

Transport and Spin Relaxation in Hot QCD

From Microscopic Collisions to Spectral Functions

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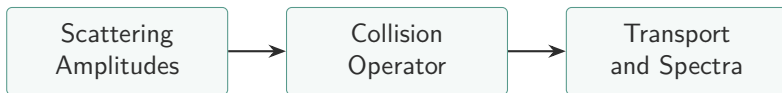
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How do Disturbances Relax?

- **Microscopic Input:** The thermal bath fixes occupations and screening; microscopic amplitudes then determine relaxation.



- **Momentum Anisotropy** $\longrightarrow$ Shear Relaxation.
- **Charge Imbalance** $\longrightarrow$ Diffusion.
- **Spin Polarization** $\longrightarrow$ Density-Matrix Evolution.

Key Point

Streaming supplies the drive; microscopic collisions set relaxation.

$$\hbar = c = k_B = 1, \quad \beta = 1/T, \quad g_{\mu\nu} = \text{diag}(-, +, +, +)$$
$$P = (E_p, \mathbf{p}), \quad K = (E_k, \mathbf{k}), \quad Q = (\omega, \mathbf{q}), \quad \mathbf{v}_p = \mathbf{p}/E_p$$
$$\int_{\mathbf{p}} \equiv \int \frac{d^3p}{(2\pi)^3}, \quad d\Pi_p = \frac{d^3p}{(2\pi)^3 2E_p}, \quad d^n p \equiv \prod_{i=1}^n dp_i$$

- **Momentum Variables:** P, K, Q : four-momenta; $\mathbf{p}, \mathbf{k}, \mathbf{q}$: three-momenta; p, k, q : their magnitudes.
- **Fourier Modes:** $e^{-i\omega t + i\mathbf{k} \cdot \mathbf{x}}$; transform: opposite phase; $\text{Im } \omega_{\text{pole}} < 0$.
- **Collision Convention:** $D_t f = -\mathcal{C}[f]$; positive relaxation operator $\mathcal{L}$.
- **Statistics Sign:** $s_a = +1$ (bosons), $s_a = -1$ (fermions).
- **Reference Plasma:** $\mu_a = 0$; chemical-potential shifts are response sources.

AMY and HT: Light Thermal Species

- **Thermal Regime:** $g \ll 1$, $p \sim T$, bare masses $\ll gT$.
Thermal masses retained in screening and collinear formation energies.

Li-Yee: Dilute Massive Quarks

- **Dilute Probe:** $m \gg gT$, occupation $\ll 1$.
Unpolarized equilibrium bath of light quarks and gluons.
- **Mass Hierarchy:** $m \gg gT$ allows relativistic motion; $m \gg T$ is not required.
- **Diluteness:** an independent assumption.



Thermal Medium and Static Response

How does a Weak Color Field Change an Equilibrium Plasma?

The Distribution Function

- **Occupation:** $f_a(t, \mathbf{x}, \mathbf{p})$: occupation per internal state of species a .

$$dN_a = \nu_a f_a \frac{d^3x d^3p}{(2\pi)^3}, \quad n_a^{\text{num}} = \nu_a \int \frac{d^3p}{(2\pi)^3} f_a$$

$$\nu_g = 2(N_c^2 - 1), \quad \nu_q = \nu_{\bar{q}} = 2N_c \quad \text{per flavor}$$

- **Momentum:** diagonal after coarse graining.
- **Internal States:** averaged for unpolarized, color-singlet matter.
- **Density and Occupation:** Number density n_a^{num} ; equilibrium occupation $f_{0a}(E)$.

Observables as Weighted Moments

Observable Moments

$$\begin{aligned}n_Q &= \sum_a \nu_a \int \frac{d^3p}{(2\pi)^3} q_a f_a, & J_Q^i &= \sum_a \nu_a \int \frac{d^3p}{(2\pi)^3} q_a v_p^i f_a \\T^{00} &= \sum_a \nu_a \int \frac{d^3p}{(2\pi)^3} E_p f_a, & T^{ij} &= \sum_a \nu_a \int \frac{d^3p}{(2\pi)^3} \frac{p^i p^j}{E_p} f_a\end{aligned}$$

Shear Stress: Momentum Flux

- **Momentum Flux:** Velocity through a y -normal surface: v_y . Momentum carried: p_x . The single-particle weight is $p_x v_y = p_x p_y / E_p$.
- **Bulk Channel:** Temperature-dependent masses add mean-field terms in the bulk channel.

Equilibrium and Quantum Statistics

- **Reference State:** Reference equilibrium and its deviation; time and position arguments suppressed:

$$f_a(\mathbf{p}) = f_{0a}(E_p) + \delta f_a(\mathbf{p})$$

Equilibrium Occupation and Its Derivative

$$f_{0a}(E) = \frac{1}{e^{\beta(E-\mu_a)} - s_a} \quad f_{0a}(E)[1 + s_a f_{0a}(E)] = -T \frac{\partial f_{0a}(E)}{\partial E}$$

Bosons

- **Bose Enhancement:** equilibrium factor $1 + f_{0a}$.

Fermions

- **Pauli Blocking:** equilibrium factor $1 - f_{0a}$.

$$E \rightarrow E + \delta E \quad \Longrightarrow \quad \delta f_a = -\frac{f_{0a}(E)[1 + s_a f_{0a}(E)]}{T} \delta E \quad (\text{fixed } T, \mu_a; \text{ first order})$$

- **Statistics:** $s_a = +1$ for bosons and $s_a = -1$ for fermions.

- **Color Multiplicities:** $SU(N_c)$: N_c quark colors, $d_A = N_c^2 - 1$ gluon colors; $N_c = 3$ in QCD.

$$\text{Quark-Gluon Vertex} \propto g\gamma^\mu t^A, \quad \text{Tr}(t^A t^B) = \frac{1}{2}\delta^{AB}$$

$$\sum_A t^A t^A = C_F \mathbf{1}, \quad C_F = \frac{N_c^2 - 1}{2N_c}, \quad C_A = N_c$$

$$C_q = C_F, \quad C_g = C_A, \quad \alpha_s = \frac{g^2}{4\pi}$$

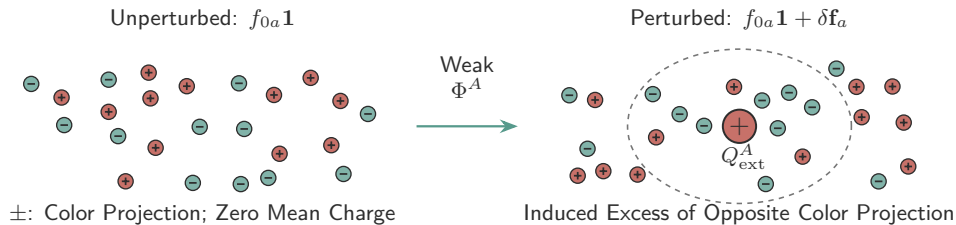
- **QCD Vertex:** g : coupling; t^A : color matrix; γ^μ : Dirac matrix.
- **Gluon Self-Interactions:** carry the adjoint color factor C_A .
- **Color Charge:** These generators fix the color charge carried by each thermal species.

How a Weak Color Potential Changes the Bath

- **Color Response:** Color matrix $\mathbf{f}_a$; averaged occupation $f_a = \text{Tr } \mathbf{f}_a / d_a$ (d_a : colors); $g \ll 1$, $\mu_a = 0$. A weak color potential shifts the energy through the generator $T_{R_a}^A$ ($T_F^A = t^A$).

$$\delta E_a = g \Phi^A T_{R_a}^A, \quad \mathbf{f}_a = f_{0a}(p) \mathbf{1} + \delta \mathbf{f}_a, \quad \delta \mathbf{f}_a = f'_{0a}(p) \delta E_a = -\frac{g}{T} f_{0a}(p) [1 + s_a f_{0a}(p)] \Phi^A T_{R_a}^A$$

- **Occupation Shift:** Positive energy shifts reduce occupation; opposite color projections are favored.



- **Induced Charge:** comes from redistributing the existing hard thermal particles.

Summing the Induced Color Charge

- **Color Trace:** Trace the color response; d_a counts colors and ν_a counts all internal states.

$$\begin{aligned}\frac{1}{d_a} \text{Tr}(T_{R_a}^A T_{R_a}^B) &= \frac{C_a}{d_A} \delta^{AB}, & \delta \mathbf{f}_a &= g f'_{0a}(p) \Phi^B T_{R_a}^B \\ \rho_{\text{ind}}^A &= g \sum_a \frac{\nu_a}{d_a} \int \frac{d^3 p}{(2\pi)^3} \text{Tr}(T_{R_a}^A \delta \mathbf{f}_a) = -m_D^2 \Phi^A, & m_D^2 &= -\frac{g^2}{d_A} \sum_a \nu_a C_a \int \frac{d^3 p}{(2\pi)^3} f'_{0a}(p) > 0 \\ \int \frac{d^3 p}{(2\pi)^3} f'_0(p) &= \frac{1}{2\pi^2} \int_0^\infty p^2 f'_0(p) dp = -2 \int \frac{d^3 p}{(2\pi)^3} \frac{f_0(p)}{p}\end{aligned}$$

Color Susceptibility

$$m_D^2 = \frac{2g^2}{d_A} \sum_a \nu_a C_a \int \frac{d^3 p}{(2\pi)^3} \frac{f_{0a}(p)}{p}$$

- **Boundary Term:** $p^2 f_0(p)$ vanishes. The occupation response determines m_D^2 .

Evaluating the Thermal Radial Integrals

- **Radial Variable:** Set $x = p/T$; the massless phase-space measure leaves one radial power of p .

$$\int \frac{d^3p}{(2\pi)^3} \frac{f_0(p)}{p} = \frac{T^2}{2\pi^2} \int_0^\infty \frac{x dx}{e^x \mp 1}$$
$$\frac{1}{e^x - 1} = \sum_{n=1}^{\infty} e^{-nx}, \quad \frac{1}{e^x + 1} = \sum_{n=1}^{\infty} (-1)^{n-1} e^{-nx}, \quad \int_0^\infty x e^{-nx} dx = \frac{1}{n^2}$$
$$\sum_{n \geq 1} \frac{1}{n^2} = \frac{\pi^2}{6}, \quad \sum_{n \geq 1} \frac{(-1)^{n-1}}{n^2} = \frac{\pi^2}{12}$$

Thermal Radial Weights

$$\int \frac{d^3p}{(2\pi)^3} \frac{f_{0g}}{p} = \frac{T^2}{12}, \quad \int \frac{d^3p}{(2\pi)^3} \frac{f_{0q}}{p} = \frac{T^2}{24}$$

- **Thermal Weights:** Both have dimension energy squared; fermions contribute half the radial weight per internal state.

The Debye Mass from Thermal Color Charges

- **Debye Mass:** Spin, color and thermal occupation weights determine the Debye mass of this bath.

$$\nu_g = 2d_A, \quad C_g = N_c, \quad \nu_{q+\bar{q}} = 4N_c N_f, \quad C_F = \frac{d_A}{2N_c}$$

$$m_D^2 = \frac{2g^2}{d_A} \left[2d_A N_c \frac{T^2}{12} + 4N_c N_f C_F \frac{T^2}{24} \right]$$

Debye Screening Scale

$$m_D^2 = g^2 T^2 \left(\frac{N_c}{3} + \frac{N_f}{6} \right), \quad m_D \sim gT$$

- **Gluons:** two physical polarizations and d_A colors.
- **Quarks:** two spins, quark/antiquark, N_c colors and N_f light flavors.
- **Soft Exchange:** At momenta $q \sim m_D$, the induced charge changes the exchanged field.

II

From Amplitudes to Collisions

How do Quantum Transition Probabilities Determine the Collision Term?

- **Moving Cell:** Follow a small phase-space cell carried by a quasiparticle:

$$\begin{aligned}\frac{d\mathbf{x}}{dt} &= \mathbf{v}_p, & \frac{d\mathbf{p}}{dt} &= \mathbf{F}_{a,\text{ext}} \\ \frac{df_a(t, \mathbf{x}(t), \mathbf{p}(t))}{dt} &= \partial_t f_a + \mathbf{v}_p \cdot \nabla_x f_a + \mathbf{F}_{a,\text{ext}} \cdot \nabla_p f_a \equiv D_t f_a \\ \nabla_x \cdot \mathbf{v}_p + \nabla_p \cdot \mathbf{F}_{a,\text{ext}} &= 0, & dN_a &= \nu_a f_a \frac{d^3x d^3p}{(2\pi)^3} \\ \text{No Collisions:} & & D_t f_a &= 0\end{aligned}$$

- **Liouville Evolution:** particle number and cell volume are preserved.
- **External Force:** $\mathbf{F}_{a,\text{ext}}$: smooth external force; scattering will change the cell's occupation.
- **Phase-Space Flow:** Fixed-mass free motion and the Lorentz force satisfy the displayed divergence condition.

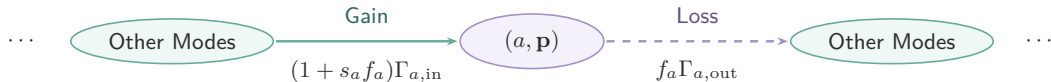
- **Operators:** Occupation: $\hat{n}_a$. Inter-mode interaction: H_I .

$$f_a = \langle \hat{n}_a \rangle, \quad \partial_t f_a|_{\text{coll}} = i \langle [H_I, \hat{n}_a] \rangle$$
$$\langle \hat{n}_a(\mathbf{p}) \hat{n}_b(\mathbf{k}) \rangle \simeq f_a(\mathbf{p}) f_b(\mathbf{k}) \quad \text{for distinct incoming modes}$$
$$\tau_{\text{event}} \ll \Delta t \ll \tau_{\text{mac}}$$

- **On-Shell Quasiparticles:** width $\ll E_p$; slow space-time variation.
- **Molecular Chaos:** neglect incoming connected correlations.
- **Markov Approximation:** local rates; no event memory.
- **Radiation Event:** the full coherent formation interval.

Collisions Change Occupation by Gain Minus Loss

- **Entry and Exit:** Along the moving cell, suppress $\mathbf{x}, \mathbf{p}$; $\Gamma_{a,\text{in/out}}$ are entry/exit rates.



$$f_a(t + \Delta t) - f_a(t) = \Delta t [(1 + s_a f_a) \Gamma_{a,\text{in}} - f_a \Gamma_{a,\text{out}}] + O(\Delta t^2)$$

$$\mathcal{C}_a[f] \equiv f_a \Gamma_{a,\text{out}} - (1 + s_a f_a) \Gamma_{a,\text{in}}$$

Boltzmann Equation

$$(\partial_t + \mathbf{v}_p \cdot \nabla_x + \mathbf{F}_{a,\text{ext}} \cdot \nabla_p) f_a = -\mathcal{C}_a[f] \quad (1)$$

- **Collision Sign:** Positive $\mathcal{C}_a$: net depletion of the selected momentum state.
- **Entry Rate:** $\Gamma_{a,\text{in}}$ excludes its final-state factor $1 + s_a f_a$.
- **Rates and Statistics:** interactions fix rates; quantum statistics fixes occupation factors.

