right] + B \ln r_h \right) \delta B$$



$$\delta\epsilon = T\delta s + \mu\delta\rho - M_B\delta B$$



$$T = -\frac{e^{\chi_0/2}}{4\pi} f' e^{-\chi/2} \Big|_{r=r_h}, \quad s = \frac{4\pi h}{r^3} \Big|_{r=r_h}$$

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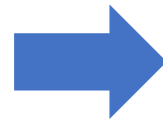
$$\langle J^t \rangle = \rho, \quad \langle J^z \rangle = -2A_{z2} = kB\mu.$$

❖ One also has

$$d\mathbf{Q} = -\xi \cdot \mathbf{L}$$

when $E(\phi)$ and $\mathbf{L}_\xi \phi = \mathbf{0}$. Integrate over Σ and apply Stokes's theorem

$$\int_{\partial\Sigma} \mathbf{Q} = - \int_\Sigma \xi \cdot \mathbf{L}$$



$$\tilde{w} = \epsilon - Ts - \mu\rho = w + kQ_{cs}$$

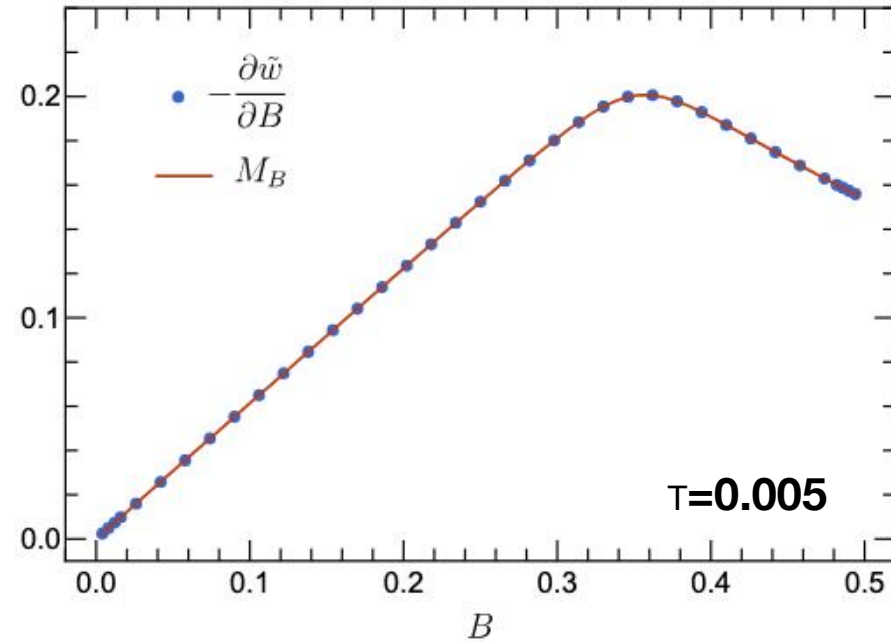
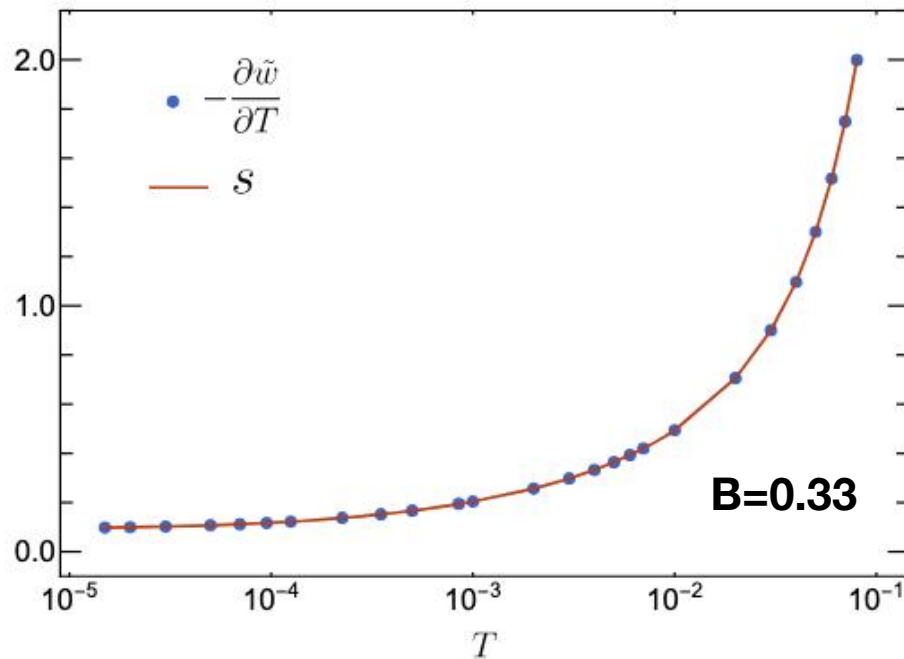
$$\delta\tilde{w} = -s\delta T - \rho\delta\mu - M_B\delta B$$



Prove the results from variation of Euclidean on-shell action !

Numerical verification

$$\delta\tilde{w} = -s\delta T - \rho\delta\mu - M_B\delta B$$



$$k = k_{\text{susy}} = \frac{2}{\sqrt{3}} \text{ and } \mu = 1$$

Recover magnetic susceptibility from hydrodynamics

$$\chi_B^{\text{hydro}} \equiv \frac{M_B}{B} = \frac{1}{4} + \frac{8h_4}{B^2} \quad M_B = \frac{\mathcal{P}_{\parallel} - \mathcal{P}_{\perp}}{B} :$$

Is this a coincidence, or is there a deeper mechanism?



Similar phenomena appear in plenty of BHs with CS interaction

✦5D Helical black holes

$$ds^2 = \frac{1}{r^2} \left[- (fe^{-\chi} - h^2 p^2) dt^2 + 2ph^2 dt dz + h^2 dz^2 + u^{-2} \omega_2^2 + u^2 (\omega_1 + \alpha dz + \beta dt)^2 + \frac{dr^2}{f} \right]$$

$$A = A_t dt + \frac{B}{2} (x dy - y dx) + b \omega_1 - A_z dz ,$$

$$\omega_1 = \cos(\lambda_L z) dx - \sin(\lambda_L z) dy , \quad \omega_2 = \sin(\lambda_L z) dx + \cos(\lambda_L z) dy , \quad \omega_3 = dz$$

geometry possessing Bianchi VII0 symmetry

$$\tilde{W} = w + k Q_{cs}$$

$$Q_{cs} = \frac{B}{6} \int_0^{r_h} (A'_z A_t - A_z A'_t) dr$$

$$\tilde{w} = \epsilon - Ts - \mu \langle J^t \rangle ,$$

$$\delta \tilde{w} = -s \delta T - \langle J^t \rangle \delta \mu - \left(2b_2 - \frac{k \lambda_L \mu \mu_b}{3} + \frac{\lambda_L^2 \mu_b}{2} \right) \delta \mu_b - M_B \delta B ,$$

$$b(r) = \mu_b + b_2 r^2 + \frac{\mu_b \lambda_L^2}{2} r^2 \ln r + \dots , \quad \text{source for helical current}$$



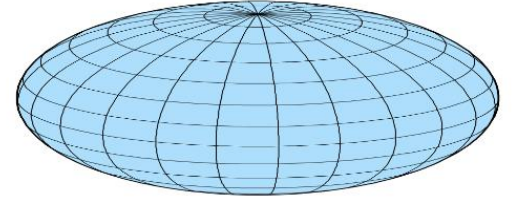
Similar phenomena appear in plenty of BHs with CS interaction

✦5D rotating black hole

$$ds^2 = -F_0(r)dt^2 + F_1(r)dr^2 + \frac{1}{4}F_2(r) (\sigma_1^2 + \sigma_2^2) + \frac{1}{4}F_3(r) (\sigma_3 + 2p(r)dt)^2$$

$$A = A_t(r)dt + \frac{B \cos \theta}{4}d\phi - \frac{A_k(r)}{2}\sigma_3,$$

$$\sigma_1 = \cos \psi d\theta + \sin \psi \sin \theta d\phi, \quad \sigma_2 = -\sin \psi d\theta + \cos \psi \sin \theta d\phi, \quad \sigma_3 = d\psi + \cos \theta d\phi$$



L=1

with a local $SU(2) \times U(1)$ symmetry

$$w = \epsilon - Ts - \mu\rho - 2\Omega_H \mathcal{J}$$

$$Q_{cs} = \frac{B}{6} \int_0^{r_h} (A'_k A_t - A_k A'_t) dr$$

$$\delta w = -s\delta T - \rho\delta\mu - 2\mathcal{J}\delta\Omega_H - M_B\delta B$$

Similar phenomena appear in plenty of BHs with CS interaction

✧7D black holes

$$S = \frac{1}{16\pi G} \int d^7x \sqrt{-g} \left(R + \frac{30}{L^2} - \frac{F^2}{4} + \frac{k}{4} \epsilon^{abcdemn} A_a F_{bc} F_{de} F_{mn} \right)$$

Dyonic black holes

$$ds^2 = \frac{1}{r^2} \left[- (f e^{-\chi} - h^2 p^2) dt^2 + 2ph^2 dt dz + dx_1^2 + dx_2^2 \right. \\ \left. + u(dx_3^2 + dx_4^2) + h^2 dz^2 + \frac{dr^2}{f} \right],$$

$$A = A_t dt - \frac{B_1}{2} x_2 dx_1 + \frac{B_1}{2} x_1 dx_2 - \frac{B_2}{2} x_4 dx_3 + \frac{B_2}{2} x_3 dx_4 - A_z dz,$$

two different magnetic fields: B1 and B2

$$\tilde{w} = \epsilon - Ts - \mu\rho = w + kQ_{cs}$$

$$Q_{cs} = 12B_1B_2 \int_0^{r_h} (A'_z A_t - A_z A'_t) dr$$

$$\left\{ \begin{array}{l} \delta\tilde{w} = -s\delta T - \rho\delta\mu - M_{B_1}\delta B_1 - M_{B_2}\delta B_2 \\ \delta\epsilon = \mu\delta\rho + T\delta s - M_{B_1}\delta B_1 - M_{B_2}\delta B_2 \end{array} \right.$$

Extend to AF^n

Key Problem: Gauge Variance of the Local CS Term

$$S = \frac{1}{16\pi G} \int d^5x \sqrt{-g} \left(R + \frac{12}{L^2} - \frac{1}{4} F_{ab} F^{ab} + \frac{k}{24} \epsilon^{abcde} A_a F_{bc} F_{de} \right)$$

The CS term under U(1) gauge transformation $A \rightarrow A + d\Lambda$

$$\mathcal{C}_5(A) = A \wedge F \wedge F \longrightarrow \mathcal{C}_5(A) + d(\Lambda F \wedge F)$$

- The local CS form is invariant only up to an exact differential.
- In the presence of magnetic flux, $\int \Lambda F \wedge F \neq 0$ on the boundary — the action is not strictly gauge invariant.
- No local boundary term can cure this? The missing contribution is nonlocal in the radial direction.

$$Q_{cs} = \frac{B}{6} \int_0^{r_h} (A'_z A_t - A_z A'_t) dr$$

The Transgression Form: Construction and Result

Transgression Form in $D=2n+1$ Dimensions:

$$\mathcal{T}_{2n+1}(A, \bar{A}) = \mathcal{C}_{2n+1}(A) - \mathcal{C}_{2n+1}(\bar{A}) - d\mathcal{B}_{2n}(A, \bar{A})$$