A Collision Changes a Many-Particle Occupation

- **Tracked Mode:** Track mode $(a, \mathbf{p})$ in $a(P)b(K) \rightarrow c(P')d(K')$; the four modes are distinct.

$$f_a(\mathbf{p}) = \sum_{\mathbf{N}} \mathbb{P}(\mathbf{N}) N_{a,p}$$

$$\Delta(N_{a,p}, N_{b,k}, N_{c,p'}, N_{d,k'}) = (-1, -1, +1, +1)$$

$$W_{\mathbf{N}'\mathbf{N}} = \frac{|\langle \mathbf{N}' | \hat{S}_{\Delta t} - \mathbf{1} | \mathbf{N} \rangle|^2}{\Delta t}, \quad \tau_{\text{event}} \ll \Delta t \ll \tau_{\text{mac}}$$

$$\partial_t f_a(\mathbf{p})|_{\text{coll}} = \sum_{\mathbf{N}, \mathbf{N}'} \mathbb{P}(\mathbf{N}) W_{\mathbf{N}'\mathbf{N}} (N'_{a,p} - N_{a,p})$$

- **Mode Occupations:** $\mathbf{N}$: integer mode occupations; $\mathbb{P}(\mathbf{N})$: their joint probability.
- **Forward Event:** one particle lost from $(a, \mathbf{p})$; reverse event: one gained.
- **Transition Probability:** We need the transition probability in these populated modes.

One Matrix Element Supplies Both Factors

- **Binary Amplitude:** Binary collisions: unit-pair amplitude S_{fi}^{box} in normalized box modes.

$$(\hat{S}_{\Delta t} - \mathbf{1})_{ab \rightarrow cd} = S_{fi}^{\text{box}} \hat{a}_{c,p'}^\dagger \hat{a}_{d,k'}^\dagger \hat{a}_{b,k} \hat{a}_{a,p}$$
$$\|\hat{a}_{a,p}|N_{a,p}\rangle\|^2 = N_{a,p}, \quad \|\hat{a}_{a,p}^\dagger|N_{a,p}\rangle\|^2 = 1 + s_a N_{a,p}$$

$$|\langle \mathbf{N}' | \hat{S}_{\Delta t} - \mathbf{1} | \mathbf{N} \rangle|^2 = |S_{fi}^{\text{box}}|^2 N_{a,p} N_{b,k} (1 + s_c N_{c,p'}) (1 + s_d N_{d,k'})$$

- **Coefficient:** interaction. Ladder operators: occupation factors.
- **Occupation Factors:** Incoming annihilation gives N ; outgoing creation gives $1 + sN$.
- **Squared Amplitude:** fermion reordering phases cancel.
- **Ensemble Average:** One quantum transition; ensemble averaging comes next.

Compute the Unit-Pair Rate from $\mathcal{M}$

- **Normalization:** QCD amplitude $\mathcal{M} \rightarrow$ box coefficient S_{fi}^{box} .

$$\begin{aligned} \langle P|K\rangle &= 2E_p(2\pi)^3\delta^3(\mathbf{p}-\mathbf{k}), & |P\rangle_{\text{box}} &= \frac{|P\rangle}{\sqrt{2E_pV}} \\ \langle P'K'|\hat{S}-1|PK\rangle &= i(2\pi)^4\delta^4(P+K-P'-K')\mathcal{M}_{cd}^{ab} \\ S_{fi}^{\text{box}} &= \frac{i\mathcal{M}_{cd}^{ab}\delta_{\mathbf{p}+\mathbf{k},\mathbf{p}'+\mathbf{k}'}}{V\sqrt{(2E_p)(2E_k)(2E_{p'})(2E_{k'})}} \int_0^{\Delta t} dt e^{i\Delta Et} \\ \frac{\left|\int_0^{\Delta t} dt e^{i\Delta Et}\right|^2}{\Delta t} &\rightarrow 2\pi\delta(\Delta E), & \Delta E &= E_{p'}+E_{k'}-E_p-E_k \\ W_{ab\rightarrow cd} &\equiv \frac{|S_{fi}^{\text{box}}|^2}{\Delta t} \rightarrow \frac{|\mathcal{M}_{cd}^{ab}|^2\delta_{\mathbf{p}+\mathbf{k},\mathbf{p}'+\mathbf{k}'}2\pi\delta(\Delta E)}{V^2(2E_p)(2E_k)(2E_{p'})(2E_{k'})} \end{aligned}$$

- **Transition Weight:** $W_{ab\rightarrow cd}$: no occupations. $\mathcal{M}$: all interfering diagrams.

Averaging the Event Gives the Kinetic Factors

- **Ensemble Average:** $f_a(\mathbf{p}) = \sum_{\mathbf{N}} \mathbb{P}(\mathbf{N}) N_{a,p}$.

$$\sum_{\mathbf{N}} \mathbb{P}(\mathbf{N}) W_{\mathbf{N}'\mathbf{N}} = W_{ab \rightarrow cd} \langle N_{a,p} N_{b,k} (1 + s_c N_{c,p'}) (1 + s_d N_{d,k'}) \rangle$$

$$\langle N_{a,p} N_{b,k} (1 + s_c N_{c,p'}) (1 + s_d N_{d,k'}) \rangle \simeq f_a(\mathbf{p}) f_b(\mathbf{k}) (1 + s_c f_c(\mathbf{p}')) (1 + s_d f_d(\mathbf{k}'))$$

- **Molecular Chaos:** factorized modes; connected occupation correlations neglected.

$$\begin{aligned} \partial_t f_a(\mathbf{p})|_{ab \leftrightarrow cd} &= W_{cd \rightarrow ab} f_c(\mathbf{p}') f_d(\mathbf{k}') (1 + s_a f_a(\mathbf{p})) (1 + s_b f_b(\mathbf{k})) \\ &\quad - W_{ab \rightarrow cd} f_a(\mathbf{p}) f_b(\mathbf{k}) (1 + s_c f_c(\mathbf{p}')) (1 + s_d f_d(\mathbf{k}')) \end{aligned}$$

- **Occupation Change:** Forward event: one particle lost. Reverse event: one gained.

Summing the Same Events Gives the Collision Integral

- **One Statistical Factor:** for $a(\mathbf{p})b(\mathbf{k}) \leftrightarrow c(\mathbf{p}')d(\mathbf{k}')$, use loss minus gain.

$$\sum_{\mathbf{k}, \mathbf{p}', \mathbf{k}'} \frac{\delta_{\mathbf{p}+\mathbf{k}, \mathbf{p}'+\mathbf{k}'}}{V^2} \rightarrow \int \frac{d^3k d^3p' d^3k'}{(2\pi)^9} (2\pi)^3 \delta^3(\mathbf{p} + \mathbf{k} - \mathbf{p}' - \mathbf{k}')$$

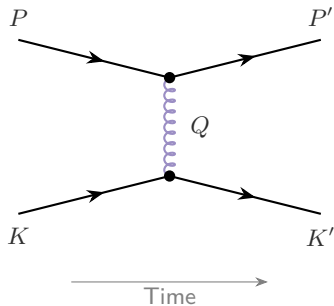
Statistical Factor and Elastic Collision Term

$$\mathcal{B}_{22} = f_a(\mathbf{p})f_b(\mathbf{k})[1 + s_c f_c(\mathbf{p}')] [1 + s_d f_d(\mathbf{k}')] - f_c(\mathbf{p}')f_d(\mathbf{k}') [1 + s_a f_a(\mathbf{p})] [1 + s_b f_b(\mathbf{k})]$$

$$\mathcal{C}_{a,2\leftrightarrow 2}(\mathbf{p}) = \frac{1}{4E_p \nu_a} \sum_{bcd} \int d\Pi_k d\Pi_{p'} d\Pi_{k'} |\mathcal{M}_{cd}^{ab}|^2 \times (2\pi)^4 \delta^4(P + K - P' - K') \mathcal{B}_{22}.$$

- **Time Reversal:** same summed $|\mathcal{M}|^2$ for loss and gain; $D_t f_a = -\mathcal{C}_a$.
- **Internal States:** amplitude squared summed; $1/\nu_a$ averages the tracked state.
- **Counting:** $1/(4E_p) = 1/(2E_p) \times 1/2$ for ordered final states.

Elastic Momentum Exchange



- **Exchange:** $Q = P - P'$ (off shell).
- **Probability:** $|\mathcal{M}|^2$.

$$P + K = P' + K'$$

The Loss Term Also Equals Density Times Cross Section

- **Classical Final States:** For one channel with $c \neq d$, set final-state occupation factors to one.

$$I_{PK} = \sqrt{(P \cdot K)^2 - m_a^2 m_b^2}, \quad v_M = \frac{I_{PK}}{E_p E_k}$$

$$d\sigma_{ab \rightarrow cd} = \frac{|\mathcal{M}_{cd}^{ab}|^2}{4I_{PK}\nu_a\nu_b} (2\pi)^4 \delta^4(P + K - P' - K') d\Pi_{p'} d\Pi_{k'}$$

$$\partial_t f_a|_{\text{loss}} = -f_a \Gamma_{a,\text{out}}, \quad \Gamma_{a,\text{out}}^{ab \rightarrow cd}(\mathbf{p}) = \nu_b \int \frac{d^3 k}{(2\pi)^3} f_b(\mathbf{k}) v_M \sigma_{ab \rightarrow cd}$$

$$\frac{\nu_b v_M}{4I_{PK}\nu_a\nu_b} \frac{d^3 k}{(2\pi)^3} = \frac{1}{2E_p \nu_a} d\Pi_k$$

- **Loss Rate:** partner density $\times$ scattering probability. v_M : Møller flux factor. $|\mathcal{M}|^2$: internal-state sum.

Lorentz Transformations Preserve the Occupation

- **Observer Change:** Change the observer; retain the same physical phase-space population.

$$x^\mu = (t, \mathbf{x}), \quad P^\mu = (E_p, \mathbf{p}), \quad P^2 = -m_a^2, \quad P^0 > 0$$

$$f'_a(x', P') = f_a(x, P)$$

$$d\Pi_p = \frac{d^4P}{(2\pi)^3} \theta(P^0) \delta(P^2 + m_a^2) = \frac{d^3p}{(2\pi)^3 2E_p}$$

- **Lorentz Scalars:** Scalar f_a ; frame-dependent energy E_p and three-volume d^3p .
- **Invariant Ingredients:** $d\Pi_p$, the conservation delta function and the summed amplitude square are invariant.
- **Mass Approximation:** Quasiparticle mass held fixed here; medium corrections are treated at their stated order.

The Covariant Equation and Its Collision Normalization

- **Coordinate-Time Rate:** $\mathcal{C}_a$ in (1) is measured per coordinate time.

$$P^\mu \partial_\mu = E_p(\partial_t + \mathbf{v}_p \cdot \nabla_x), \quad \mathcal{C}_a^{\text{cov}}[f] \equiv E_p \mathcal{C}_a[f]$$

Covariant Collision Equation

$$P^\mu \partial_\mu f_a = -\mathcal{C}_a^{\text{cov}}[f] \quad (\text{No External Force}) \quad (2)$$

$$\mathcal{C}_{a,2\leftrightarrow 2}^{\text{cov}} = \frac{1}{4\nu_a} \sum_{bcd} \int d\Pi_k d\Pi_{p'} d\Pi_{k'} |\mathcal{M}_{cd}^{ab}|^2 \times (2\pi)^4 \delta^4(P + K - P' - K') \mathcal{B}_{22}.$$

$$[\mathcal{C}_a] = \text{energy}, \quad [\mathcal{C}_a^{\text{cov}}] = \text{energy}^2$$

- **Same Statistical Factor:** $\mathcal{B}_{22}$ is defined on p. 22.
- **Three-Dimensional Form:** divide by $P^0 = E_p$.

- **External Field:** An external electromagnetic field is $\mathcal{F}^{\mu\nu}$; q_a is the particle charge.

$$\begin{aligned} \frac{dx^\mu}{d\lambda} &= P^\mu, & \frac{dP^\mu}{d\lambda} &= q_a \mathcal{F}^{\mu\nu} P_\nu, & \frac{d(P^2)}{d\lambda} &= 2q_a P_\mu \mathcal{F}^{\mu\nu} P_\nu = 0 \\ \mathcal{F}^{0i} &= E_i, & \mathcal{F}^{ij} &= \epsilon^{ijk} B_k, & \mathbf{F}_{a,\text{ext}} &= q_a (\mathbf{E} + \mathbf{v}_p \times \mathbf{B}) \\ & & \left[P^\mu \partial_\mu + q_a \mathcal{F}^{i\nu} P_\nu \frac{\partial}{\partial p^i} \right] f_a &= -\mathcal{C}_a^{\text{cov}}[f] \\ & & \frac{q_a \mathcal{F}^{i\nu} P_\nu}{E_p} \frac{\partial f_a}{\partial p^i} &= \mathbf{F}_{a,\text{ext}} \cdot \nabla_p f_a \end{aligned}$$

- **On-Shell Variables:** Only the three p^i are independent; $P^0 = E_p$ lies on the mass shell.

Local Equilibrium is Set by the Fluid Rest Frame

- **Local Energy:** selected by the fluid velocity u^μ .

$$\begin{aligned}u^\mu u_\mu &= -1, & E_{\text{rest}} &= -u_\mu P^\mu \\f_{0a}^{\text{loc}}(x, P) &= \frac{1}{\exp\{\beta(x)[-u_\mu(x)P^\mu - \mu_a(x)]\} - s_a} \\f_a &= f_{0a}^{\text{loc}} + \delta f_a, & \delta f_a &= f_{0a}^{\text{loc}}(1 + s_a f_{0a}^{\text{loc}})\phi_a \\u^\mu &= (1, \mathbf{u}) + O(u^2) \implies E_{\text{rest}} = E_p - \mathbf{p} \cdot \mathbf{u} + O(u^2)\end{aligned}$$

- **Drive and Relaxation:** $T, \mu_a, \mathbf{u}$ gradients: driving. Collisions: relaxation of δf_a .

Detailed Balance

- **Local Balance:** compare the same event and its reverse at equilibrium.

Recall Equilibrium Occupation; p. 8

$$f_{0a}(E) = [e^{\beta(E-\mu_a)} - s_a]^{-1}$$

$$\frac{f_{0a}}{1 + s_a f_{0a}} = e^{-\beta(E-\mu_a)}$$

Forward and Reverse Balance

$$\frac{f_{0a}(E_p) f_{0b}(E_k) [1 + s_c f_{0c}(E_{p'})] [1 + s_d f_{0d}(E_{k'})]}{f_{0c}(E_{p'}) f_{0d}(E_{k'}) [1 + s_a f_{0a}(E_p)] [1 + s_b f_{0b}(E_k)]} = e^{\beta(\Delta E - \Delta \mu)} = 1$$

$$\Delta E = E_{p'} + E_{k'} - E_p - E_k = 0, \quad \Delta \mu = \mu_c + \mu_d - \mu_a - \mu_b = 0$$

Key Point

Detailed balance gives $\mathcal{B}_{22}[f_0] = 0$ and hence $\mathcal{C}[f_0] = 0$.

Why a Collision Invariant has a Vanishing Moment

- **Invariant:** $\chi_a(P) + \chi_b(K) = \chi_c(P') + \chi_d(K')$ in each event.
- **First Interchange:** exchange the incoming legs, keeping the same $\mathcal{B}_{22}$ from p. 22.

$$\sum_a \nu_a \int d\Pi_p \chi_a(P) \mathcal{C}_{a,2\leftrightarrow 2}^{\text{cov}} = \frac{1}{4} \sum_{abcd} \int d\Pi_p d\Pi_k d\Pi_{p'} d\Pi_{k'} |\mathcal{M}_{cd}^{ab}|^2 \\ \times (2\pi)^4 \delta^4(P + K - P' - K') \mathcal{B}_{22} \chi_a(P).$$

Average over the Two Incoming Legs

$$\sum_a \nu_a \int d\Pi_p \chi_a(P) \mathcal{C}_{a,2\leftrightarrow 2}^{\text{cov}} = \frac{1}{8} \sum_{abcd} \int d\Pi_p d\Pi_k d\Pi_{p'} d\Pi_{k'} |\mathcal{M}_{cd}^{ab}|^2 \\ \times (2\pi)^4 \delta^4(P + K - P' - K') \mathcal{B}_{22} [\chi_a(P) + \chi_b(K)].$$

Reversing the Event Completes the Conservation Proof

- **Event Reversal:** exchange $(a, P), (b, K)$ with $(c, P'), (d, K')$.
- **Statistical Sign:** the same loss-minus-gain factor obeys $\mathcal{B}_{22} \rightarrow -\mathcal{B}_{22}$.

Vanishing Invariant Moment

$$\sum_a \nu_a \int d\Pi_P \chi_a(P) \mathcal{C}_{a,2 \leftrightarrow 2}^{\text{cov}} = \frac{1}{16} \sum_{abcd} \int d\Pi_P d\Pi_K d\Pi_{P'} d\Pi_{K'} |\mathcal{M}_{cd}^{ab}|^2 \\ \times (2\pi)^4 \delta^4(P + K - P' - K') \mathcal{B}_{22} \times [\chi_a(P) + \chi_b(K) - \chi_c(P') - \chi_d(K')] = 0.$$

- **Reason for Zero:** the last line vanishes event by event for an invariant.
- **Normalization:** incoming average $1/4 \rightarrow 1/8$; reversal average $1/8 \rightarrow 1/16$.

Conservation Laws are Moments of the Collision Term

- **Collision Invariant:** obeys $\chi_a(P) + \chi_b(K) = \chi_c(P') + \chi_d(K')$.

Recall Collision Normalization and Measure; p. 26, p. 3

$$\mathcal{C}_a^{\text{cov}} = E_p \mathcal{C}_a, \quad d\Pi_p = \frac{d^3p}{(2\pi)^3 2E_p}$$

Conservation from Collision Moments

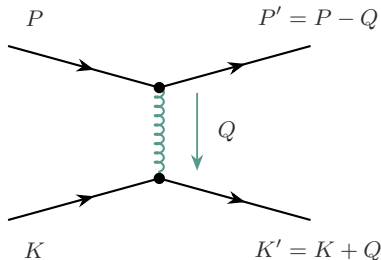
$$\begin{aligned} \sum_a \nu_a \int d\Pi_p \chi_a(P) \mathcal{C}_a^{\text{cov}}[f] &= 0, \quad \chi_a = q_a \text{ or } P^\nu \\ J_Q^\mu &= 2 \sum_a \nu_a q_a \int d\Pi_p P^\mu f_a, \quad T^{\mu\nu} = 2 \sum_a \nu_a \int d\Pi_p P^\mu P^\nu f_a \\ \partial_\mu J_Q^\mu &= 0, \quad \partial_\mu T^{\mu\nu} = 0 \quad (\text{No External Force}) \\ J_Q^0 &= n_Q, \quad T^{00} = \sum_a \nu_a \int \frac{d^3p}{(2\pi)^3} E_p f_a \end{aligned} \tag{3}$$

- **Conserved Moments:** Relabeling the legs gives $\Delta\chi = 0$. The factor 2 compensates $1/(2E_p)$ in $d\Pi_p$. Only conserved charges survive number-changing reactions.

Which Momenta Control a Collision?

- **Incoming and Outgoing Momenta:** The two incoming particles carry P, K ; the outgoing particles carry P', K' .

$$\begin{aligned} P &= (E_p, \mathbf{p}), & K &= (E_k, \mathbf{k}) \\ p &= |\mathbf{p}| \sim T, & k &= |\mathbf{k}| \sim T \\ Q &= P - P' = K' - K = (\omega, \mathbf{q}) \\ \omega &= E_p - E_{p'}, & \mathbf{q} &= \mathbf{p} - \mathbf{p}' \\ q &= |\mathbf{q}| \ll T, & \theta &\simeq \frac{|\mathbf{q}_\perp|}{p} \ll 1 \end{aligned}$$



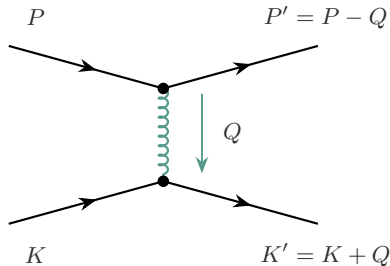
From a Gluon Exchange to a Collision Rate

- **Hard Kinematics:** For massless particles with generic incoming directions, $|P \cdot K| \sim T^2$.

$$(P - P')^2 = q^2 - \omega^2 > 0$$

$$|\mathcal{M}| \sim \frac{g^2 |P \cdot K|}{(P - P')^2} \sim \frac{g^2 T^2}{q^2 - \omega^2}$$

$$\frac{d\sigma}{d[(P - P')^2]} \sim \frac{|\mathcal{M}|^2}{[(P + K)^2]^2} \sim \frac{g^4}{(q^2 - \omega^2)^2}$$



- **Generic Transfer:** For generic transfer directions, $q^2 - \omega^2 \sim q^2$; hence

Differential Collision Rate

$$d\sigma \sim g^4 \frac{dq}{q^3}, \quad d\Gamma \sim n_{\text{bath}} v_M d\sigma \sim g^4 T^3 \frac{dq}{q^3} \quad (n_{\text{bath}} \sim T^3, v_M \sim 1)$$

- **Soft Screening:** For $q \sim m_D$, the previously derived medium response must dress this exchange.

III

Medium Corrections to Soft Exchange

How does the Plasma Response Enter a Scattering Amplitude?

Thermal Loops Give the Medium Self-Energy

- **Gluon Self-Energy:** hard internal lines; thermal average with H_0 .

$$\langle \mathcal{O} \rangle_T = \frac{\text{Tr}(e^{-H_0/T} \mathcal{O})}{\text{Tr}(e^{-H_0/T})}, \quad f_{0g}(k) = \frac{1}{e^{k/T} - 1}, \quad f_{0q}(k) = \frac{1}{e^{k/T} + 1}$$

$$\langle a^\dagger a \rangle_T = f_{0a}(k), \quad \langle aa^\dagger \rangle_T = 1 + s_a f_{0a}(k), \quad E_k = k, \quad \mu_a = 0$$

- **HTL Scales:** Hard thermal loops (HTL): internal $k \sim T$; external $|\omega|, q \ll T$.

$$\Pi_{\text{HTL}} = \left[\text{Quark} + \text{Gluon} + \text{Ghost} + \text{Tadpole} \right] \text{Leading Thermal}$$

The diagram shows four Feynman diagrams for the HTL self-energy Π_{HTL} . Each diagram consists of a horizontal external line (green wavy) with two black vertices. The diagrams are: 1. Quark: a black loop connecting the two vertices. 2. Gluon: a purple loop connecting the two vertices. 3. Ghost: a black dashed loop connecting the two vertices. 4. Tadpole: a purple loop attached to one of the vertices. The diagrams are summed together and enclosed in large square brackets. To the right of the brackets is the text 'Leading Thermal'.

- **Thermal Weights:** Energy integral $\longrightarrow f_{0q}, f_{0g}$ weights $\longrightarrow$ static m_D^2 :

$$\Pi_{\text{el}}(0, q) = m_D^2 = 4g^2 \int \frac{d^3k}{(2\pi)^3} \frac{N_c f_{0g}(k) + N_f f_{0q}(k)}{k}$$

$$-f'_{0a}(k) = \frac{f_{0a}(k)[1 + s_a f_{0a}(k)]}{T}, \quad \rho_{\text{ind}}^A = -m_D^2 \Phi^A \quad (\omega = 0)$$

From Self-Energy Insertions to a Screened Propagator

- **Electric Propagator:** Π_{el} is the self-energy insertion; D_{00}^R is the full electric propagator.

$$D_{00,0}^R(0, q) = \frac{1}{q^2}, \quad \Pi_{\text{el}}(0, q) = m_D^2$$

$$D_{00}^R \text{ (Full)} = D_{00,0}^R \text{ (Bare)} - \text{wavy line with } \Pi_{\text{el}} + \text{wavy line with } \Pi_{\text{el}} \Pi_{\text{el}} - \dots$$

Resummed Electric Propagator

$$D_{00}^R = D_{00,0}^R - D_{00,0}^R \Pi_{\text{el}} D_{00}^R$$

$$D_{00}^R(0, q) = \frac{1}{[D_{00,0}^R(0, q)]^{-1} + \Pi_{\text{el}}(0, q)} = \frac{1}{q^2 + m_D^2}$$

$$\frac{\Pi_{\text{el}}(0, q)}{q^2} = \frac{m_D^2}{q^2} \sim 1 \quad (q \sim gT)$$

- **Soft Power Counting:** At the soft scale, every number of self-energy insertions contributes at the same order.

Gauss Law Gives the Same Static Propagator

Recall Induced Charge; p. 11

$$\rho_{\text{ind}}^A = -m_D^2 \Phi^A$$

$$-\nabla^2 \Phi^A = \rho_{\text{ext}}^A + \rho_{\text{ind}}^A = \rho_{\text{ext}}^A - m_D^2 \Phi^A$$

Screened Field

$$(q^2 + m_D^2) \Phi^A(\mathbf{q}) = \rho_{\text{ext}}^A(\mathbf{q}), \quad D_{00}^R(0, \mathbf{q}) = \frac{\Phi^A}{\rho_{\text{ext}}^A} = \frac{1}{q^2 + m_D^2}$$

$$\rho_{\text{ext}}^A(\mathbf{x}) = Q_{\text{ext}}^A \delta^3(\mathbf{x}) \implies \Phi^A(r) = \frac{Q_{\text{ext}}^A}{4\pi r} e^{-m_D r}, \quad \lambda_D = m_D^{-1}$$

- **Screened Potential:** Outside the source, $(r\Phi)'' = m_D^2(r\Phi)$; decay selects $e^{-m_D r}$ and Gauss flux fixes its coefficient.

Energy Transfer Requires the Dynamic Self-Energy

- **Energy Transfer:** ω : collisionless HTL response.

Recall Collisionless Streaming and Isotropy; p. 15, p. 8

$$D_t f = 0, \quad D_t = \partial_t + \mathbf{v} \cdot \nabla_x + \mathbf{F}_{a,\text{ext}} \cdot \nabla_p, \quad \nabla_p f_{0a} = f'_{0a}(p) \hat{\mathbf{p}}$$

$$(\partial_t + \mathbf{v} \cdot \nabla_x) \delta \mathbf{f}_a = -g f'_{0a}(p) \mathbf{v} \cdot \mathbf{E}^A T_{Ra}^A, \quad \mathbf{v} = \hat{\mathbf{p}}$$

Retarded Occupation Response

$$\delta \mathbf{f}_a(\omega, \mathbf{q}; \mathbf{p}) = \frac{-ig f'_{0a}(p) \mathbf{v} \cdot \mathbf{E}^A(\omega, \mathbf{q}) T_{Ra}^A}{\omega - \mathbf{v} \cdot \mathbf{q} + i0^+}$$

$$j_{\text{ind}}^{\mu,A} = g \sum_a \frac{\nu_a}{d_a} \int \frac{d^3 p}{(2\pi)^3} v^\mu \text{Tr}(T_{Ra}^A \delta \mathbf{f}_a), \quad v^\mu = (1, \mathbf{v})$$

- **Retarded Prescription:** $+i0^+$.
- **Isotropy:** gives $(\mathbf{v} \times \mathbf{B}) \cdot \nabla_p f_{0a} = 0$.
- **Static Limit:** At $\omega = 0$, $\mathbf{E}^A = -i\mathbf{q}\Phi^A$ recovers $\delta \mathbf{f}_a = g f'_{0a} \Phi^A T_{Ra}^A$.

The Angular Integral Gives the Electric Response

- **Coulomb Gauge:** $\mathbf{E}_L = -i\mathbf{q}\Phi$.

Recall Radial Color Susceptibility; p. 11

$$m_D^2 = -\frac{g^2}{d_A} \sum_a \nu_a C_a \int \frac{d^3p}{(2\pi)^3} f'_{0a}(p)$$

$$\rho_{\text{ind}} = -\Pi_{\text{el}}\Phi, \quad \frac{q\mu}{\omega - q\mu + i0^+} = \frac{\omega}{\omega - q\mu + i0^+} - 1, \quad \mu = \hat{\mathbf{v}} \cdot \hat{\mathbf{q}}$$

$$\frac{1}{2} \int_{-1}^1 \frac{d\mu}{\omega - q\mu + i0^+} = \frac{1}{2q} \log \frac{\omega + q + i0^+}{\omega - q + i0^+}$$

Electric HTL Response

$$\Pi_{\text{el}}(\omega, q) = m_D^2 \left[1 - \frac{\omega}{2q} \log \frac{\omega + q + i0^+}{\omega - q + i0^+} \right], \quad D_{00}^R = \frac{1}{q^2 + \Pi_{\text{el}}}$$

$$\omega = 0 : \quad \Pi_{\text{el}} = m_D^2$$

- **Radial and Angular Integrals:** Radial integral: m_D^2 . Angular integral: frequency dependence.