[M. Nakahara, Geometry, topology and physics (2003)]

$$\mathcal{C}_{2n+1}(A) = A \wedge F^n \qquad \mathcal{B}_{2n}(A, \bar{A}) = A \wedge \bar{A} \wedge \sum_{j=0}^{n-1} F^{n-1-j} \wedge \bar{F}^j$$

$\bar{A}$ is a **reference gauge** potential carrying the same magnetic flux as A , with $\bar{F} = d\bar{A}$.

$$\bar{A}_{hel} = \frac{B}{2}(xdy - ydx), \quad \bar{A}_{rot} = \frac{B}{4} \cos \theta d\varphi$$

$$\bar{A}_{7D} = \frac{B_1}{2}(x_1 dx_2 - x_2 dx_1) + \frac{B_2}{2}(x_3 dx_4 - x_4 dx_3)$$

Under simultaneous U(1) transformations $A \rightarrow A + d\Lambda, \bar{A} \rightarrow \bar{A} + d\Lambda$

$$\delta_{\Lambda} \mathcal{T}_{2n+1} = 0$$

The transgression form is **strictly invariant**, unlike the local CS representative

The Transgression Form: Construction and Result

Replace $S_{CS} = \frac{k}{6} \int \mathcal{C}_5(A)$ with $S_{\mathcal{T}_5} = \frac{k}{6} \int \mathcal{T}_5(A, \bar{A})$, choosing:

$$\bar{A} = \frac{B}{2}(x dy - y dx)$$

Result:

$$w_{\mathcal{T}} = w_{QSR} + kQ_{cs} = w$$

where the missing finite contribution is:

On the quantum statistical relation and the covariant phase space Hamiltonian

[José Luis V. Cerdeira, Tomás Ortín](#) (Oct 9, 2026)

e-Print: [2610.13377](#) [hep-th]

$$Q_{cs} = \frac{B}{6} \int_0^{r_h} (A'_z A_t - A_z A'_t) dr$$

- Q_{cs} is a **nonlocal radial integral** from horizon to boundary.
- It is automatically generated by the transgression form — not added by hand.
- With $w = w_{QSR} + kQ_{cs}$, the first law $\delta w = -s\delta T - \rho\delta\mu - M_B\delta B$ and the hydrodynamic susceptibility are **exactly restored**.

Universality Across Backgrounds:

Background	Q_{cs} Structure	Consistent?
5D planar dyonic BH	$\frac{B}{6} \int_0^{r_h} (A'_z A_t - A_z A'_t) dr$	✓
5D helical BH (Bianchi VII ₀)	same form	✓
5D rotating BH (squashed S^3)	same form	✓
<u>5D scalarized BH</u>	same form (independent of $V(\phi), Z(\phi)$)	✓
<u>5D Gauss–Bonnet BH</u>	same form (Wald entropy)	✓
7D with B_1, B_2	$12B_1 B_2 \int_0^{r_h} (A'_z A_t - A_z A'_t) dr$	✓

Key Physical Implications:

- **Gauge invariance** is a physical requirement. The gauge variance of the local CS term prevents a well-posed Euclidean path integral; the thermodynamic inconsistency is its direct manifestation.
- QSR remains valid — provided it is applied to the gauge-invariant **transgression** action, not the local CS representative.
- Contrast with pure CS AdS gravity (transgression form was a regularization tool). Here, transgression completion supplies a **finite global contribution** for thermodynamic consistency. The **first example** where transgression modifies a finite BH thermodynamic potential.
- Holographic consequences: relevant to **quantum critical** materials and strongly coupled QCD matter.

Outline

- Introduction
- Thermodynamics for CS black holes
- **Application: quantum criticality**
- Summary

Quantum Criticality

➤ **Quantum phase transitions** (QPTs) occur at $T=0$, driven by **quantum fluctuations**

➤ **Quantum critical point** (QCP):

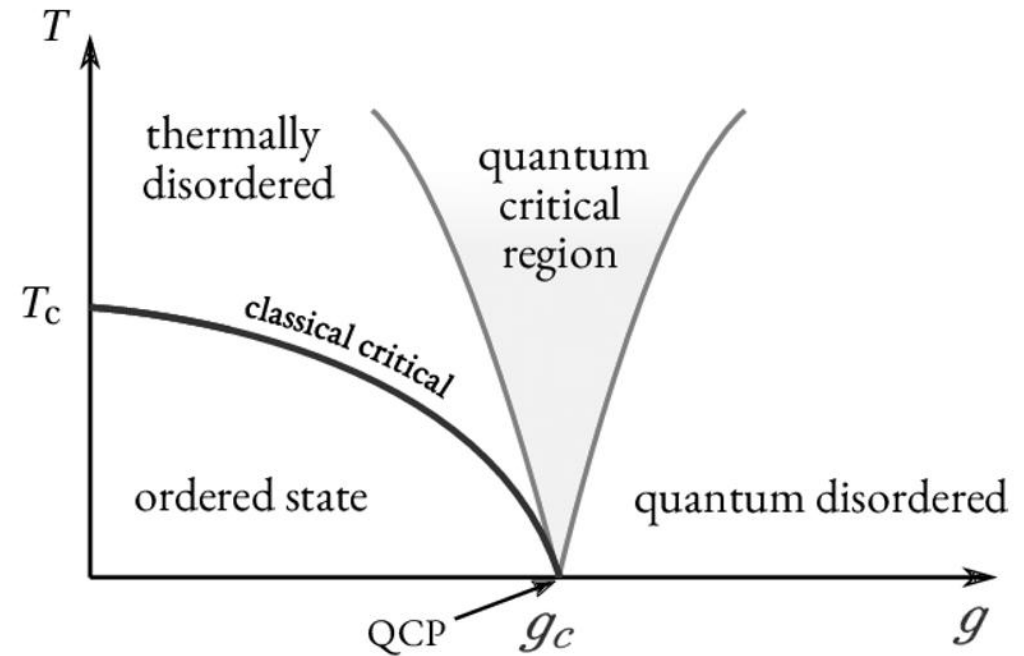
– Boundary between distinct phases

– **Scale invariance** $\xi \sim |g - g_c|^{-\nu}$

– Low-energy excitations **beyond quasiparticle** picture

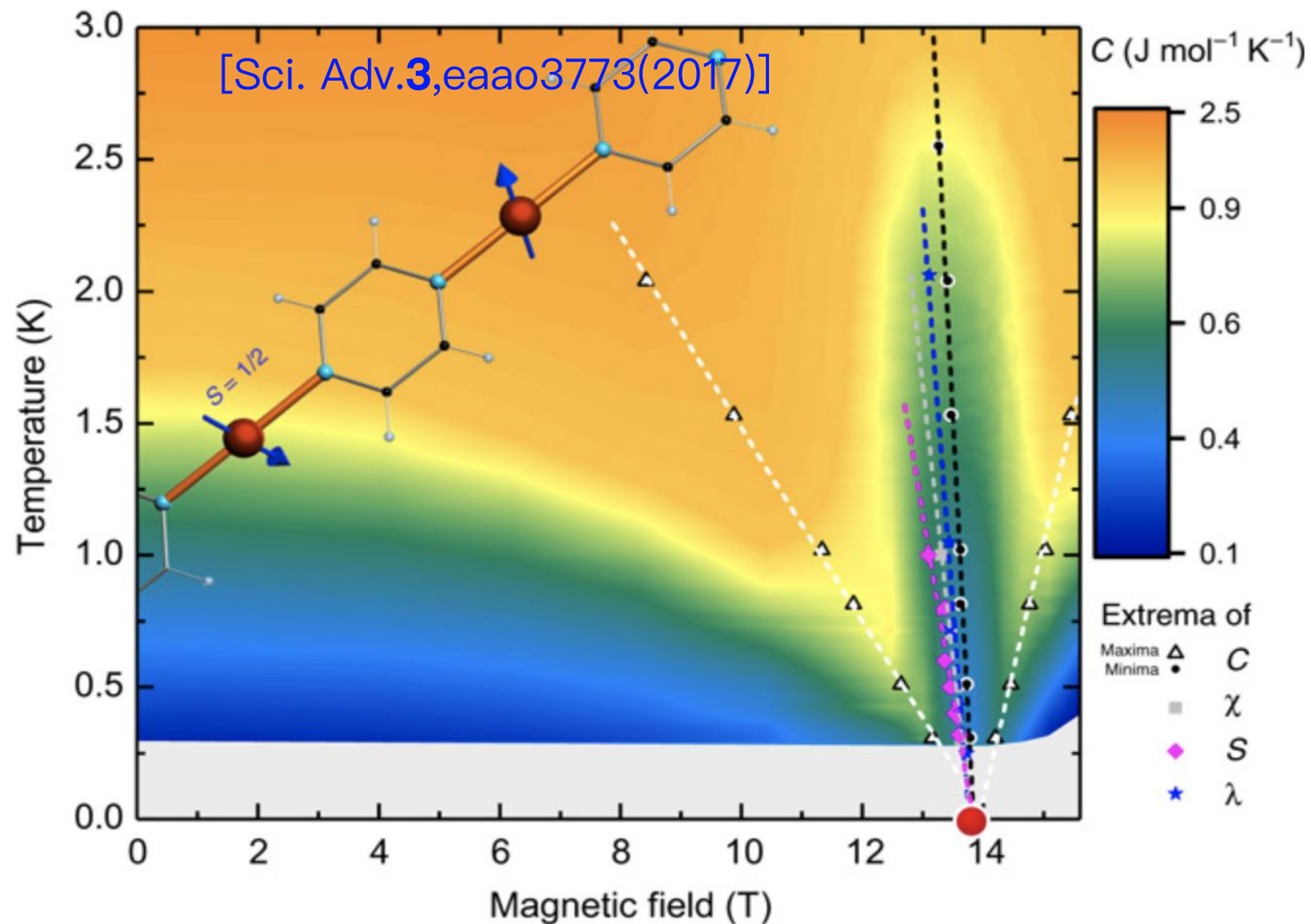
➤ **Quantum critical regime** (QCR):