The Same Particles Respond to Transverse Fields

- **Transverse Projection:** Choose $\mathbf{q} \parallel \hat{\mathbf{z}}$ and $\mathbf{E} \parallel \hat{\mathbf{x}}$. The azimuthal average gives $\langle v_x^2 \rangle_\varphi = (1 - \mu^2)/2$.

$$E_x = i\omega A_x, \quad j_{\text{ind}}^x = -\Pi_T A_x$$

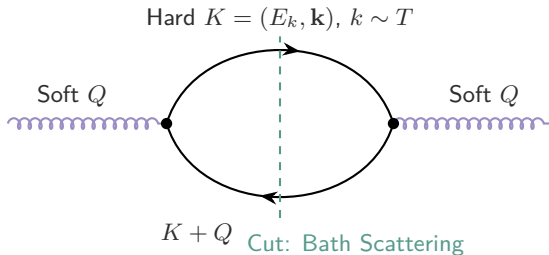
Transverse HTL Self-Energy

$$\Pi_T(\omega, q) = \frac{m_D^2 \omega}{4} \int_{-1}^1 \frac{(1 - \mu^2) d\mu}{\omega - q\mu + i0^+} = \frac{m_D^2}{2} \left[\frac{\omega^2}{q^2} + \frac{\omega}{2q} \left(1 - \frac{\omega^2}{q^2} \right) \log \frac{\omega + q + i0^+}{\omega - q + i0^+} \right]$$

$$\Pi_T(0, q) = 0, \quad D_T^R = \frac{1}{(\omega + i0^+)^2 - q^2 - \Pi_T}$$

- **Magnetic Response:** HTL: no static transverse mass. Nonperturbative magnetic scale $g^2 T$: beyond HTL.

The Angular Pole Explains Landau Damping



Landau Resonance

$$\text{Im} \frac{1}{\omega - \mathbf{v} \cdot \mathbf{q} + i0^+} = -\pi \delta(\omega - \mathbf{v} \cdot \mathbf{q})$$

- **Resonant Particles:** exchange energy with the field: $|\omega| < q$ for $|\mathbf{v}| = 1$.
- **Landau Damping:** This produces Landau damping in both electric and transverse channels.

Response Also Determines Spontaneous Thermal Fluctuations

- **Thermal Fluctuations:** Even with zero mean color field, $\langle A(t)A(0) \rangle$ is nonzero.

$$G^>(\omega) = \int dt e^{i\omega t} \langle A(t)A(0) \rangle, \quad G^<(\omega) = \int dt e^{i\omega t} \langle A(0)A(t) \rangle$$

$$\omega = E_m - E_n : \quad \frac{e^{-\beta E_n}}{e^{-\beta E_m}} = e^{\beta\omega} \quad \Longrightarrow \quad G^> = e^{\beta\omega} G^<$$

Thermal Fluctuation Relation

$$\rho^{AA} = G^> - G^< = -2 \text{Im} D^R, \quad G^< = f_{0g}(\omega) \rho^{AA}, \quad G^> = [1 + f_{0g}(\omega)] \rho^{AA}$$

$$f_{0g}(\omega) = (e^{\beta\omega} - 1)^{-1}, \quad G^> \simeq \frac{T}{\omega} \rho^{AA} \quad (|\omega| \ll T)$$

- **Thermal Correlator:** Here f_{0g} is the Bose function at signed frequency, and $D^R(t) = -i\theta(t)\langle [A(t), A(0)] \rangle$. These bath fluctuations supply momentum kicks during radiation.

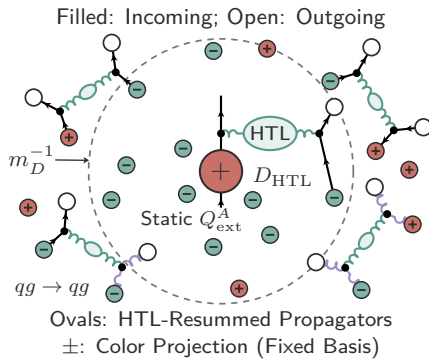
Screening and Hard Thermal Loops

- **Medium Response:** modifies test fields and thermal-particle exchanges.

$$D_{00}^R(0, \mathbf{q}) \propto \frac{1}{q^2 + m_D^2}, \quad m_D^2 = g^2 T^2 \left(\frac{N_c}{3} + \frac{N_f}{6} \right)$$

Hard Thermal Loop (HTL)

- **Hard Particles:** $k \sim T$.
 - **Soft Exchange:** $|\omega|, q \sim gT$.
 - **Resummation:** $\Pi \sim g^2 T^2 \sim q^2$.
-
- **Longitudinal Static Fields:** Debye screening.
 - **Static Transverse Fields:** no HTL mass.
 - **Landau Damping (LO):** both HTL sectors.



Return the Response to the Collision Amplitude

- **Interparticle Exchange:** propagation dressed by the bath response.

Recall From Probability to Rate (Massless Scaling); p. 24, p. 34

$$d\Gamma \sim n_{\text{bath}} v_M d\sigma, \quad \frac{d\sigma}{d[(P - P')^2]} \sim \frac{|\mathcal{M}|^2}{[(P + K)^2]^2}$$

Screened Scattering Amplitude

$$\mathcal{M}_{\text{soft}} \propto g^2 J_1^\mu D_{\mu\nu}^{\text{HTL}}(\omega, \mathbf{q}) J_2^\nu \quad (\text{Color Indices Suppressed})$$

- **Current Vertices:** J_1, J_2 : particle current vertices; no artificial source.
- **Thermal and Hard Collisions:** Thermal collisions; early hard collisions produce jets and heavy probes.
- **Static Limit:** Debye screening. Dynamic exchange: both HTL sectors.
- **Collision Input:** Rates enter the collision operator and soft-radiation kernel.
- **Transport Scales:** Next: resolved soft kicks and cumulative angular relaxation.

Why a Collision is Not a Transport Relaxation

- **Longitudinal Loss:** Longitudinal momentum loss per kick: $p(1 - \cos \theta)$.

$$1 - \cos \theta \simeq \frac{\theta^2}{2}, \quad \theta \sim \frac{q}{T}$$

Transverse Broadening and Transport Weighting

$$\hat{q} \equiv \frac{d\langle \Delta \mathbf{p}_{\perp}^2 \rangle}{dt} = \int q_{\perp}^2 d\Gamma, \quad \Gamma_{\text{tr}} \sim \frac{\hat{q}}{p^2} \quad (p \sim T) \quad \Gamma_{\text{tr}} \sim \int (1 - \cos \theta) d\Gamma \sim \int \frac{q^2}{T^2} d\Gamma$$

$$\langle \Theta_N^2 \rangle \sim N\theta^2 \sim 1 \quad \implies \quad N \sim \theta^{-2}$$

- **Accumulated Deflection:** $q \sim gT$: $O(g)$ per kick; $O(1/g^2)$ independent kicks for an $O(1)$ deflection.

Collision Rate versus Transport Rate

Recall Differential Rate, Screening and Transport Weight; p. 34, p. 13, p. 46

$$d\Gamma \sim g^4 T^3 \frac{dq}{q^3}, \quad m_D \sim gT, \quad d\Gamma_{\text{tr}} \sim \frac{q^2}{T^2} d\Gamma$$

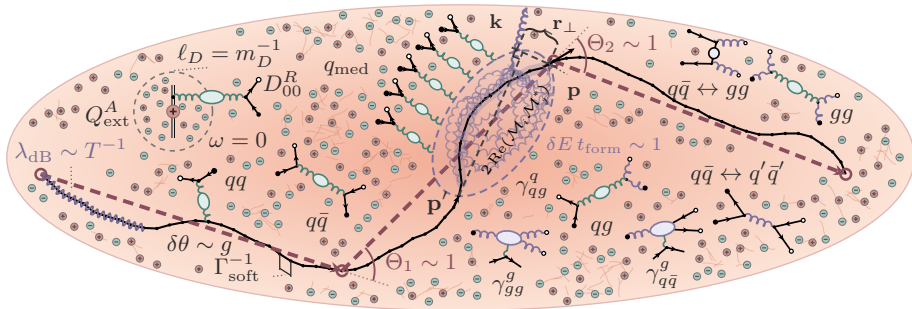
Collision and Transport Rates

$$\Gamma_{\text{soft}} \sim g^4 T^3 \int_{m_D}^T \frac{dq}{q^3} = \frac{g^4 T^3}{2} \left(\frac{1}{m_D^2} - \frac{1}{T^2} \right) \sim g^2 T$$

$$\Gamma_{\text{tr}} \sim g^4 T \int_{m_D}^T \frac{dq}{q} \sim g^4 T \log(1/g)$$

- **Soft-Collision Rate:** Γ_{soft} : resolved $q \gtrsim m_D$ kicks; waiting time $\sim 1/(g^2 T)$.
- **Broadening:** $\hat{q} \sim T^2 \Gamma_{\text{tr}} \sim g^4 T^3 \log(1/g)$; equal weight per log interval.
- **Screening Region:** $q \sim m_D$: full HTL required. Sharp cutoff: scaling estimate only.

Separated Time Scales

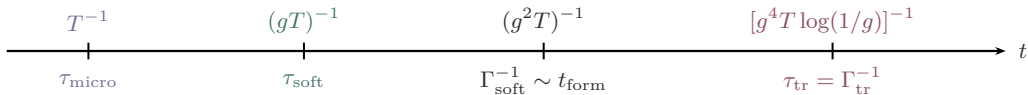


$\odot D_{\text{HTL}}$

$\circ S_{\text{HTL}}$

γ_{bc}^a : LPM; Curves: Alternative Histories

$\pm$: Color Projection.



A Screened Transport Integral

- **Screened Model:** logarithm plus a kernel-dependent constant.

Recall Static Model Inputs; p. 38, p. 46

$$D_{00}^R(0, q) = \frac{1}{q^2 + m_D^2}, \quad d\Gamma_{\text{tr}} \sim \frac{q^2}{T^2} d\Gamma, \quad d(q^2) = 2q dq$$

$$I = \int_0^T dq \frac{q^3}{(q^2 + m_D^2)^2}, \quad u = q^2 + m_D^2, \quad q^3 dq = \frac{1}{2}(u - m_D^2) du$$

$$I = \frac{1}{2} \int_{m_D^2}^{T^2 + m_D^2} du \left(\frac{1}{u} - \frac{m_D^2}{u^2} \right)$$

Logarithm and Model Constant

$$I = \frac{1}{2} \left[\log \frac{T^2 + m_D^2}{m_D^2} + \frac{m_D^2}{T^2 + m_D^2} - 1 \right] = \log \frac{T}{m_D} - \frac{1}{2} + O(m_D^2/T^2)$$

- **Leading Logarithm:** LL: universal logarithm. Constant $-1/2$: this model, not QCD.

Complete Leading-Order Structure

$$\mathcal{L}_{\text{LO}} = g^4 T [\mathcal{A} \log(T/m_D) + \mathcal{B}] \quad \text{up to higher powers of } g$$

- **LO Operators:** $\mathcal{A}$: logarithmic coefficient; $\mathcal{B}$: finite LO operator.
- **LL:** retain the logarithm; relative error $O[1/\log(1/g)]$.
- **Complete LO:** include the constant at the same power of g .
- **NLL:** next order in the inverse-log expansion.
- **NLO:** higher order in g .

Key Point

Inverting the complete LO operator retains all inverse-log orders.

IV

Effective Kinetic Theory and Radiation

Which Elastic and Radiative Processes Contribute at Leading Order?

Two Collision Sectors at Leading Order

- **Elastic and Collinear Sectors:** Hard elastic events use the rate derived above; collinear formation requires coherent resummation.

Leading-Order Kinetic Equation

$$(\partial_t + \hat{\mathbf{p}} \cdot \nabla_x) f_a = -\mathcal{C}_{a,2\leftrightarrow 2}[f] - \mathcal{C}_{a,1\leftrightarrow 2}[f] \quad (4)$$

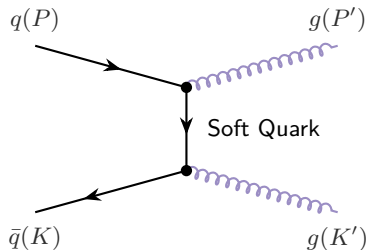
$2 \leftrightarrow 2$: Scattering and Conversion

- **Elastic Processes:** Elastic, annihilation and Compton processes; HTL treatment of soft exchange.

Effective $1 \leftrightarrow 2$: Collinear Splitting

- **Collinear Processes:** $q \leftrightarrow qg$, $\bar{q} \leftrightarrow \bar{q}g$, $g \leftrightarrow gg$, $g \leftrightarrow q\bar{q}$.
The bath supplies soft momentum during formation.

Light-Quark Conversion at Leading Log



- **Soft-Quark Exchange:** A soft exchanged quark turns a hard quark into a hard gluon, or the reverse.
- **Species Change:** small deflection.
- **LL Conversion:** LL quark–gluon conversion.
- **Massive Quarks:** A quark mass $m \gg gT$ removes this soft-fermion logarithm in LY.

From the Propagator Pole to a Thermal Energy Shift

Recall Transverse Response and Thermal Integrals; p. 41, p. 12

$$\Pi_T = \frac{m_D^2}{2} \left[x^2 + \frac{x(1-x^2)}{2} \log \frac{x+1+i0^+}{x-1+i0^+} \right], \quad x = \omega/p$$

$$\int \frac{d^3p}{(2\pi)^3} \frac{f_{0g}}{p} = \frac{T^2}{12}, \quad \int \frac{d^3p}{(2\pi)^3} \frac{f_{0q}}{p} = \frac{T^2}{24}$$

- **Quasiparticle Energy:** Particle energy: pole of the retarded propagator.

$$E_p^2 - p^2 - \text{Re} \Pi_T(E_p, p) = 0, \quad \lim_{\omega \rightarrow p^+} \Pi_T(\omega, p) = \frac{m_D^2}{2} \implies m_{\infty, g}^2 = \frac{m_D^2}{2}$$

On-Shell One-Loop Quark Mass and Hard-Particle Energy

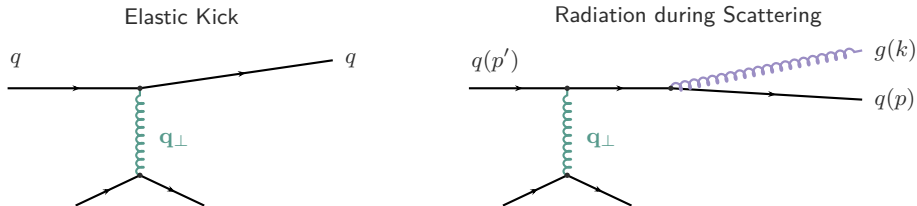
$$m_{\infty, q}^2 = 2g^2 C_F \int \frac{d^3p}{(2\pi)^3} \frac{f_{0g}(p) + f_{0q}(p)}{p} = 2g^2 C_F \left(\frac{T^2}{12} + \frac{T^2}{24} \right) = \frac{C_F g^2 T^2}{4}$$

$$E_p = \sqrt{p^2 + m_{\infty}^2} = p + \frac{m_{\infty}^2}{2p} + O(m_{\infty}^4/p^3)$$

- **Matching and Formation:** Gluon matching region: $gT \ll p \ll T$. Collinear cancellation retains the $O(g^2 T)$ shift.