– Scaling behavior extends to finite T



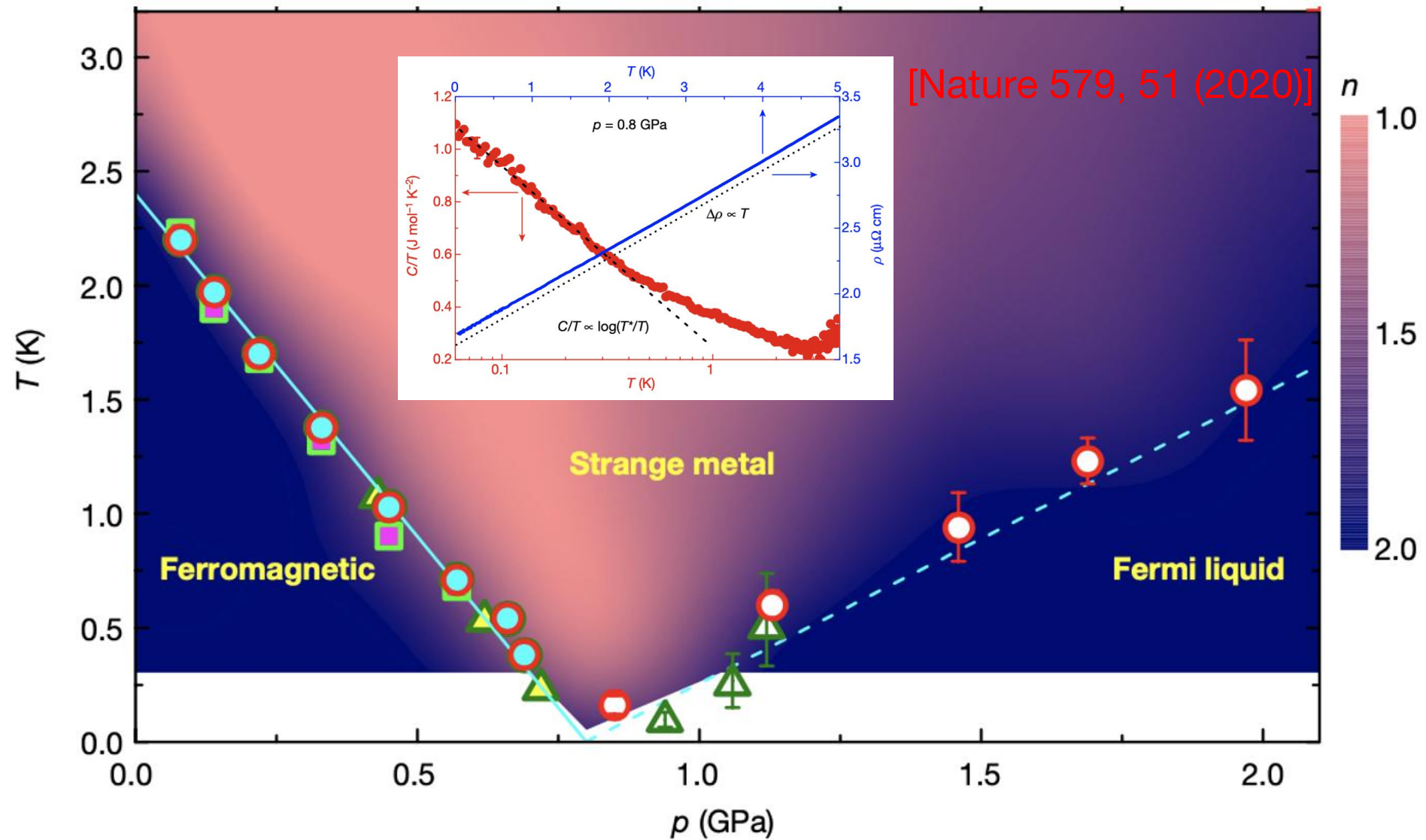
Quantum criticality in quantum magnets

T – B phase diagram of **CuPzN** (硝酸铜吡嗪)



Quantum criticality in heavy-fermion materials

T - p phase diagram of **CeRh6Ge4**



Quantum Criticality

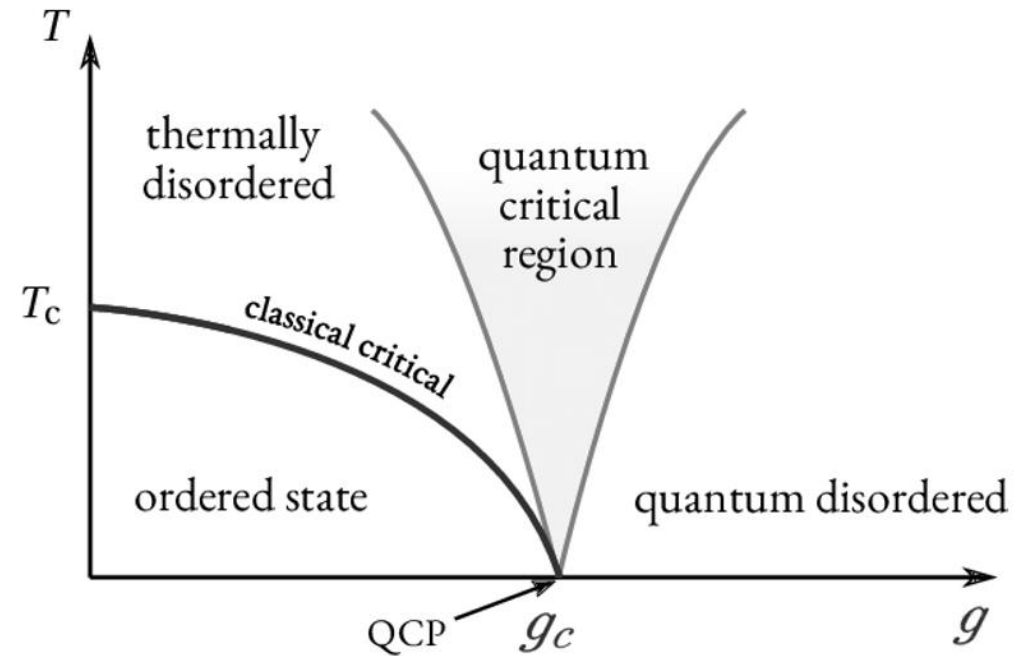
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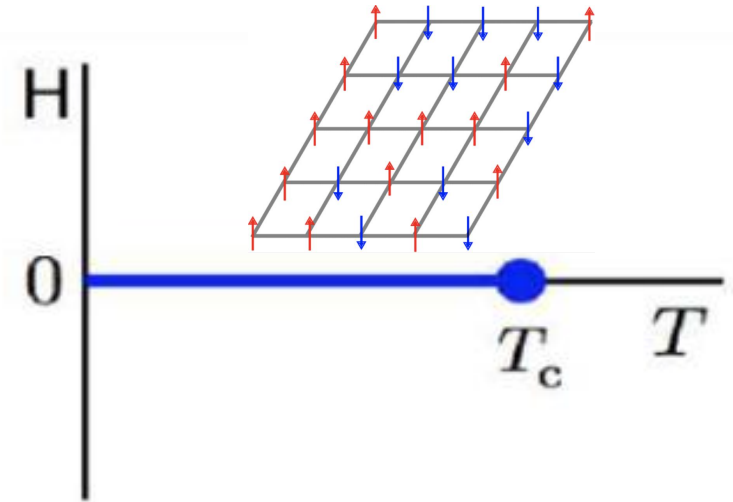
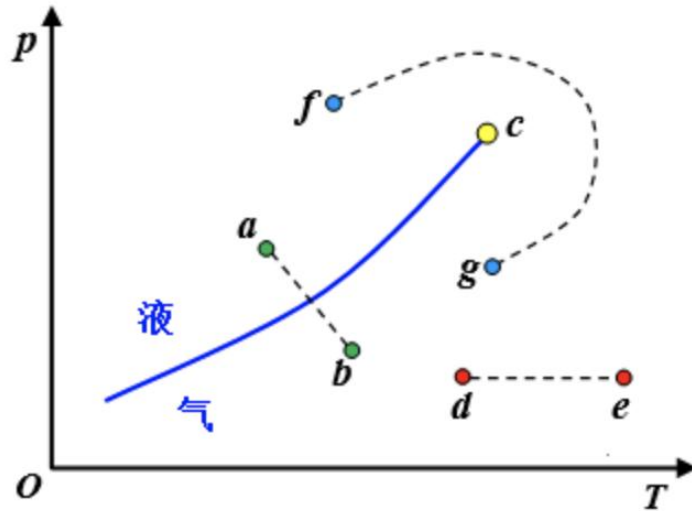
- Scaling behavior extends to finite T
- **Universal critical exponents: universal class**



liquid–gas phase transition

vs

3D Ising ferromagnet system



Despite **distinct microscopic** interactions, macroscopic critical behaviors—including **critical exponents** and scaling laws—are **identical** near the critical point.

Determined solely by the **spatial dimensionality** and the order–parameter **symmetry**.

Grüneisen Ratio

$$\Gamma_g = \frac{1}{T} \left(\frac{\partial T}{\partial g} \right)_s$$

Adiabatic temperature change per unit field variation

Universal divergence near QCP: $\Gamma_g \propto T^{-1/z\nu}$

Why is it important?

- ✓ Sensitive probe of QCP
- ✓ Determines critical exponents
- ✓ Measures enhanced **magnetocaloric effect**

Experimental observations:

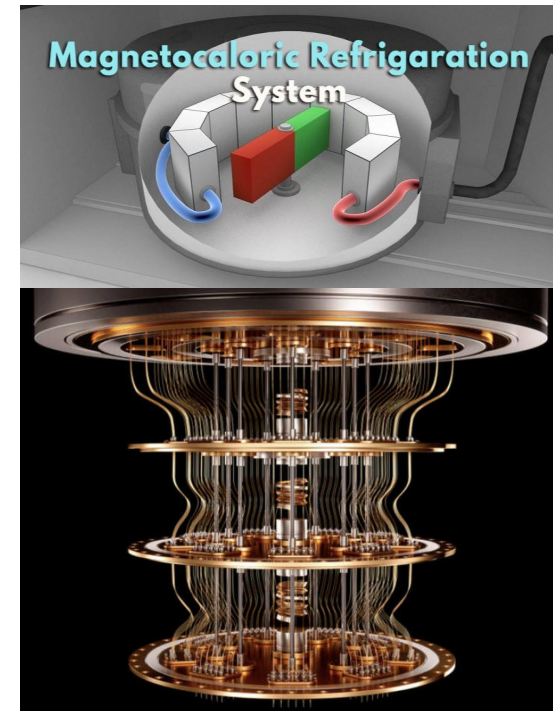
- $\text{Cu}(\text{C}_4\text{H}_4\text{N}_2)(\text{NO}_3)_2$ [Sci. Adv. 2017]
- $\text{YbNi}_4(\text{P}_{1-x}\text{As}_x)_2$ [Science 2013]
- $\text{CeCu}_{6-x}\text{Ag}_x$ [PRL 2003, 2004]
- YbRh_2Si_2 [PRL 2009]
- CeRh_6Ge_4 [PRL 2025]

Heavy
fermions

$$\xi \sim |g - g_c|^{-\nu}$$

$$\omega \sim k^z$$

$$\tau \propto \xi^z$$



Grüneisen Ratio

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- YbRh_2Si_2 [PRL 2009]
- **CeRh_6Ge_4 : $\Gamma_P \sim T^{-2/3}$ [PRL 2025] (novel QCP with $z=3$)**

Heavy
fermions

metallic quantum
criticality !

$$\xi \sim |g - g_c|^{-\nu}$$

$$\omega \sim k^z$$

$$\tau \propto \xi^z$$



Hertz–Millis Theory: Limitations

Effective action:

(metallic quantum criticality)

$$S_{\text{HM}} = \sum_{\mathbf{k}, i\omega_n} \left(\delta + \mathbf{k}^2 + \frac{|\omega_n|}{\Gamma_k} \right) |\phi_{\mathbf{k}, i\omega_n}|^2 + u \int_0^{1/T} d\tau \int d^d \mathbf{r} \phi(\mathbf{r}, \tau)^4 \quad \Gamma_k = \Gamma_0 \mathbf{k}^{z-2}$$

ϕ : order parameter

Γ_k : Landau damping rate

δ : tuning parameter

u : strength of the interaction

- Key assumption: integrate out fermionic degrees
- ✗ Ignores backreaction of order parameter fluctuations on fermions
- ✗ "Dangerous" neglect in high–energy sector

- Physics it cannot capture:
 - Non–Fermi liquid behavior
 - QCPs beyond Landau paradigm

Quantum Monte Carlo: suffer from the sign problem at finite density

Tensor network: struggle to capture highly entangled nature of systems

- Motivation for **holography**:

Naturally handles strongly coupled systems + No quasiparticle assumption required

Holographic Duality : A New Perspective

Quantum Many-Body System (d dims)