A Momentum Kick and an Emitted Gluon are Different Events

- **Final State:** For a tagged quark, first specify the final particles.



Momentum Transfer versus Particle Production

$$\text{Kick: } \Delta \mathbf{p}_\perp = \mathbf{q}_\perp, \quad q_\perp \sim gT, \quad \delta\theta \sim q_\perp/p \sim g$$

$$\text{Radiation: } q(p') \longrightarrow q(p) + g(k), \quad p' = p + k, \quad p, p', k \sim T$$

- **Exchange and Radiation:** Green: internal soft exchange. Purple: an outgoing radiated mode.
One kick does not require the production of one gluon.

- **Transverse CM:** $p' = p + k$; $\mathbf{p}_\perp = -\mathbf{k}_\perp$ (daughter q, g).

$$\mathbf{h} \equiv k\mathbf{p}_\perp - p\mathbf{k}_\perp = p'\mathbf{p}_\perp = -p'\mathbf{k}_\perp$$

$$E \simeq p_z + \frac{p_\perp^2 + m_\infty^2}{2p_z}, \quad \delta E = E_q(p) + E_g(k) - E_q(p')$$

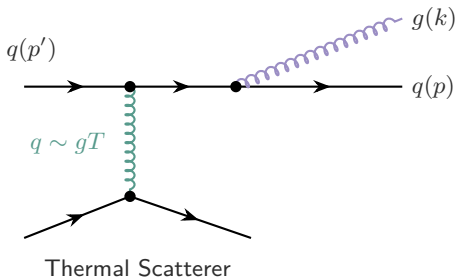
Formation Energy and Time

$$\delta E = \frac{m_{\infty,g}^2}{2k} + \frac{m_{\infty,q}^2}{2p} - \frac{m_{\infty,q}^2}{2p'} + \frac{h^2}{2pkp'} \quad (5)$$

$$\delta E t_{\text{form}} \sim 1 \quad \Rightarrow \quad t_{\text{form}} \sim \frac{1}{g^2 T} \quad (p, p', k \sim T)$$

- **Formation Criterion:** The emission becomes resolved when its relative phase reaches order one.

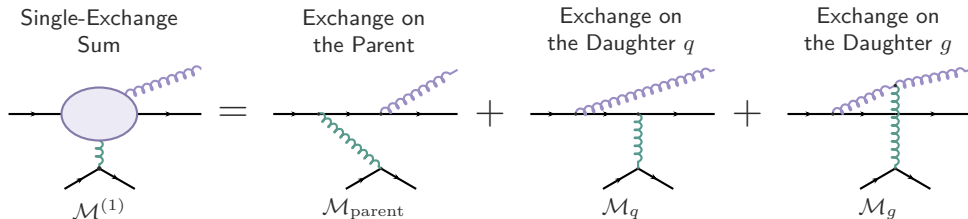
Collision-Induced Radiation



- **Soft Exchange:** with the bath.
- **Daughters:** hard, nearly collinear.
- **Interference:** sum amplitudes.
- **Amplitude:** one contributing graph.

One Soft Exchange Contributes Several Radiation Amplitudes

- **Common External States:** Fix the same incoming and outgoing momenta, colors and medium recoil.



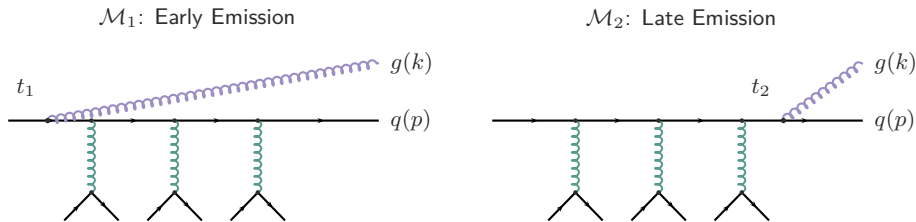
Add the Allowed Attachments before Squaring

$$\mathcal{M}^{(1)} = \mathcal{M}_{\text{parent}} + \mathcal{M}_q + \mathcal{M}_g, \quad P^{(1)} \propto |\mathcal{M}^{(1)}|^2$$

- **Single-Exchange Amplitude:** Purple blob: the single-exchange amplitude; LPM includes all exchange orders.
Each term has one final hard gluon; the last term uses a three-gluon vertex.

Early and Late Emission can Reach the Same Final State

- **Interfering Alternatives:** Both panels describe alternatives for the same measured $q(p)g(k)$.



Coherent Alternatives; Incoherent Medium Trace

$$\mathcal{M}_{FI} = \mathcal{M}_{1,FI} + \mathcal{M}_{2,FI} + \cdots, \quad P \propto \frac{1}{Z} \sum_{I,F} e^{-\beta E_I} |\mathcal{M}_{FI}|^2$$

- **Medium Average:** I, F : initial/final medium states; $Z = \sum_I e^{-\beta E_I}$.
Add amplitudes at fixed I, F ; sum probabilities over unobserved F .

Formation Time Comes from the Relative Phase

- **Two Histories:** During $\tau = t_2 - t_1$, one history contains qg and the other still contains q .

Recall The Two Intermediate Energies; p. 56

$$\delta E = E_q(p) + E_g(k) - E_q(p')$$

$$\frac{e^{-i[E_q(p)+E_g(k)]\tau}}{e^{-iE_q(p')\tau}} = e^{-i\delta E\tau}, \quad \Delta\phi = \int_{t_1}^{t_2} dt \delta E(t)$$

The Time Window for Coherent Emission

$$|\Delta\phi| \lesssim 1 : \text{Retain the Interference,} \quad t_{\text{form}} \sim \delta E^{-1}$$

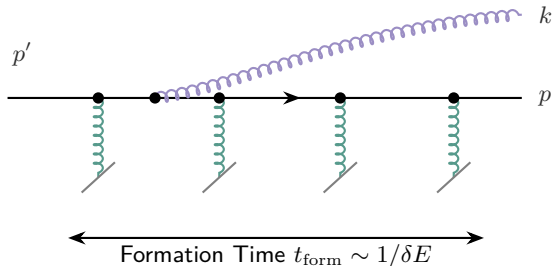
- **Phase Averaging:** Large relative phases oscillate away under the emission-time integrals. Formation is the loss of sensitivity to widely separated emission times, not a second gluon emission.

Interference during Formation

Recall Competing Time Scales; p. 56, p. 47

$$t_{\text{form}} \sim \frac{1}{g^2 T}, \quad \Gamma_{\text{soft}}^{-1} \sim \frac{1}{g^2 T}$$

Emission Amplitude



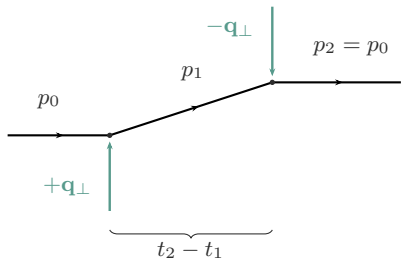
Interference during Formation

$$\left| \sum_i \mathcal{M}_i \right|^2 = \sum_i |\mathcal{M}_i|^2 + \sum_{i \neq j} \mathcal{M}_i \mathcal{M}_j^*$$

- **LPM Resummation:** Landau–Pomeranchuk–Migdal (LPM) resummation sums all such amplitudes.

Two Opposite Kicks Give Opposite Radiation Currents

- **Abelian Example:** two kicks return a charged particle to its initial momentum.



- **Soft Photon:** momentum K , polarization ϵ . The charge couples with e :

$$j(p) = e \frac{p \cdot \epsilon}{p \cdot K}, \quad a = j(p_1) - j(p_0)$$

$$p_2 = p_0 : \quad j(p_2) - j(p_1) = -a$$

- **Interference:** combine the two currents with their relative phase.

How Interference Suppresses Independent Radiation

- **Input:** opposite currents from p. 62.
- **Coherent Limit:** a small phase difference between the two kicks.

The Two Kicks Radiate Coherently

$$\mathcal{M} = a - e^{i\Delta\phi} a, \quad \underbrace{\frac{|\mathcal{M}|^2}{2|a|^2}}_{\text{Independent Kicks}} = 1 - \cos \Delta\phi \simeq \frac{\Delta\phi^2}{2} \ll 1$$

- **Phase Average:** restores the independent result.
- **QCD:** adds color rotation and gluon rescattering.
- **Scope:** LPM example; arbitrary histories need not interfere destructively.

Why Arbitrarily Many Soft Exchanges Enter at Leading Order

- **Expansion Parameter:** The relevant expansion parameter is a collision rate times the available time.

Recall Soft Collision and Formation Scales; p. 47, p. 56

$$\Gamma_{\text{soft}} \sim g^2 T, \quad t_{\text{form}} \sim (g^2 T)^{-1}$$

- **Ordered Times:** Ordered time volume, with phase/color factors suppressed only for power counting:

$$\int_{0 < t_1 < \dots < t_N < t_{\text{form}}} \prod_{j=1}^N (\Gamma_{\text{soft}} dt_j) = \frac{(\Gamma_{\text{soft}} t_{\text{form}})^N}{N!}$$

Soft Collisions and Hard Radiation have Different Counts

$$\langle N_{\text{soft}} \rangle \sim \Gamma_{\text{soft}} t_{\text{form}} = O(1), \quad P_{\text{split}}(t_{\text{form}}) \sim \alpha_s \ll 1$$

- **All Exchange Orders:** Every fixed N can contribute at the same power of g ; the calculation sums all N .
Five drawn exchanges are a schematic choice, not five emissions or a predicted collision count.

Splitting Contributes at Leading Order

- **Emission Probability:** Probability per formation time: $\alpha_s \sim g^2$.

Recall Formation Time and Hard Elastic Transfers; p. 56, p. 34

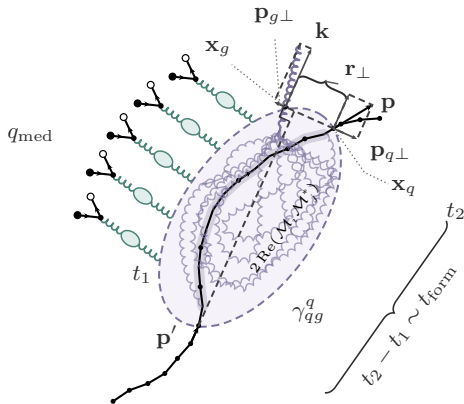
$$t_{\text{form}} \sim (g^2 T)^{-1}, \quad \Gamma_{q \sim T} \sim g^4 T^3 \int_{cT}^T \frac{dq}{q^3} \sim g^4 T \quad (0 < c < 1)$$

Leading-Order Power Counting

$$\Gamma_{\text{split}} \sim \frac{\alpha_s}{t_{\text{form}}} \sim g^4 T, \quad \Gamma_{q \sim T} \sim g^4 T$$

- **Radiation Enhancement:** collinear phase space; nearly on-shell propagators.
- **Both Rates:** the same power of g .
- **Elastic Leading Log:** Soft elastic transport adds the LL enhancement $\log(1/g)$.

A Symbolic View of the Coherent Formation Region



- **Coherent Formation**

$$\Delta\phi \simeq \delta E(t_2 - t_1) \lesssim 1$$

- **Three-Line Color Interference**

$$\mathbf{T}_q + \mathbf{T}_g + \mathbf{T}_{\bar{q}} = 0$$

- **Momentum Exchange**

$$\mathbf{p} + \mathbf{k} = \mathbf{p}' + \mathbf{Q}_{\text{med}}$$

- **Transverse Separation**

$$\mathbf{r}_{\perp} = \mathbf{x}_q - \mathbf{x}_g = \mathbf{p}'\mathbf{b}$$

Color Pairs Fix the Transverse Momentum Shifts

- **Interference:** daughter q , daughter g , conjugate parent $\bar{q}$.

$$\mathbf{T}_q + \mathbf{T}_g + \mathbf{T}_{\bar{q}} = 0, \quad c_{ij} = \frac{1}{2}(C_i + C_j - C_\ell), \quad \{i, j, \ell\} = \{q, g, \bar{q}\}$$

$$p\mathbf{x}_q + k\mathbf{x}_g = p'\mathbf{x}_{\bar{q}}, \quad \mathbf{b} = \frac{\mathbf{x}_q - \mathbf{x}_g}{p'}, \quad \mathbf{F}(\mathbf{h}) = \int d^2b e^{-i\mathbf{h}\cdot\mathbf{b}} \widetilde{\mathbf{F}}(\mathbf{b})$$

Pair	Separation	Shift Coefficient r	Weight c_r
$q, \bar{q}$	$k\mathbf{b}$	k	$C_F - C_A/2$
g, q	$-p'\mathbf{b}$	$-p'$	$C_A/2$
$\bar{q}, g$	$p\mathbf{b}$	p	$C_A/2$

- **Fourier Translation:** $e^{ir\mathbf{q}_\perp \cdot \mathbf{b}} \widetilde{\mathbf{F}} \longleftrightarrow \mathbf{F}(\mathbf{h} - r\mathbf{q}_\perp)$.
- **Vertex:** $\mathbf{F}$ carries transverse radiation amplitude; $[h] = E^2$, $[b] = E^{-2}$.

Soft Fields Sampled along a Fast Trajectory

Recall Thermal Fluctuations; p.43

$$G_{\mu\nu}^> = [1 + f_{0g}(\omega)]\rho_{\mu\nu} \simeq \frac{T}{\omega}\rho_{\mu\nu}, \quad \rho_{\mu\nu} = G_{\mu\nu}^> - G_{\mu\nu}^< = -2\text{Im } D_{\mu\nu}^R$$

$$G_{\mu\nu}^>(Q) = \int d^4x e^{i\omega t - i\mathbf{q}\cdot\mathbf{x}} \langle A_\mu(x) A_\nu(0) \rangle_T, \quad G_{\mu\nu}^<(Q) = \int d^4x e^{i\omega t - i\mathbf{q}\cdot\mathbf{x}} \langle A_\nu(0) A_\mu(x) \rangle_T$$

- **Eikonal Coupling:** $g v^\mu A_\mu$; $v^\mu = (1, 0, 0, 1)$; $v \cdot A = -A^0 + A^z$.
- **Trajectory:** $\mathbf{x}(t) \simeq \hat{\mathbf{z}}t$; phase $(\omega - q_z)t$.

$$\lim_{\tau \rightarrow \infty} \frac{1}{\tau} \left| \int_0^\tau dt e^{i(\omega - q_z)t} \right|^2 = 2\pi\delta(\omega - q_z)$$

Thermal Input for a Transverse Exchange

$$\mathcal{A}_\perp(q_\perp) = \int \frac{d\omega dq_z}{(2\pi)^2} 2\pi\delta(\omega - q_z) v^\mu v^\nu G_{\mu\nu}^>(Q)$$

- **Spacelike Exchange:** $Q^2 = q_\perp^2 > 0$; soft ω versus emitted energy k .