- Strongly coupled
- No quasiparticles
- Critical behavior

■ AdS radius r : RG energy scale

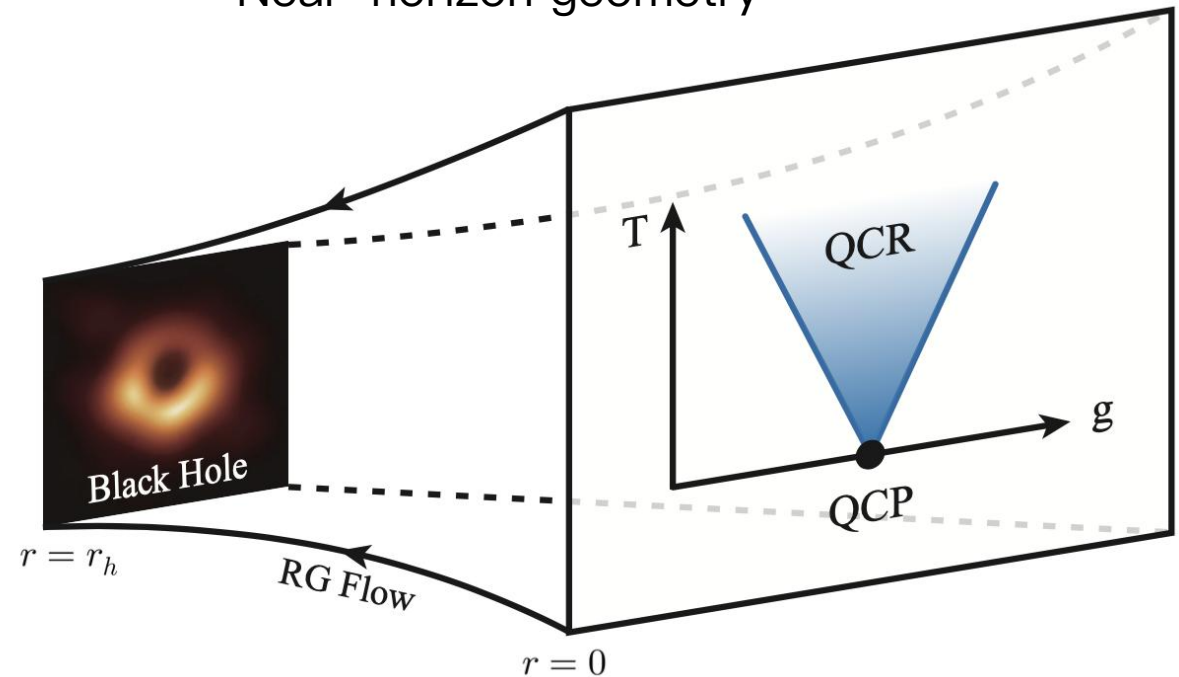
$r \rightarrow 0$: UV (short distance)

$r \rightarrow \infty$: IR (long distance, low energy)

QPTs realized by tuning IR geometries

Gravity Theory (d+1 dims)

- Classical gravity
- Black hole
- Near-horizon geometry



Holographic model

Einstein–Maxwell–Chern–Simons Theory

[D'Hoker and Kraus, 2012]

$$S = \frac{1}{16\pi G_N} \int d^5x \sqrt{-g} \left(R + \frac{12}{L^2} - \frac{1}{4} F_{ab} F^{ab} + \frac{k}{24} \epsilon^{abcde} A_a F_{bc} F_{de} \right)$$

➤ Why this model?

Boundary QPT with chiral anomalous: $\partial_\mu J^\mu \sim k \mathbf{E} \cdot \mathbf{B}$

✓ Simple (metric + gauge field only)

✓ CS term encodes chiral anomaly

✓ **Magnetic–field–driven QCP**

➤ Parameter choices:

$k_{susy} = 2/\sqrt{3}$ (minimal supergravity) $\rightarrow z=3$

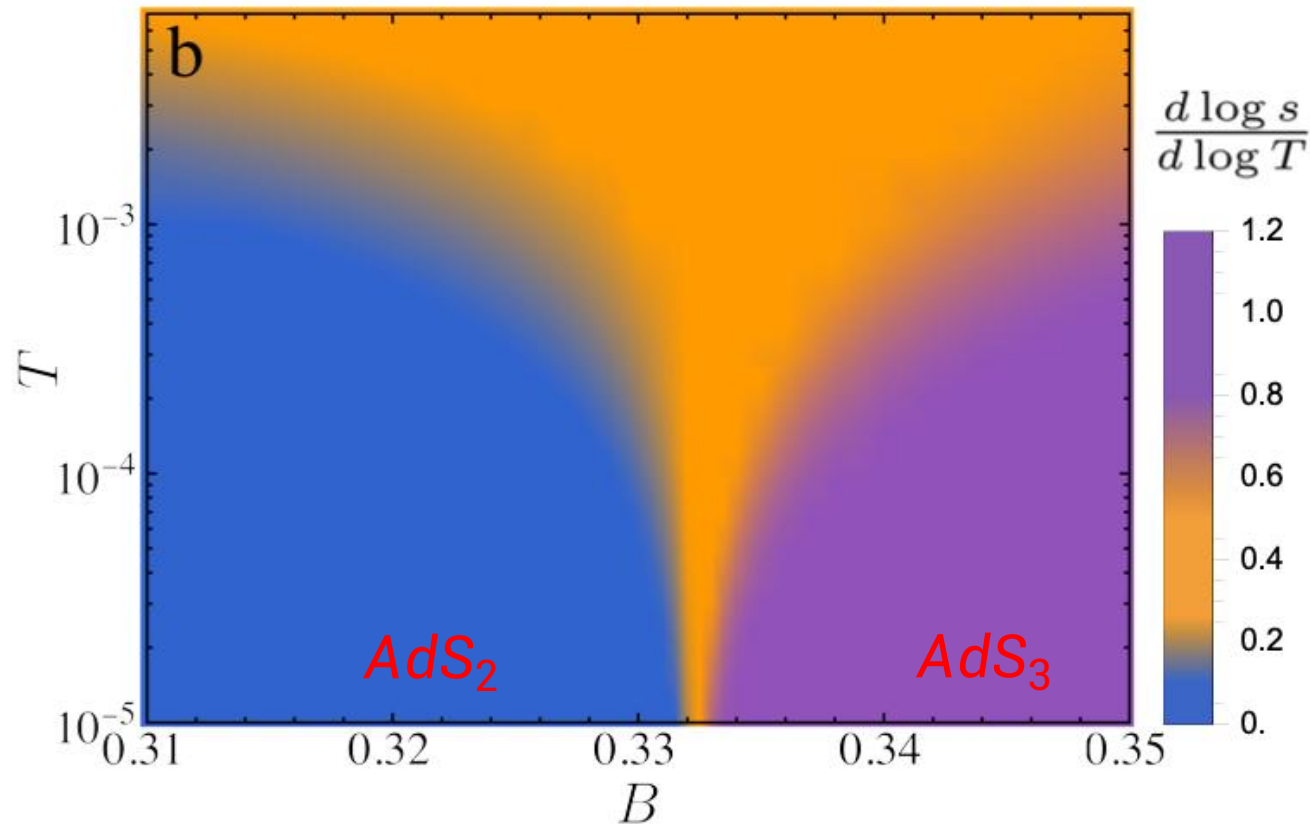
$z = 3$ for $k > 3/4$

$k = 2/3 \rightarrow z=2$ (third–order transition)

$z = k/(1 - k)$ for $1/2 < k < 3/4$

Key feature: $z=3$ emerges from IR dynamics, not put in by hand

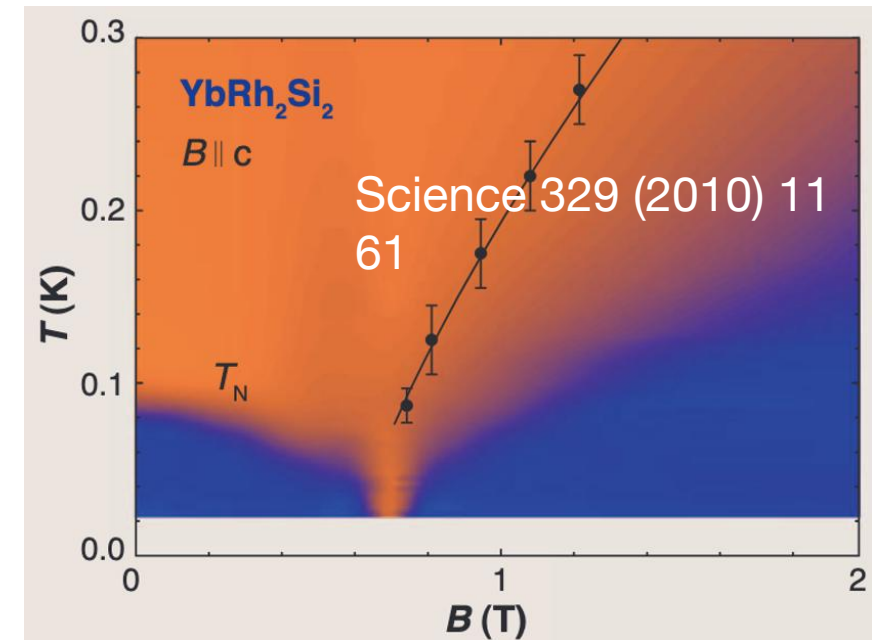
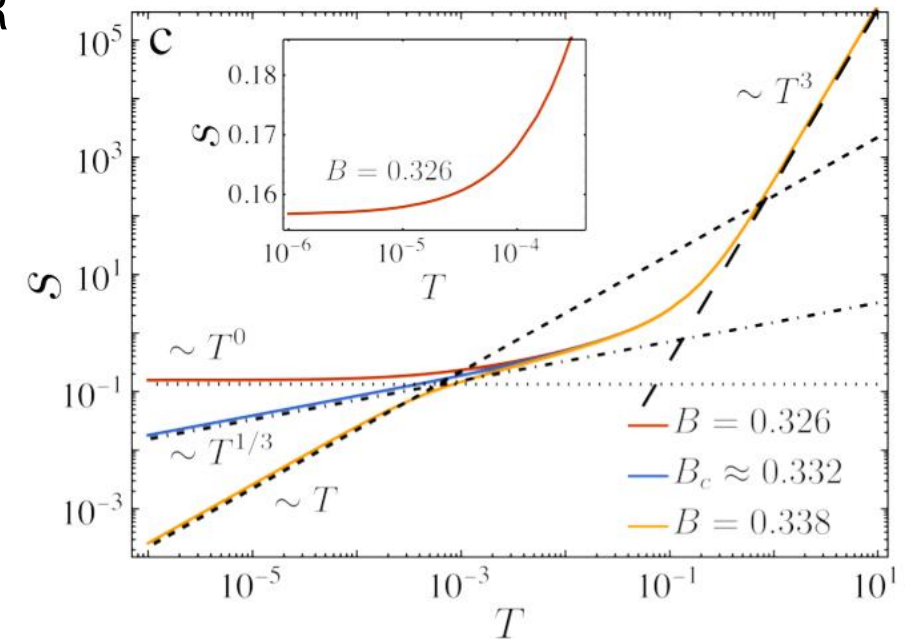
T - B Phase Diagram and QCR



Near-horizon **Schrodinger geometry** at B_c

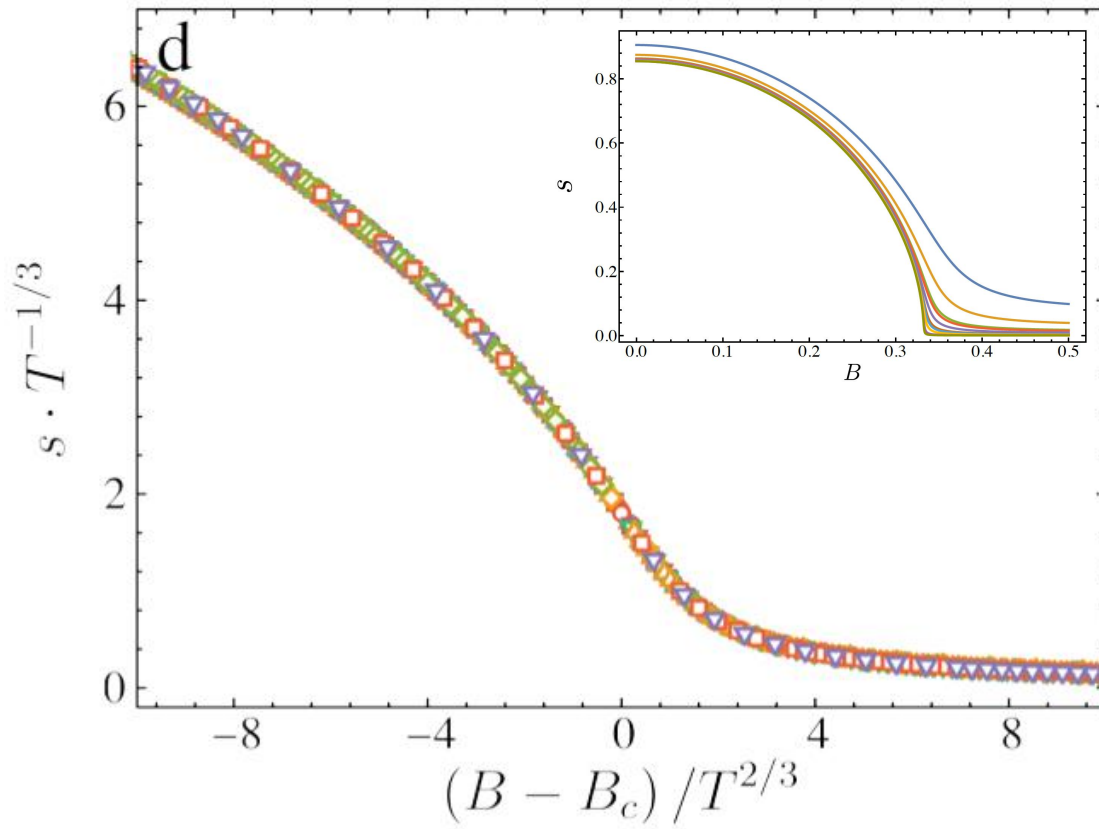
$$ds^2 = \frac{dr^2}{12r^2} + 4\sqrt{3}rdtdx_3 - \frac{2e_0^2 r^{2k}}{k(2k-1)}dt^2 + dx_1^2 + dx_2^2$$

effective **space dimension** $d = 1$



Entropy Scaling: $z=3, \nu=1/2$

■ Critical scaling form: $s \sim T^{d/z} \phi_s \left((B - B_c) T^{-1/z\nu} \right)$



■ From data collapse:

$$-d/z = 1/3 \rightarrow z = 3d$$

$$-1/(z\nu) = 2/3 \rightarrow z\nu = 3/2$$

■ With $d=1$ (effective spatial dimension):

$$z=3, \nu=1/2 \rightarrow \text{cubic dispersion: } \omega \sim k^3$$

■ Connection to experiments:

CeRh₆Ge₄ also exhibits $z=3$, cubic dispersion

Specific Heat Scaling

Isothermal specific heat c vs magnetic field

■ Vertical dashed line: QCP ($B_c = 0.332$)

■ Peak grows as T decreases

Scaling function $\phi_c(x)$ from data collapse

■ Single-peak structure

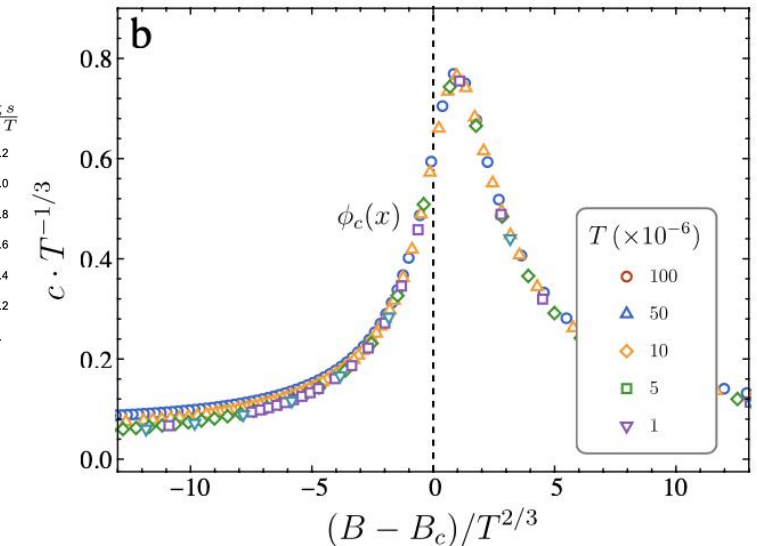
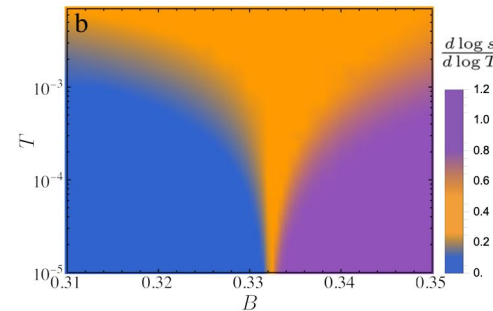
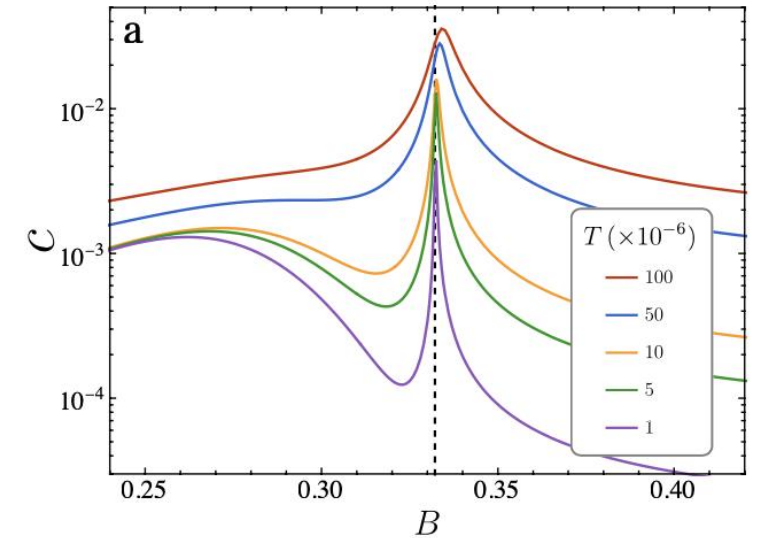
■ Different from conventional double-peak structure

■ Key observations:

– Only right peak identifies QCR boundary

$$B - B_c \propto T^{1/\nu z} = T^{2/3}$$

– Left boundary cannot be determined from c alone

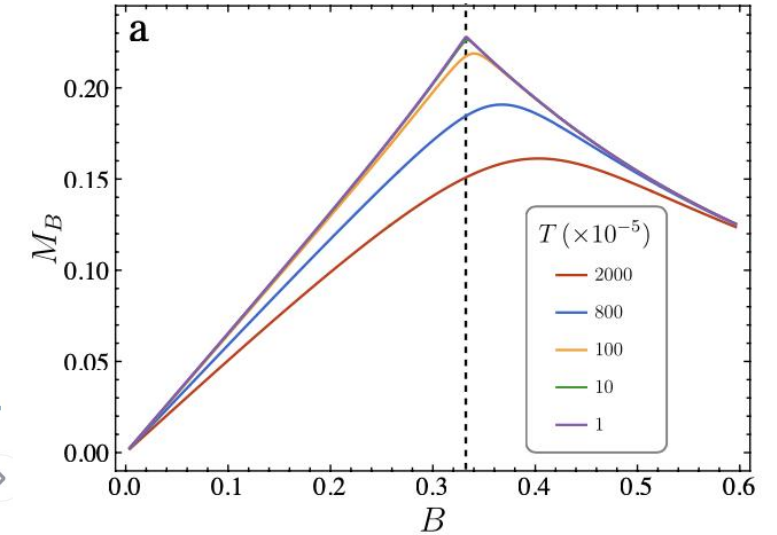
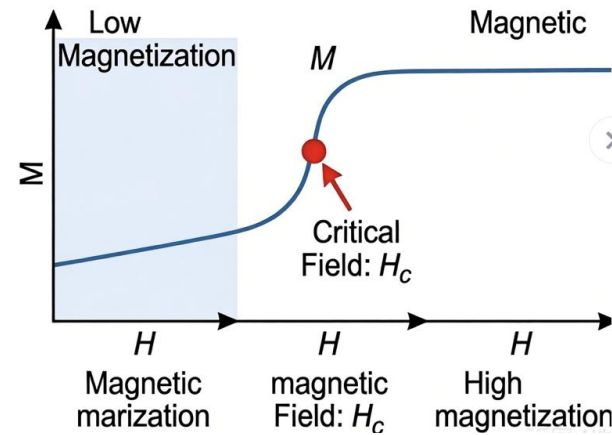


Magnetization and Susceptibility

Magnetization M_B vs B at different T

- Non-monotonic behavior
- Cusp singularity near QCP at low T
- No rapid increase (unlike conventional **metamagnets**)

[Millis et al, PRL, 2001]



Magnetization and Susceptibility

Magnetization M_B vs B at different T

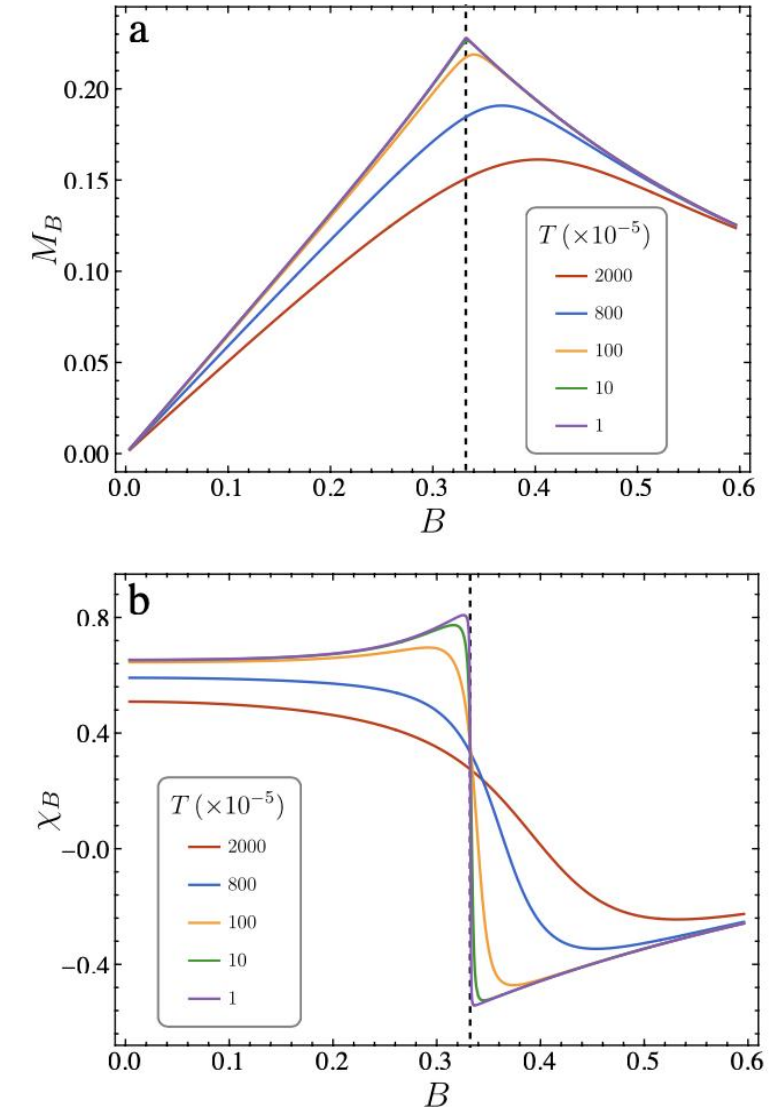
- Non-monotonic behavior
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Magnetic susceptibility χ_B vs B