Evaluating the Trajectory Integral with HTL

- **Fixed $q_\perp$:** $q_z = \omega$; $x = \omega/q$; $\Pi_L \equiv (1 - x^2)\Pi_{\text{el}}$.

$$q^2 = \frac{q_\perp^2}{1 - x^2}, \quad \frac{d\omega}{\omega}(1 - x^2) = \frac{dx}{x}, \quad -1 < x < 1$$

$$v^\mu v^\nu D_{\mu\nu}^R = (1 - x^2) \left[\frac{1}{q_\perp^2 + \Pi_L(x)} - \frac{1}{q_\perp^2 + \Pi_T(x)} \right]$$

$$\frac{\mathcal{A}_\perp}{T} = \frac{2}{\pi} \int_0^1 \frac{dx}{x} \text{Im} \left[\frac{1}{q_\perp^2 + \Pi_T(x)} - \frac{1}{q_\perp^2 + \Pi_L(x)} \right]$$

- **Causality:** subtracted dispersion relation; no upper-half-plane poles.

The Spectral Integral Reduces to HTL Endpoints

$$\frac{2}{\pi} \int_0^1 \frac{dx}{x} \text{Im} \frac{1}{q_\perp^2 + \Pi_\alpha(x)} = \frac{1}{q_\perp^2 + \Pi_\alpha(0)} - \frac{1}{q_\perp^2 + \Pi_\alpha(\infty)}, \quad \alpha = L, T$$

$$\Pi_L(0) = m_D^2, \quad \Pi_T(0) = 0, \quad \Pi_L(\infty) = \Pi_T(\infty) = m_D^2/3$$

- **HTL Limits:** p. 40, p. 41; ∞ in complex x .

From the HTL Integral to Color-Pair Weights

$$\frac{\mathcal{A}_\perp}{T} = \underbrace{\left[\frac{1}{q_\perp^2 + m_D^2/3} - \frac{1}{q_\perp^2 + m_D^2} \right]}_{\text{Longitudinal}} + \underbrace{\left[\frac{1}{q_\perp^2} - \frac{1}{q_\perp^2 + m_D^2/3} \right]}_{\text{Transverse}}$$

Bath Fluctuations Weighted by Each Color Pair

$$\mathcal{A}_\perp(q_\perp) = T \left(\frac{1}{q_\perp^2} - \frac{1}{q_\perp^2 + m_D^2} \right) = \frac{T m_D^2}{q_\perp^2 (q_\perp^2 + m_D^2)}$$

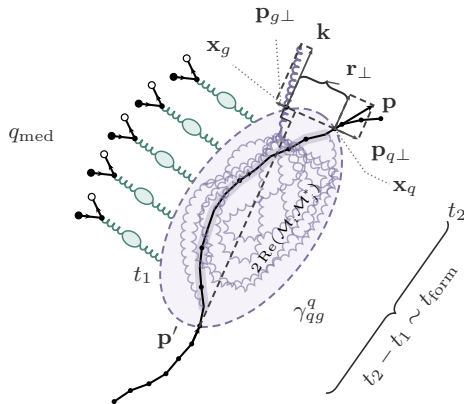
$$W_r(\mathbf{q}_\perp) \equiv \frac{g^2 c_r}{(2\pi)^2} \mathcal{A}_\perp(q_\perp), \quad r \in \{k, -p', p\}$$

- **Pair Weights:** c_r from p. 67; W_r enters $\mathcal{C}_\perp$ on the next page.
- **Single Line** $a = q, g$: replace c_r by C_a to obtain a positive probability.

$$dP_a = \frac{g^2 C_a}{(2\pi)^2} \mathcal{A}_\perp(q_\perp) dq_x dq_y dt, \quad d^2 q_\perp \equiv dq_x dq_y$$

- **Interference:** $c_k = -1/(2N_c) < 0$; individual W_r are not probabilities.

One Formation Step: Phase and Transverse Exchange



- **Fixed Kinematics:** (p', p, k) .
- **Soft Exchange:** $\mathbf{h} \rightarrow \mathbf{h} + r\mathbf{q}_\perp$.
- **Formation Phase:** $e^{-i\delta E \delta t}$.
- **Local Step:** typical $q \sim gT$.

$$\tau_{\text{bath}} \sim (gT)^{-1} \ll \delta t \ll t_{\text{form}}$$

- **Virtual Subtraction:** preserves a constant transverse amplitude.

$$\mathcal{C}_\perp(\text{constant}) = 0$$

Exchange and Virtual Subtraction

$$(\mathcal{C}_\perp \mathbf{F})(\mathbf{h}) = \sum_r \int d^2 q_\perp W_r(\mathbf{q}_\perp) [\mathbf{F}(\mathbf{h}) - \mathbf{F}(\mathbf{h} - r\mathbf{q}_\perp)]$$

From the Relative Phase to Interference Evolution

- **Interference:** qg in the amplitude, q in its conjugate; fixed (p', p, k) .

Recall Free Phase and Transverse Exchange; p. 60, p. 71

$$e^{-i\delta E(\mathbf{h})\tau}, \quad \tau = t_2 - t_1 \geq 0, \quad (\delta E \mathbf{F})(\mathbf{h}) \equiv \delta E(\mathbf{h})\mathbf{F}(\mathbf{h})$$

$$(\mathcal{C}_\perp \mathbf{F})(\mathbf{h}) = \sum_r \int d^2 q_\perp W_r(\mathbf{q}_\perp) [\mathbf{F}(\mathbf{h}) - \mathbf{F}(\mathbf{h} - r\mathbf{q}_\perp)]$$

- **Operator Action:** δE multiplies; $\mathcal{C}_\perp$ mixes transverse momenta.
- **AMY Matching:** self-energies and exchange ladders give this $\mathcal{C}_\perp$.
- **Propagator $\mathcal{U}$:** evolves the averaged interference; acts on vertex functions of $\mathbf{h}$.

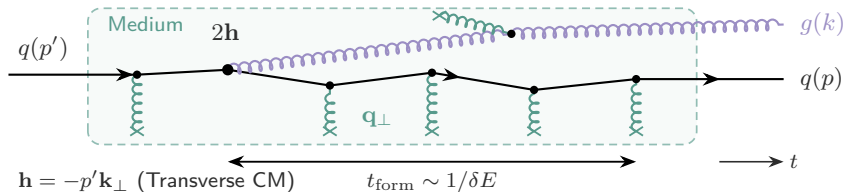
$$\tau_{\text{bath}} \ll \delta t \ll t_{\text{form}}, \quad \mathcal{U}(\delta t) = 1 - \delta t(i\delta E + \mathcal{C}_\perp) + O(\delta t^2)$$

Local Evolution after Medium Averaging

$$\partial_\tau \mathcal{U}(\tau) = -(i\delta E + \mathcal{C}_\perp)\mathcal{U}(\tau), \quad \mathcal{U}(0) = 1$$

- **Validity:** weak coupling; nearly collinear motion; bath stationary during formation.

Integrating the Formation History Gives the LPM Equation



- **Source Vertex:** $2\mathbf{h}$ in AMY normalization; $\mathcal{U}$ acts on this function of $\mathbf{h}$.
- **Effective Vertex:** $\mathbf{F}_\epsilon$ integrates the propagation from p. 72 over $\tau \geq 0$.

$$\mathbf{F}_\epsilon = \int_0^\infty d\tau e^{-\epsilon\tau} \mathcal{U}(\tau) 2\mathbf{h}, \quad \epsilon > 0$$

$$(\epsilon + i\delta E + \mathcal{C}_\perp) \mathbf{F}_\epsilon = - \left[e^{-\epsilon\tau} \mathcal{U}(\tau) \right]_{\tau=0}^{\tau \rightarrow \infty} 2\mathbf{h} = 2\mathbf{h}$$

- **Boundary Terms:** ϵ damps the upper endpoint; $\mathcal{U}(0) = 1$ supplies $2\mathbf{h}$.

The Source is the Zero-Time Boundary Term

$$\mathbf{F} = \lim_{\epsilon \rightarrow 0^+} \mathbf{F}_\epsilon, \quad (i\delta E + \mathcal{C}_\perp) \mathbf{F} = 2\mathbf{h}$$

The Same Equation Sums Every Collision Insertion

Recall Integrated Formation Equation; p. 73

$$(\epsilon + i\delta E + \mathcal{C}_\perp)\mathbf{F}_\epsilon = 2\mathbf{h}, \quad \epsilon \rightarrow 0^+$$

- **Iteration:** start with free propagation; apply $\mathcal{C}_\perp$ and propagate again.

$$\mathbf{F}_\epsilon^{(0)}(\mathbf{h}) = \frac{2\mathbf{h}}{\epsilon + i\delta E(\mathbf{h})}, \quad \mathbf{F}_\epsilon^{(n+1)}(\mathbf{h}) = -\frac{(\mathcal{C}_\perp \mathbf{F}_\epsilon^{(n)})(\mathbf{h})}{\epsilon + i\delta E(\mathbf{h})}$$

Formal Sum with the Operator Order Retained

$$\mathbf{F}_\epsilon^{\text{formal}} = \sum_{n=0}^{\infty} \mathbf{F}_\epsilon^{(n)} = \left[(\epsilon + i\delta E)^{-1} - (\epsilon + i\delta E)^{-1} \mathcal{C}_\perp (\epsilon + i\delta E)^{-1} \right. \\ \left. + (\epsilon + i\delta E)^{-1} \mathcal{C}_\perp (\epsilon + i\delta E)^{-1} \mathcal{C}_\perp (\epsilon + i\delta E)^{-1} - \dots \right] 2\mathbf{h}.$$

- **Each Insertion:** exchange plus virtual subtraction; operators act from right to left.
- **All Orders:** $\Gamma_{\text{soft}} t_{\text{form}} \sim 1$; no extra power of g suppresses another insertion.
- **Computation:** solve the full equation; no assumed series convergence.

The Explicit Equation for the Splitting Vertex

Recall Formation Energy; p. 56

$$\delta E = \frac{m_{\infty,g}^2}{2k} + \frac{m_{\infty,q}^2}{2p} - \frac{m_{\infty,q}^2}{2p'} + \frac{h^2}{2pkp'}, \quad p' = p + k$$

The Three Color Pairs in One Integral Equation

$$2\mathbf{h} = i\delta E \mathbf{F}(\mathbf{h}) + g^2 \int \frac{d^2 q_{\perp}}{(2\pi)^2} \mathcal{A}_{\perp}(q_{\perp}) \left\{ (C_F - \tfrac{1}{2}C_A)[\mathbf{F}(\mathbf{h}) - \mathbf{F}(\mathbf{h} - k\mathbf{q}_{\perp})] \right. \\ \left. + \tfrac{1}{2}C_A[\mathbf{F}(\mathbf{h}) - \mathbf{F}(\mathbf{h} + p'\mathbf{q}_{\perp})] + \tfrac{1}{2}C_A[\mathbf{F}(\mathbf{h}) - \mathbf{F}(\mathbf{h} - p\mathbf{q}_{\perp})] \right\}$$

- **Inputs:** thermal masses, $\mathcal{A}_{\perp}$, fixed (p', p, k) .
- **Unknown:** $\mathbf{F}(\mathbf{h})$; coherent transverse radiation vertex.
- **Three Pairs:** the same $W_k, W_{-p'}, W_p$ from p. 70.
- **Negative Weight:** positivity belongs to the complete color sum.

The Complete Color Sum is Nonnegative

- **Color State:** $\langle c|c\rangle = 1$; $\sum_i \mathbf{T}_i|c\rangle = 0$.
- **Pair Weight:** $c_{ij} = -\langle c|\mathbf{T}_i \cdot \mathbf{T}_j|c\rangle = c_r$.

Positive Interference Sum

$$\sum_{i<j} c_{ij} [1 - \cos(\mathbf{q}_\perp \cdot (\mathbf{x}_i - \mathbf{x}_j))] = \frac{1}{2} \left\| \sum_i e^{i\mathbf{q}_\perp \cdot \mathbf{x}_i} \mathbf{T}_i|c\rangle \right\|^2 \geq 0$$

$$\sum_i C_i + 2 \sum_{i<j} \langle \mathbf{T}_i \cdot \mathbf{T}_j \rangle = 0$$

$$\mathcal{C}_\perp(\mathbf{b}) = \frac{g^2}{2} \int \frac{d^2 q_\perp}{(2\pi)^2} \mathcal{A}_\perp(q_\perp) \left\| \sum_i e^{i\mathbf{q}_\perp \cdot \mathbf{x}_i} \mathbf{T}_i|c\rangle \right\|^2 \geq 0$$

- **Norm Expansion:** diagonal charges cancel the constant pair sum.
- **Position Space:** multiplication by $\mathcal{C}_\perp(\mathbf{b}) \geq 0$.
- **Momentum Space:** Fourier transformation preserves the quadratic form.

The LPM Solution Gives a Nonnegative Radiation Weight

- **Radiation Weight:** interference of bare vertex $2\mathbf{h}$ with dressed vertex $\mathbf{F}$.

$$(i\delta E + \mathcal{C}_\perp)\mathbf{F} = 2\mathbf{h}$$

Nonnegative Interference Integral

$$\text{Re} \int \frac{d^2h}{(2\pi)^2} \mathbf{F}^* \cdot 2\mathbf{h} = \int \frac{d^2h}{(2\pi)^2} \mathbf{F}^* \cdot \mathcal{C}_\perp \mathbf{F} \geq 0$$

- **Phase Term:** purely imaginary in this quadratic form.
- **No Kicks, $\delta E > 0$:** $\mathbf{F} = 2\mathbf{h}/(i\delta E)$; zero induced rate.
- **Next:** spin/color prefactors; then statistical occupations.

From the LPM Vertex to the Splitting Kernel

- **Channel** $a = q, g$: solve $\mathbf{F}_a$ at fixed (p', p, k) .

$$d_q = N_c, \quad d_g = d_A \quad (\text{Color Multiplicities})$$

Channel Splitting Kernels

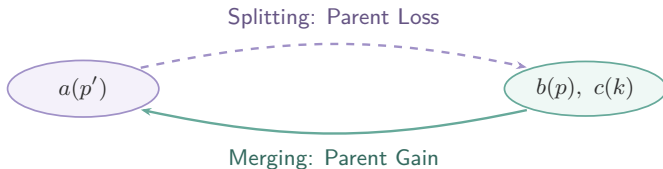
$$\mathcal{F}_a(p', p, k) = \frac{d_a C_a \alpha_s}{2(2\pi)^3} \int \frac{d^2 h}{(2\pi)^2} 2\mathbf{h} \cdot \text{Re } \mathbf{F}_a(\mathbf{h}; p', p, k)$$

$$\gamma_{qg}^q = \frac{p'^2 + p^2}{p'^2 p^2 k^3} \mathcal{F}_q, \quad \gamma_{gg}^g = \frac{p'^4 + p^4 + k^4}{p'^3 p^3 k^3} \mathcal{F}_g$$

$$\gamma_{q\bar{q}}^g(p'; p, k) = \frac{p^2 + k^2}{k^2 p^2 p'^3} \mathcal{F}_q(k, -p, p')$$

- $\mathbf{F}_a$: transverse vertex; $\mathcal{F}_a$: its integrated interference weight.
- γ_{bc}^a : spin/color-summed kernel, $[\gamma] = E^2$; probability on p. 80.
- **Crossing**: $\mathcal{F}_q(k, -p, p')$ reuses the quark vertex with crossed momenta.

Splitting Gain and Loss



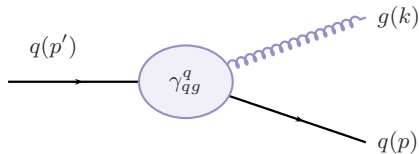
One Statistical Factor for Splitting and Merging

$$\mathcal{B}_{bc}^a(p'; p, k) = f_a(p')[1 + s_b f_b(p)][1 + s_c f_c(k)] - [1 + s_a f_a(p')] f_b(p) f_c(k)$$

$$\mathcal{C}_{a,1 \leftrightarrow 2}(p') \Big|_{\text{parent}} = \frac{(2\pi)^3}{2p'^2 \nu_a} \sum_{bc} \int_0^\infty dp dk \delta(p' - p - k) \times \gamma_{bc}^a(p'; p, k) \mathcal{B}_{bc}^a(p'; p, k).$$

- **Collinear Momenta:** p', p, k share the same direction; the superscript labels the parent.
- **Parent Depletion:** positive $\mathcal{B}_{bc}^a$ gives $D_t f_a = -\mathcal{C}_a < 0$.
- **Detailed Balance:** $p' = p + k$ and $\mu_a = \mu_b + \mu_c$ give $\mathcal{B}_{bc}^a[f_0] = 0$.

From the Splitting Kernel to a Quark Probability



Medium Interactions Included

- **Parent:** fixed p' ; $p = p' - k$.
- **Resolved Daughters:** $p, k \gg gT$.
- **Quark Average:** $\nu_q = 2N_c$.
- **Counting:** two qg orderings cancel $1/2$.

Probability Density in Emitted Energy and Time

$$dP_{q \rightarrow qg} \equiv \mathcal{R}_{q \rightarrow qg}(k; p') dk dt$$

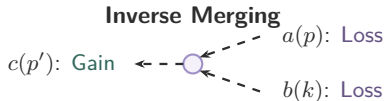
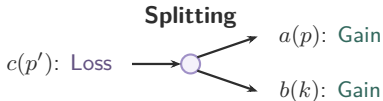
$$\mathcal{R}_{q \rightarrow qg}(k; p') = \frac{(2\pi)^3}{p'^2 \nu_q} \gamma_{qg}^q(p'; p' - k, k) [1 - f_{0q}(p' - k)] [1 + f_{0g}(k)]$$

$$\Gamma_{\text{split}}(p') = \int_{\text{resolved}} dk \mathcal{R}_{q \rightarrow qg}(k; p'), \quad [\gamma] = E^2, \quad [\mathcal{R}] = 1, \quad [\Gamma_{\text{split}}] = E$$

- **Inclusive γ :** all formation exchanges already included.

The Same Splitting Also Feeds Its Daughters

- **Tracked Daughter:** $a(p)$ in $c(p') \leftrightarrow a(p)b(k); p' = p + k$.



Daughter Gain and Inverse Merging

$$\mathcal{C}_{a,1\leftrightarrow 2}(p)|_{\text{daughter}} = -\frac{(2\pi)^3}{p^2\nu_a} \sum_{bc} \int_0^\infty dk dp' \delta(p+k-p') \times \gamma_{ab}^c(p'; p, k) \mathcal{B}_{ab}^c(p'; p, k).$$

$$\mathcal{C}_{a,1\leftrightarrow 2} = \mathcal{C}_{a,\text{parent}} + \mathcal{C}_{a,\text{daughter}}$$

- **Same Factor:** $\mathcal{B}_{ab}^c$ is defined on p. 79; c is the parent and a, b are the daughters.
- **Daughter Gain:** the overall minus sign makes splitting a gain under $D_t f = -\mathcal{C}$.
- **Counting:** either daughter can be tracked; no additional 1/2.

The Full Splitting Term Conserves Energy

- **Energy Moment:** add the parent and both daughter contributions.
- **Collinear Accuracy:** use $E_p = p$ in this splitting term.

Splitting Conserves Energy

$$\sum_a \nu_a \int \frac{d^3 p}{(2\pi)^3} p \mathcal{C}_{a,1\leftrightarrow 2} = \frac{1}{2} \sum_{abc} \int d\Omega \int_0^\infty dp' dp dk \delta(p' - p - k) \\ \times \gamma_{bc}^a(p'; p, k) \mathcal{B}_{bc}^a(p'; p, k) \underbrace{(p' - p - k)}_0 = 0.$$

- **Same Factor:** $\mathcal{B}_{bc}^a$ from p. 79 includes splitting and inverse merging.
- **Counting:** parent coefficient $1/2$; daughter terms averaged over the two orderings.
- **Conservation:** the same event argument applies to allowed flavor charges.
- **Particle Number:** $1 - 1 - 1 \neq 0$; number need not be conserved.

Quark Daughters with One Statistical Factor

- **Quark Daughter:** $q_i(p)$; parents $q_i(p')$ and $g(p')$, $i = 1, \dots, N_f$.
- **Common Arguments:** every γ and $\mathcal{B}$ below is evaluated at $(p'; p, k)$; $\mathcal{B}$ is defined on p. 79.

Both Quark-Daughter Channels

$$\mathcal{C}_{q_i, 1 \leftrightarrow 2}(p)|_{\text{daughter}} = -\frac{(2\pi)^3}{p^2 \nu_{q_i}} \int_0^\infty dk dp' \delta(p + k - p') \left[\gamma_{q_i g}^{q_i} \mathcal{B}_{q_i g}^{q_i} + \gamma_{q_i \bar{q}_i}^g \mathcal{B}_{q_i \bar{q}_i}^g \right].$$

$$q_i(p') \leftrightarrow q_i(p)g(k), \quad g(p') \leftrightarrow q_i(p)\bar{q}_i(k)$$

- **Statistics:** $s_q = s_{\bar{q}} = -1$, $s_g = +1$ in the single definition of $\mathcal{B}$.
- **Inverse Processes:** each $\mathcal{B}$ already contains inverse merging.

Antiquark Daughters with One Statistical Factor

- **Antiquark Daughter:** $\bar{q}_i(p)$; parents $\bar{q}_i(p')$ and $g(p')$; quark and antiquark occupations remain independent.
- **Common Arguments:** every γ and $\mathcal{B}$ below is evaluated at $(p'; p, k)$; $\mathcal{B}$ is defined on p. 79.

Both Antiquark-Daughter Channels

$$\mathcal{C}_{\bar{q}_i, 1 \leftrightarrow 2}(p)|_{\text{daughter}} = -\frac{(2\pi)^3}{p^2 \nu_{\bar{q}_i}} \int_0^\infty dk dp' \delta(p + k - p') \left[\gamma_{\bar{q}_i g}^{\bar{q}_i} \mathcal{B}_{\bar{q}_i g}^{\bar{q}_i} + \gamma_{\bar{q}_i q_i}^g \mathcal{B}_{\bar{q}_i q_i}^g \right].$$

$$\bar{q}_i(p') \leftrightarrow \bar{q}_i(p)g(k), \quad g(p') \leftrightarrow \bar{q}_i(p)q_i(k)$$

- **Daughter Order:** $\gamma_{\bar{q}_i q_i}^g(p'; p, k) = \gamma_{q_i \bar{q}_i}^g(p'; k, p)$.
- **Gain Sign:** positive $\mathcal{B}$ feeds the daughter under $D_t f = -\mathcal{C}$.

Gluon Daughters with One Statistical Factor

- **Gluon Daughter:** include $g \rightarrow gg$ and radiation from every $q_i, \bar{q}_i$.
- **Common Arguments:** every γ and $\mathcal{B}$ below is evaluated at $(p'; p, k)$; $\mathcal{B}$ is defined on p. 79.

All Gluon-Daughter Channels

$$\mathcal{C}_{g,1\leftrightarrow 2}(p)|_{\text{daughter}} = -\frac{(2\pi)^3}{p^2\nu_g} \int_0^\infty dk dp' \delta(p+k-p') \left\{ \gamma_{gg}^g \mathcal{B}_{gg}^g + \sum_{i=1}^{N_f} \left[\gamma_{gq_i}^{q_i} \mathcal{B}_{gq_i}^{q_i} + \gamma_{g\bar{q}_i}^{\bar{q}_i} \mathcal{B}_{g\bar{q}_i}^{\bar{q}_i} \right] \right\}.$$

- **Daughter Order:** $\gamma_{gq_i}^{q_i}(p'; p, k) = \gamma_{q_i g}^{q_i}(p'; k, p)$; likewise for $\bar{q}_i$.
- **Counting:** keep quarks and antiquarks separate; no extra 1/2.

Quantum Formation Fits Inside a Local Kinetic Collision

- **Kinetic Interval:** resolve a completed splitting.
- **Background Time:** τ_{medium} .

Recall The Probability of Completing a Splitting; p. 64, p. 65

$$\Gamma_{\text{split}} \sim \alpha_s / t_{\text{form}}, \quad \tau_{\text{split}} = \Gamma_{\text{split}}^{-1} \sim (g^4 T)^{-1}$$

A Time Interval that Resolves Completed Splittings

$$t_{\text{form}} \ll \Delta t \ll \min(\tau_{\text{split}}, \tau_{\text{medium}})$$

$$\text{Coherent Formation} \longrightarrow \gamma_{qg}^q \longrightarrow \mathcal{C}_{1 \leftrightarrow 2}[f]$$

- **Formation:** coherent sum over emission and soft-scattering histories.
- **Coarse Graining:** completed splittings become gain/loss events.
- **Successive Splittings:** repeated kinetic events.

Consistent Treatment of Overlapping Regions

- **Overlap:** nearly on-shell exchange and successive splittings.

Recall Asymptotic Energy Shift; p. 54

$$E_p = p + \frac{m_\infty^2}{2p} + \dots, \quad m_{\infty,g}^2 = \frac{m_D^2}{2}, \quad m_{\infty,q}^2 = \frac{C_F g^2 T^2}{4}$$

AMY Prescription at LO

- **External Kinematics:** Massless external kinematics in $2 \leftrightarrow 2$.
- **Soft Propagators:** HTL self-energies when $P - P'$ or $P - K'$ is soft.
- **Formation Energies:** Asymptotic masses in collinear formation energies.
- **Other Prescriptions:** explicit overlap subtraction required.

- **Weak Coupling:** hard quasiparticles dominate.
- **Smooth Momentum Distributions:** slow variation during formation.
- **Internal States:** Color-singlet, spin-unpolarized occupations.
- **Background Isotropy:** Approximately isotropic backgrounds; generic large anisotropy can cause soft instabilities.
- **Controlled Application:** near-equilibrium transport.

Constructing the Collision Operator

1. **Particle Content:** species and equilibrium occupations.
2. **Elastic Input:** Screening masses and $2 \leftrightarrow 2$ amplitudes.
3. **Radiation Input:** Transverse LPM equation and splitting kernels γ_{bc}^a .
4. **Quantum Statistics:** Quantum gain–loss factors for both sectors.
5. **Transport Input:** Source, conserved modes and measured moment.

V

Transport Coefficients

How does the Collision Operator Determine Diffusion and Viscosity?

- **Frame:** local rest frame.

Recall Microscopic Current and Stress; p. 7

$$J_Q^i = \sum_a \nu_a \int \frac{d^3p}{(2\pi)^3} q_a v_p^i f_a, \quad T^{ij} = \sum_a \nu_a \int \frac{d^3p}{(2\pi)^3} \frac{p^i p^j}{E_p} f_a$$

Constitutive Relations

$$\begin{aligned} J_Q^i &= \sigma_Q E^i, & J_Q^i &= -D \partial_i n_Q \\ \delta T^{ij} &= -\eta \Sigma^{ij} - \zeta \delta^{ij} \nabla \cdot \mathbf{u}, & \Sigma^{ij} &= \partial_i u_j + \partial_j u_i - \frac{2}{3} \delta^{ij} \nabla \cdot \mathbf{u} \end{aligned}$$

- **Charge Transport:** σ_Q : conductivity; D : conserved-charge diffusion.
- **Viscous Transport:** η : shear viscosity; ζ : bulk viscosity.
- **Complete LO:** AMY-T computes η , D , σ_Q at complete LO.

The Linearized Collision Operator

- **Reference:** $\mathcal{C}[f_0] = 0$; streaming of local equilibrium supplies the drive.

Recall Kinetic Equation and Perturbation; p. 17, p. 8

$$D_t f_a = -\mathcal{C}_a[f], \quad \delta f_a(\mathbf{p}) = f_{0a}(E_p)[1 + s_a f_{0a}(E_p)]\phi_a(\mathbf{p})$$

$$(\mathcal{L}\phi)_a(\mathbf{p}) = \frac{1}{f_{0a}(E_p)[1 + s_a f_{0a}(E_p)]} \times \frac{d}{d\epsilon} \mathcal{C}_a \left[\{f_{0b}(E_k) + \epsilon f_{0b}(E_k)[1 + s_b f_{0b}(E_k)]\phi_b(\mathbf{k})\}_b \right] \Big|_{\epsilon=0}. \quad (6)$$

Driven Linearized Kinetics

$$(\partial_t + \mathbf{v}_p \cdot \nabla_x)\phi_a + (\mathcal{L}\phi)_a = \mathcal{S}_a, \quad \mathcal{S}_a = -\frac{D_t f_{0a}(E_p)}{f_{0a}(E_p)[1 + s_a f_{0a}(E_p)]} \quad (7)$$

- **Linear Response:** neglect products of gradients with ϕ .
- **Stationary Transport:** $\mathcal{L}\phi = \mathcal{S}$; collisions mix the species b .