- Paramagnetic ($\chi_B > 0$) for $B < B_c$
- Diamagnetic ($\chi_B < 0$) for $B > B_c$
- Phase transition: Paramagnetic $\rightarrow$ Diamagnetic (**second order**)
 $\chi_B \propto T^{-\alpha/z\nu}$
- Critical **exponent** $\alpha = 0$ from χ_B

Check quantum hyperscaling relation

$$2 - \alpha = \nu(d + z) \rightarrow 2 = \frac{1}{2} \times (1 + 3) = 2 \quad \checkmark$$



EMCS Cubic Universality Class: Critical Exponents

Critical exponent	α	β^*	γ^*	δ^*	η^*	ν	z	d
EMCS theory	0	1	0	1	2	$\frac{1}{2}$	3	1

$$G(r) \propto r^{-(d-2+\eta)}$$

$$\chi \propto |g - g_c|^{-\gamma}$$

$$\phi \propto |g - g_c|^\beta$$

at T=0

quantum hyperscaling

$$\alpha + (d + z)\nu = 2$$

Widom

$$\gamma = \beta(\delta - 1)$$

Fisher

$$\gamma = \nu(2 - \eta)$$

Rushbrooke

$$\alpha + 2\beta + \gamma = 2$$

- $\beta^*, \gamma^*, \delta^*, \eta^*$: critical exponents for **unconventional order parameters** (η/s or ρ_{frac}/ρ)
- This universality class is **novel**: not previously identified in literature
- **Robust** across different k values with same $z=3$ (**$z = 3$ for $k > 3/4$**)

Unconventional Order Parameters

■ Why unconventional?

- No symmetry breaking at QCP
- No local order parameter ϕ from Landau paradigm

■ Order parameter criteria:

- ✓ Finite in one phase
- ✓ Vanishes in the other
- ✓ Exhibits critical scaling near QCP $\phi \propto |B - B_c|^\beta$

Unconventional Order Parameters

■ Why unconventional?

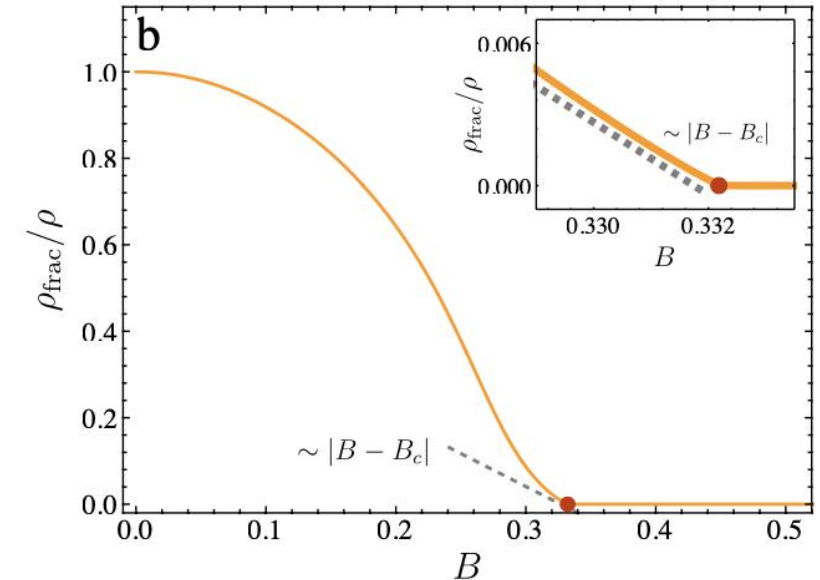
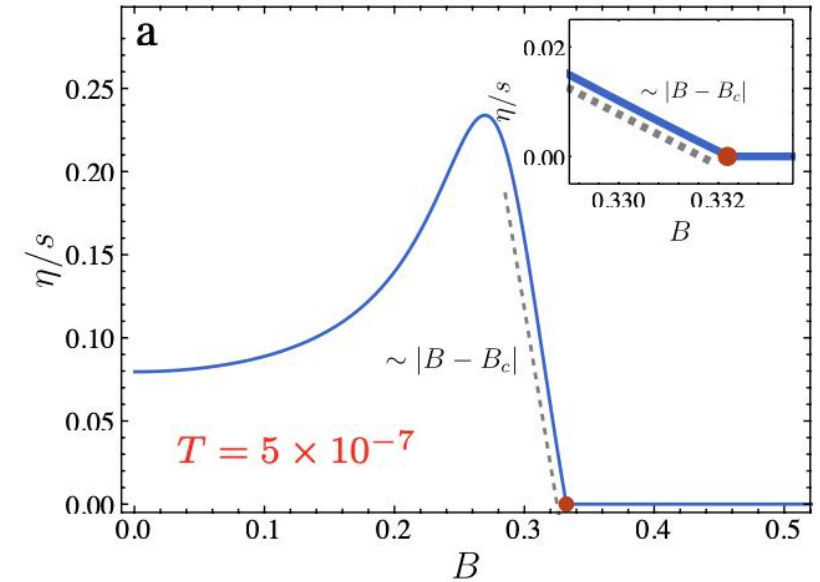
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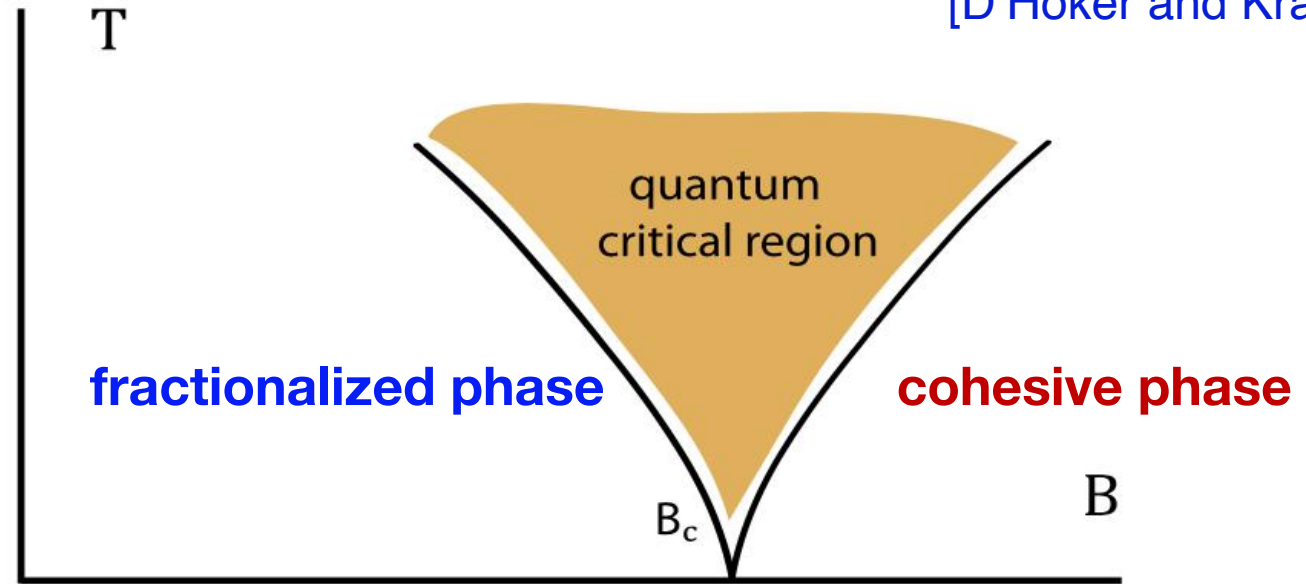
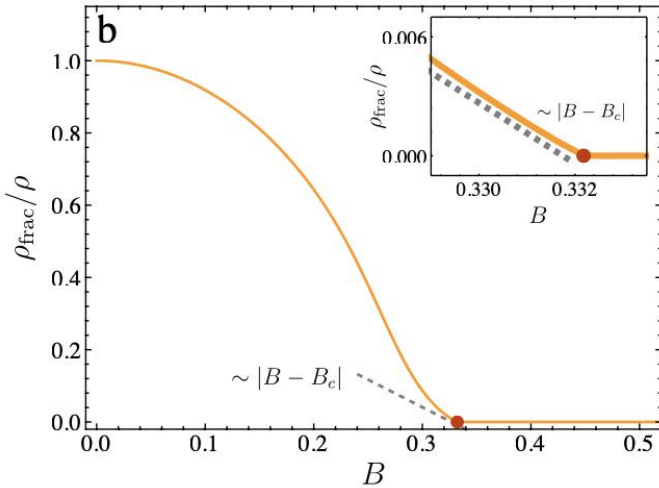
■ Two candidates:

1. η/s (shear viscosity to entropy density ratio)
2. ρ_{frac}/ρ (fractionalized charge fraction)



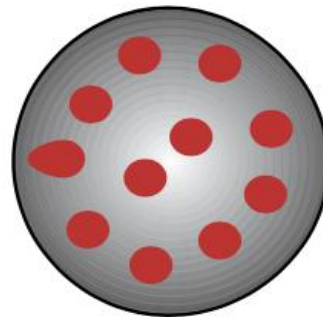
[D'Hoker and Kraus, 2012]

G
A
U
G
E
T
H
E
O
R
Y

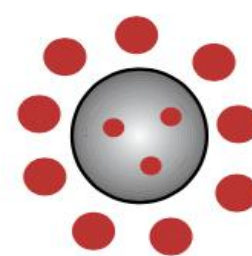


QCP at which all electric charge resides outside the horizon of the black brane

G
R
A
V
I
T
Y



all charge inside
the black hole



all charge outside
the black hole

Order Parameter Scaling

■ Both η/s and ρ_{frac}/ρ exhibit:

- Finite for $B < B_c$
- Linear decrease near QCP
- Zero for $B > B_c$

■ Using scaling relations:

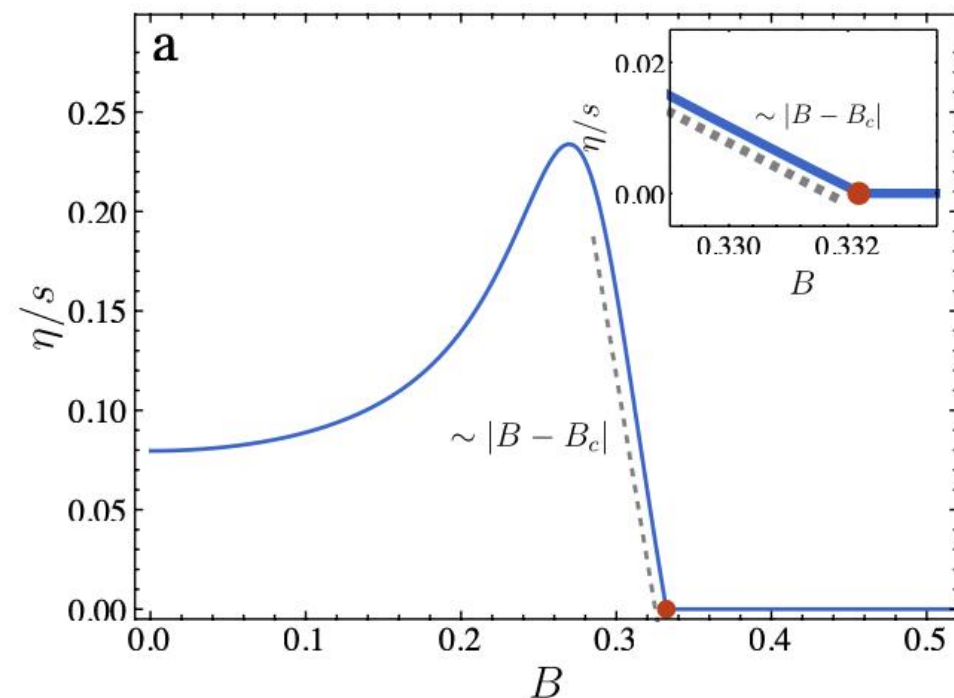
$$\gamma^* = \beta^*(\delta^* - 1) \rightarrow \delta^* = 2$$

$$\gamma^* = \nu(2 - \eta^*) \rightarrow \eta^* = 1$$

■ Transport coefficient as order parameter:

Novel feature beyond Landau paradigm

order parameter $\propto |B - B_c| \rightarrow \beta^* = 1$



[Jun-Kun Zhao, Li Li, JHEP 10 (2025) 131]

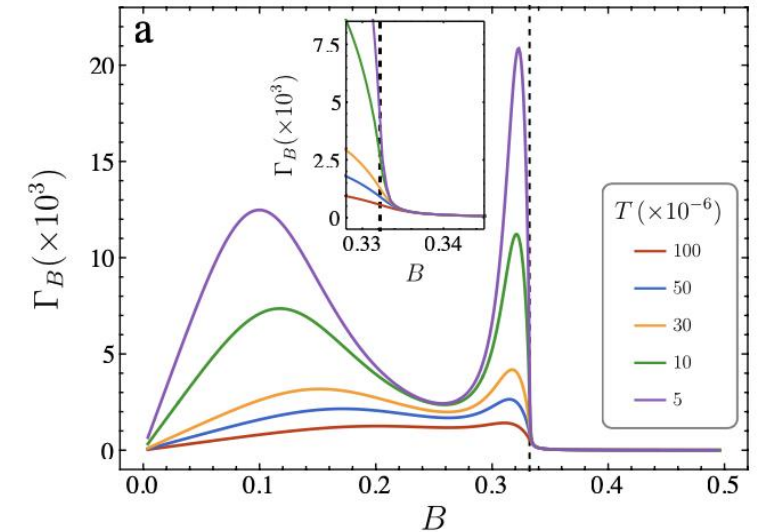
Grüneisen Ratio at z=3 QCP

Magnetic Grüneisen ratio

$$\Gamma_B = \frac{1}{T} \left(\frac{\partial T}{\partial B} \right)_S = - \frac{(\partial S / \partial B)_T}{T(\partial S / \partial T)_B}$$

Γ_B vs B at different T

- Single peak grows and diverges as $T \rightarrow 0$
- No sign reversal across QCP
- Consistent with monotonic isentropes



Grüneisen Ratio at z=3 QCP

Magnetic Grüneisen ratio

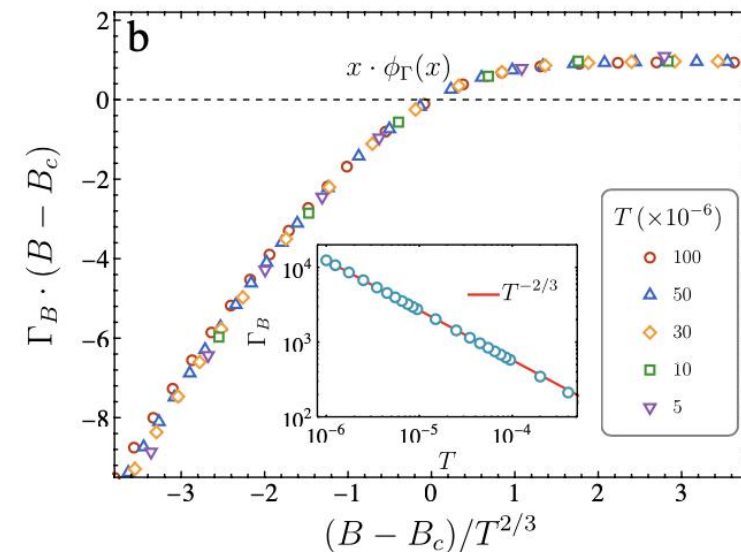
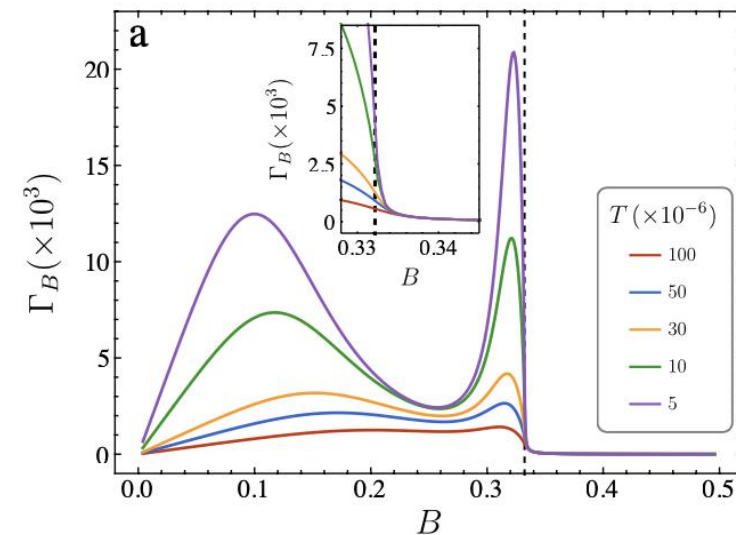
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Scaling function $\phi_\Gamma(x)$ from data collapse

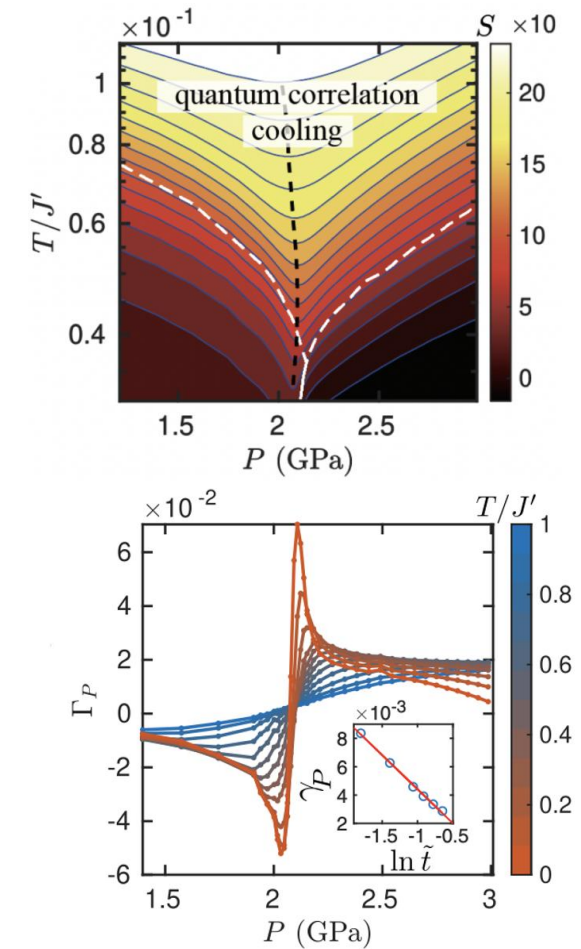
- $\Gamma_B = T^{-\frac{2}{3}} \phi_\Gamma((B - B_c)/T^{2/3})$
- Inset: Γ_B at $B = B_c$ confirms universal divergence



No Sign Reversal: What Does It Mean?

■ **Conventional QPTs:** Γ changes sign across QCP (**peak–dip structure**)

- Entropy is maximal at QCP
- Isentropes bend toward lower T at QCP



Shastry–Sutherland model
[PRL, 2023]

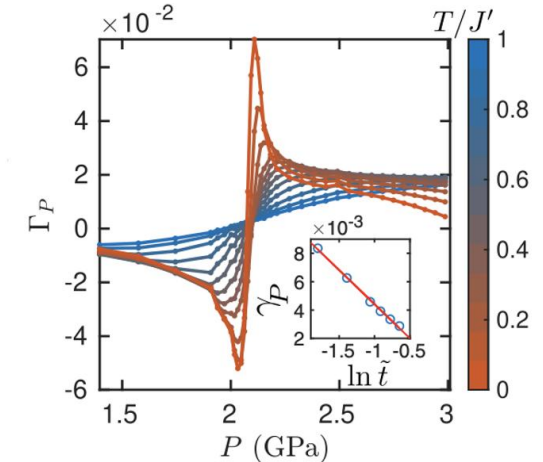
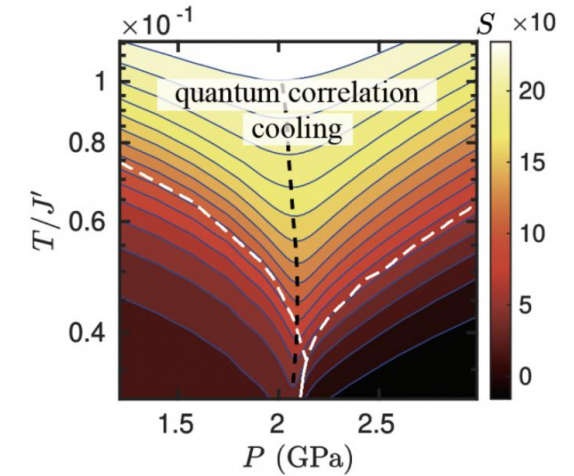
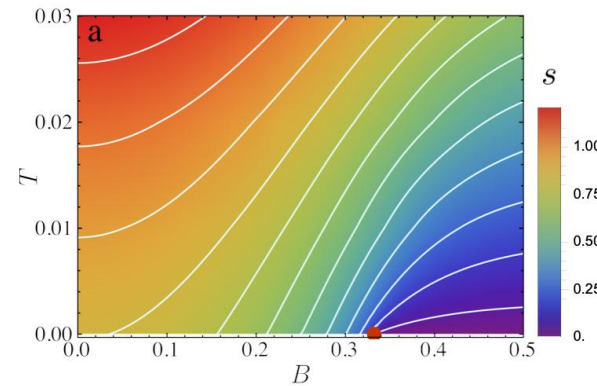
No Sign Reversal: What Does It Mean?

■ **Conventional QPTs:** Γ changes sign across QCP (**peak–dip structure**)

- Entropy is maximal at QCP
- Isentropes bend toward lower T at QCP

■ **Our model:** Γ remains positive everywhere

- Entropy decreases more slowly along $B = B_c$
- Isentropes monotonic with B



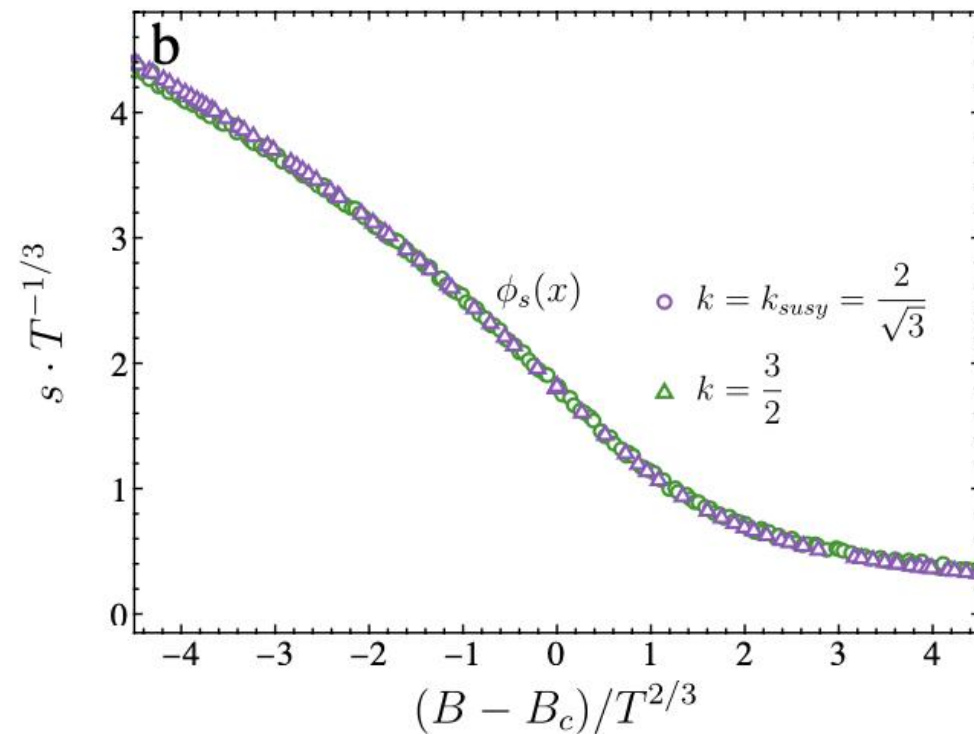
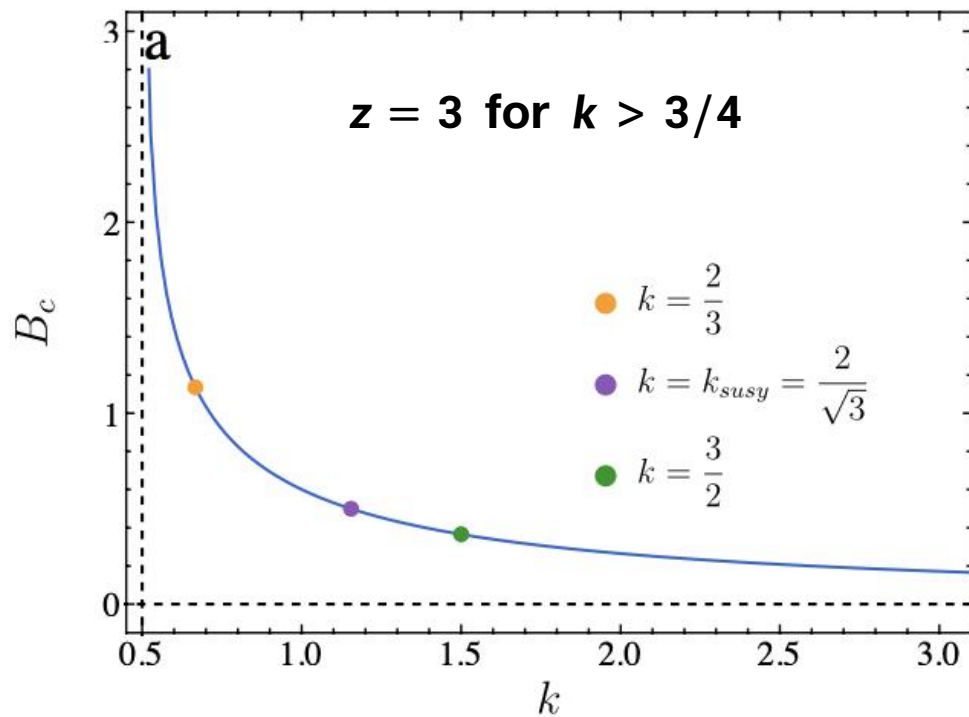
Shastry–Sutherland model
[PRL, 2023]

■ Similar observation: 3D XY model of pressure–induced QPT [PRL 2026]

■ **New universality class:** Different from conventional quantum critical systems

Universality of the $z = 3$ QCP

$$\frac{k}{24} \epsilon^{abcde} A_a F_{bc} F_{de}$$



Quantum critical scaling functions for different CS coupling fall onto a **single curve**.

The $z = 3$ QCPs are universal across different values of k and belong to the **same universality class**.

z=2 QPT: 3rd Phase Transition

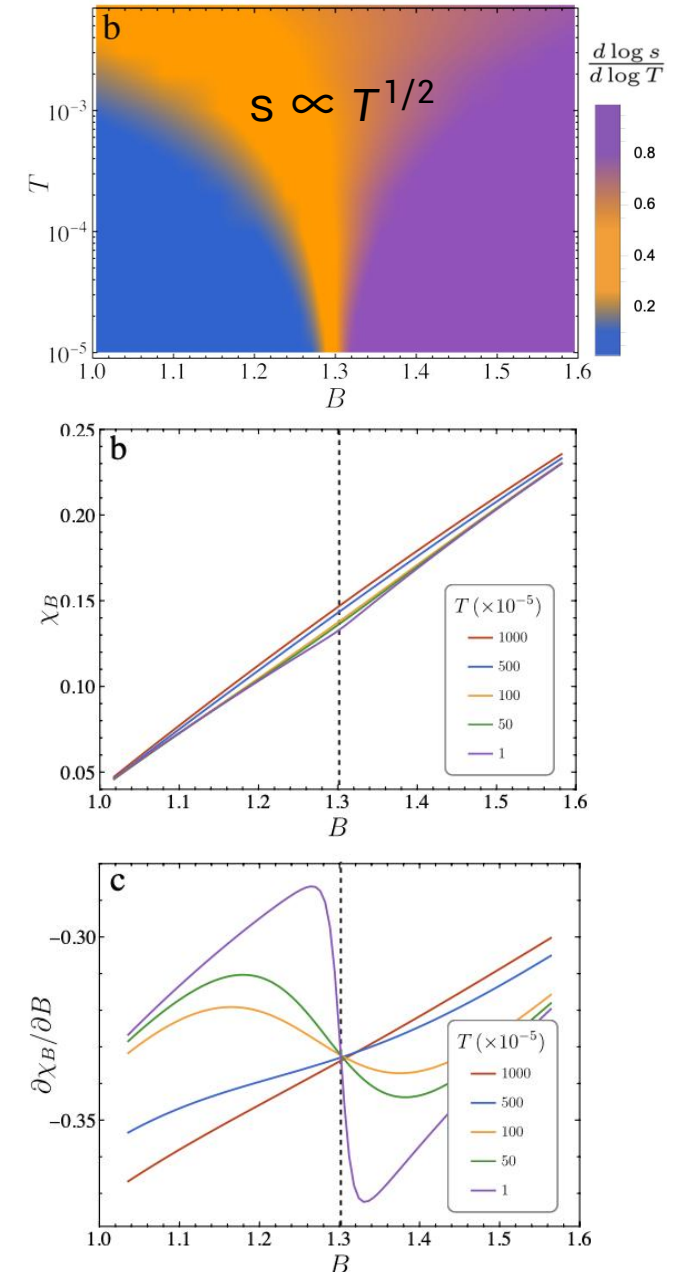
■ Chern–Simons coupling: $k=2/3 \rightarrow z=2$

■ Key observations:

- Entropy: $s \propto T^{1/2}$ at QCP
- Magnetization and susceptibility: continuous
- $\partial\chi_B/\partial B$ becomes discontinuous at low T

■ **Third–order transition:** No symmetry breaking +Both phases paramagnetic ($\chi_B > 0$)

■ Critical exponents: Only z and ν can be extracted
+Standard scaling relations not applicable



Gruneisen Ratio at z=2 QPT

Γ_B vs B at different T

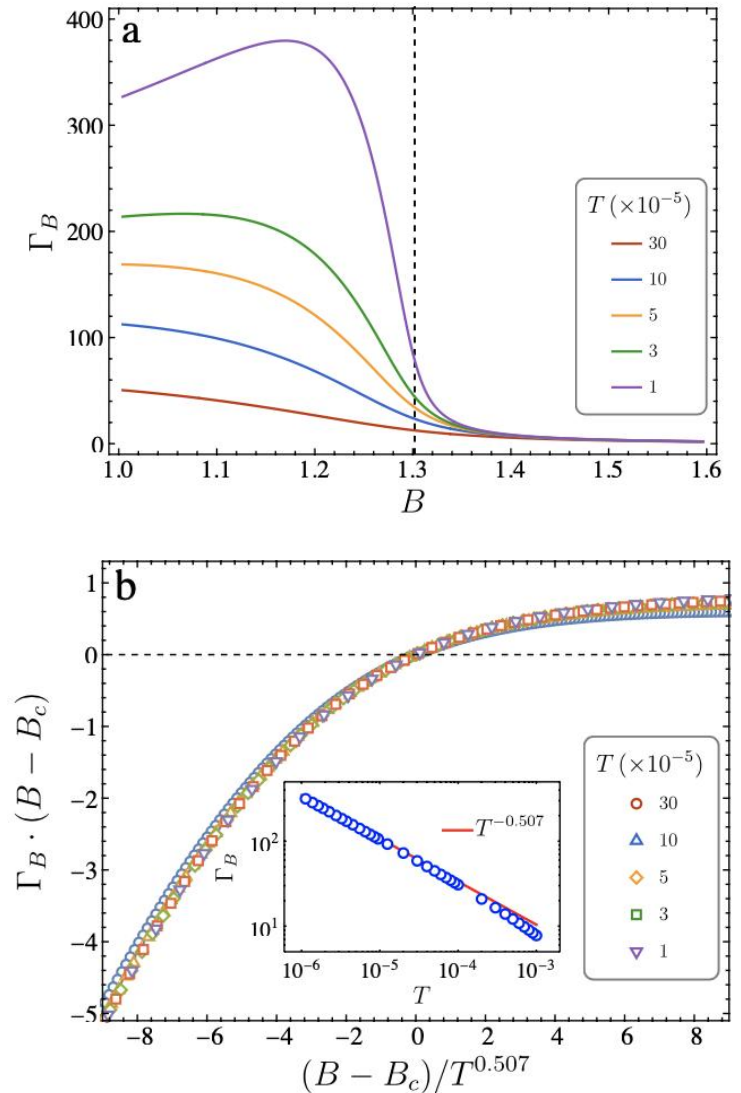
- Diverges near QCP as T decreases
- Remains positive (no sign reversal)

Data collapse of Γ_B

- $x = (B - B_c)/T^{0.507}$
- Universal scaling function $\phi_\Gamma(x)$

Key conclusion: Universal scaling of Gruneisen ratio extends to third-order QPTs

This is a new result !



Outline

- Introduction
- Thermodynamics for CS black holes
- Application: quantum criticality
- **Summary**

Key Takeaways

- ◆ General principle: BH thermodynamics in any theory with CS-type interactions requires the corresponding **transgression completion**.
- ◆ BH thermodynamics provides a natural framework for studying strongly coupled **quantum criticality** without quasiparticle assumptions.

Thank You !