- **One Statistical Factor:** linearize $\mathcal{B}_{22}$ from p. 22; $\mathcal{B}_{22}[f_0] = 0$.

$$\delta \log \frac{f_a(\mathbf{p})}{1 + s_a f_a(\mathbf{p})} = \left[\frac{1}{f_{0a}(E_p)} - \frac{s_a}{1 + s_a f_{0a}(E_p)} \right] \delta f_a(\mathbf{p}) = \frac{\delta f_a(\mathbf{p})}{f_{0a}(E_p)[1 + s_a f_{0a}(E_p)]} = \phi_a(\mathbf{p}).$$

Linearize the Collision Statistical Factor

$$\delta \mathcal{B}_{22} = f_{0a}(E_p) f_{0b}(E_k) [1 + s_c f_{0c}(E_{p'})][1 + s_d f_{0d}(E_{k'})] \times [\phi_a(\mathbf{p}) + \phi_b(\mathbf{k}) - \phi_c(\mathbf{p}') - \phi_d(\mathbf{k}')].$$

- **Cancellation:** differentiate the two products in $\mathcal{B}_{22}$ and use detailed balance.
- **Equilibrium Product:** the positive coefficient is shown explicitly; $\mathcal{B}_{22}[f_0]$ itself is zero.
- **Invariant:** an event-conserved perturbation makes the last bracket vanish.

Deriving the Shear Source

- **Shear Flow:** $u_x(y)$ near $\mathbf{u} = 0$; $\Sigma_{xy} = \partial_y u_x$.
- **Reference:** $\delta f_a = f_a - f_{0a}^{\text{loc}}$.

$$\partial_E f_{0a}(E) = -\beta f_{0a}(E)[1 + s_a f_{0a}(E)]$$

$$f_{0a}^{\text{loc}} = f_{0a}(E_p - \mathbf{p} \cdot \mathbf{u}), \quad v_y \partial_y f_{0a}^{\text{loc}} = \beta f_{0a}(E_p)[1 + s_a f_{0a}(E_p)] \frac{p_x p_y}{E_p} \Sigma_{xy}$$

Shear Driving and Measurement

$$(\mathcal{L}\phi)_a(\mathbf{p}) = -\beta \frac{p_x p_y}{E_p} \Sigma_{xy}$$

$$\delta T^{xy} = \sum_a \nu_a \int \frac{d^3 p}{(2\pi)^3} \frac{p_x p_y}{E_p} f_{0a}(E_p)[1 + s_a f_{0a}(E_p)] \phi_a(\mathbf{p}) = -\eta \Sigma_{xy}$$

- **Same Momentum Factor:** $p_x p_y / E_p$ drives and measures shear stress.

- **Electric Force:** $\mathbf{F}_{a,\text{ext}} = q_a \mathbf{E}$.

$$\nabla_p f_{0a}(E_p) = -\beta f_{0a}(E_p)[1 + s_a f_{0a}(E_p)] \mathbf{v}_p, \quad \mathcal{S}_a = \beta q_a \mathbf{v}_p \cdot \mathbf{E}$$

$$\mathcal{S}_a = -\beta q_a \mathbf{v}_p \cdot \nabla \mu_Q, \quad \chi_Q = \frac{\partial n_Q}{\partial \mu_Q} = \beta \sum_a \nu_a \int \frac{d^3 p}{(2\pi)^3} f_{0a}(E_p)[1 + s_a f_{0a}(E_p)] q_a^2$$

Einstein Relation

$$\mathbf{J}_Q = -\sigma_Q \nabla \mu_Q = -D \nabla n_Q \implies \sigma_Q = \chi_Q D$$

- **Same Vector Sector:** electric field and chemical-potential gradient.

Detailed Balance and Positive Dissipation

- **Detailed Balance:** $\mathcal{L}$ is self-adjoint; the equilibrium product multiplies a nonnegative event square.

Positive Collision Quadratic Form

$$\begin{aligned}
 & \sum_a \nu_a \int \frac{d^3 p}{(2\pi)^3} f_{0a}(E_p) [1 + s_a f_{0a}(E_p)] \phi_a^*(\mathbf{p}) (\mathcal{L}_{2 \leftrightarrow 2} \phi)_a(\mathbf{p}) \\
 &= \frac{1}{8} \sum_{abcd} \int d\Pi_p d\Pi_k d\Pi_{p'} d\Pi_{k'} (2\pi)^4 \delta^4(P + K - P' - K') |\mathcal{M}_{cd}^{ab}|^2 \\
 &\quad \times f_{0a}(E_p) f_{0b}(E_k) [1 + s_c f_{0c}(E_{p'})] [1 + s_d f_{0d}(E_{k'})] \times |\phi_a(\mathbf{p}) + \phi_b(\mathbf{k}) - \phi_c(\mathbf{p}') - \phi_d(\mathbf{k}')|^2 \geq 0.
 \end{aligned}$$

- **Splitting:** the corresponding positive square is $|\phi_a - \phi_b - \phi_c|^2$.

$$\begin{aligned}
 & \frac{1}{2} \partial_t \sum_a \nu_a \int \frac{d^3 p}{(2\pi)^3} f_{0a}(E_p) [1 + s_a f_{0a}(E_p)] |\phi_a|^2 \\
 &= - \sum_a \nu_a \int \frac{d^3 p}{(2\pi)^3} f_{0a}(E_p) [1 + s_a f_{0a}(E_p)] \phi_a^* (\mathcal{L} \phi)_a \leq 0.
 \end{aligned}$$

Conserved Modes and Matching

Recall Conserved Collision Moments; p. 32

$$\sum_a \nu_a \int \frac{d^3 p}{(2\pi)^3} \chi_a \mathcal{C}_a[f] = 0, \quad \chi_a = E_p, p_i, q_a$$

Conserved Zero Modes

$$\mathcal{L}E_p = 0, \quad \mathcal{L}p_i = 0, \quad \mathcal{L}q_a = 0$$

- **Equilibrium Shifts:** Temperature, velocity and conserved-charge shifts move equilibrium.
- **Projection:** Project conserved modes out before inversion.
- **Stationary Solution:** source orthogonal to every zero mode.

Transport Sectors

- **Shear:** traceless rank two.
- **Charge Current at $\mu = 0$:** charge-conjugation odd; orthogonal to momentum.

Fix the Local Equilibrium before Inverting Collisions

- **Matching:** fix local $T, \mathbf{u}, \mu_Q$ by setting conserved deviations to zero.

$$\delta e = \sum_a \nu_a \int \frac{d^3p}{(2\pi)^3} f_{0a}(E_p) [1 + s_a f_{0a}(E_p)] E_p \phi_a = 0$$

$$\delta P_i = \sum_a \nu_a \int \frac{d^3p}{(2\pi)^3} f_{0a}(E_p) [1 + s_a f_{0a}(E_p)] p_i \phi_a = 0$$

$$\delta n_Q = \sum_a \nu_a \int \frac{d^3p}{(2\pi)^3} f_{0a}(E_p) [1 + s_a f_{0a}(E_p)] q_a \phi_a = 0$$

Source Solvability

$$\mathcal{L}\phi = \mathcal{S}, \quad \mathcal{L}z_\alpha = 0 \quad \implies \quad \sum_a \nu_a \int \frac{d^3p}{(2\pi)^3} f_{0a}(E_p) [1 + s_a f_{0a}(E_p)] z_{\alpha a}^* \mathcal{S}_a = 0$$

- **Reason:** self-adjointness moves $\mathcal{L}$ onto the conserved mode z_α .
- **Ambiguity:** adding a conserved mode changes the equilibrium parameters.

Project Conserved Modes Out of the Inverse

- **Normalized Conserved Modes:** choose z_α with $\mathcal{L}z_\alpha = 0$.

$$\sum_a \nu_a \int \frac{d^3p}{(2\pi)^3} f_{0a}(E_p) [1 + s_a f_{0a}(E_p)] z_{\alpha a}^*(\mathbf{p}) z_{\beta a}(\mathbf{p}) = \delta_{\alpha\beta}$$

Explicit Orthogonal Projection

$$(P_\perp u)_a(\mathbf{p}) = u_a(\mathbf{p}) - \sum_\alpha z_{\alpha a}(\mathbf{p}) \times \sum_b \nu_b \int \frac{d^3k}{(2\pi)^3} f_{0b}(E_k) [1 + s_b f_{0b}(E_k)] z_{\alpha b}^*(\mathbf{k}) u_b(\mathbf{k}).$$
$$\phi = P_\perp \phi$$

- **Inverse:** $\mathcal{L}^{-1}$ acts on this nonconserved subspace.
- **Species and Momentum:** both are summed inside the projection integral.

- **Isotropy:** independent angular sectors ℓ .

$$\phi_a(\mathbf{p}) = \sum_{\ell m} \phi_{a,\ell m}(p) Y_{\ell m}(\hat{\mathbf{p}})$$

Sector	Angular Structure	Disturbance
$\ell = 0$	Scalar	Temperature, Density, Bulk
$\ell = 1$	$\hat{p}_i$	Current, Momentum
$\ell = 2$	$\hat{p}_i \hat{p}_j - \delta_{ij}/3$	Shear Stress

- **Perturbation Wave Number k :** $k = 0$ gives radial equations.
- **Streaming:** finite k couples adjacent ℓ .

The Shear Angular Factor

- **Rotational Symmetry:** rank-four angular average.

Recall Single-Particle Shear Moment; p. 94

$$\frac{p_x p_y}{E_p} = \frac{p^2}{E_p} \hat{p}_x \hat{p}_y$$

$$\langle \hat{p}_i \hat{p}_j \hat{p}_k \hat{p}_l \rangle_\Omega = a(\delta_{ij} \delta_{kl} + \delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk})$$

$$\sum_{i,k} \langle \hat{p}_i^2 \hat{p}_k^2 \rangle_\Omega = 1 = a(9 + 3 + 3) \implies a = \frac{1}{15}$$

Shear Angular Normalization

$$\langle \hat{p}_x^2 \hat{p}_y^2 \rangle_\Omega = \frac{1}{15}, \quad \left\langle \left(\frac{p_x p_y}{E_p} \right)^2 \right\rangle_\Omega = \frac{p^4}{15 E_p^2} \quad (x \neq y)$$

- **Radial Reduction:** shear angular factor 1/15.

Viscosity as an Explicit Momentum Integral

- **Inverse:** restrict $\mathcal{L}^{-1}$ to nonconserved modes; the source has components $p_x p_y / E_p$ for each species.

$$\phi_{\text{shear},a}(\mathbf{p}) = -\beta \Sigma_{xy} \left[\mathcal{L}^{-1} \left(\frac{p_x p_y}{E_p} \right) \right]_a (\mathbf{p})$$
$$\delta T^{xy} = \sum_a \nu_a \int \frac{d^3 p}{(2\pi)^3} f_{0a}(E_p) [1 + s_a f_{0a}(E_p)] \frac{p_x p_y}{E_p} \phi_{\text{shear},a}(\mathbf{p}) = -\eta \Sigma_{xy}$$

Viscosity from the Full Stress Moment

$$\eta = \beta \sum_a \nu_a \int \frac{d^3 p}{(2\pi)^3} f_{0a}(E_p) [1 + s_a f_{0a}(E_p)] \frac{p_x p_y}{E_p} \left[\mathcal{L}^{-1} \left(\frac{p_x p_y}{E_p} \right) \right]_a (\mathbf{p}) \quad (8)$$

- **Species Mixing:** brackets denote the a component after applying the inverse operator.
- **Dimension:** $[\eta] = T^3$.

Conductivity as an Explicit Momentum Integral

- **Electric Response:** invert the source $\beta q_a \mathbf{v}_p \cdot \mathbf{E}$.

$$\phi_{E,a}(\mathbf{p}) = \beta E_j \left[\mathcal{L}^{-1}(q_b v_k^j) \right]_a (\mathbf{p})$$

$$J_Q^i = \beta E_j \sum_a \nu_a \int \frac{d^3 p}{(2\pi)^3} f_{0a}(E_p) [1 + s_a f_{0a}(E_p)] q_a v_p^i \left[\mathcal{L}^{-1}(q_b v_k^j) \right]_a (\mathbf{p}) = \sigma_Q E_i$$

Conductivity and Diffusion

$$\sigma_Q = \frac{\beta}{3} \sum_{i=1}^3 \sum_a \nu_a \int \frac{d^3 p}{(2\pi)^3} f_{0a}(E_p) [1 + s_a f_{0a}(E_p)] q_a v_p^i \left[\mathcal{L}^{-1}(q_b v_k^i) \right]_a (\mathbf{p}), \quad D = \frac{\sigma_Q}{\chi_Q}$$

- **Operator Input:** $q_b v_k^i$ is a function of input species b and momentum $\mathbf{k}$.
- **Isotropy:** average over three spatial directions; $[\sigma_Q] = T$, $[D] = T^{-1}$.

Viscosity in a Relaxation-Time Model

- **Model:** massless gas; constant τ ; shear operator $\mathcal{L} = 1/\tau$.

$$\eta = \frac{\beta\tau}{15} \sum_a \nu_a \int \frac{d^3p}{(2\pi)^3} p^2 f_{0a}(p) [1 + s_a f_{0a}(p)]$$
$$f_{0a}(p) [1 + s_a f_{0a}(p)] = -T \partial_p f_{0a}(p)$$
$$\int_0^\infty dp p^4 f_{0a}(p) [1 + s_a f_{0a}(p)] = -T [p^4 f_{0a}(p)]_0^\infty + 4T \int_0^\infty dp p^3 f_{0a}(p)$$

Relaxation-Time Normalization

$$\eta = \frac{4e\tau}{15}, \quad e = 3P, \quad \frac{\eta}{e + P} = \frac{\tau}{5}$$

- **Scope:** one relaxation time; AMY keeps the full collision operator.

The Variational Principle

- **Shear Example:** real trial functions $\psi_a(\mathbf{p})$; conserved modes excluded.

$$\mathcal{Q}[\psi] = \sum_a \nu_a \int \frac{d^3p}{(2\pi)^3} f_{0a}(E_p) [1 + s_a f_{0a}(E_p)] \left[\psi_a(\mathbf{p}) \frac{p_x p_y}{E_p} - \frac{1}{2} \psi_a(\mathbf{p}) (\mathcal{L}\psi)_a(\mathbf{p}) \right]$$

$$\delta \mathcal{Q} = \sum_a \nu_a \int \frac{d^3p}{(2\pi)^3} f_{0a}(E_p) [1 + s_a f_{0a}(E_p)] \delta \psi_a(\mathbf{p}) \left[\frac{p_x p_y}{E_p} - (\mathcal{L}\psi)_a(\mathbf{p}) \right]$$

$$\delta \mathcal{Q} = 0 \quad \Rightarrow \quad \psi_{*,a}(\mathbf{p}) = \left[\mathcal{L}^{-1} \left(\frac{p_x p_y}{E_p} \right) \right]_a(\mathbf{p})$$

Complete the Square

$$\begin{aligned} \mathcal{Q}[\psi] &= \frac{1}{2} \sum_a \nu_a \int \frac{d^3p}{(2\pi)^3} f_{0a}(E_p) [1 + s_a f_{0a}(E_p)] \frac{p_x p_y}{E_p} \left[\mathcal{L}^{-1} \left(\frac{p_x p_y}{E_p} \right) \right]_a \\ &\quad - \frac{1}{2} \sum_a \nu_a \int \frac{d^3p}{(2\pi)^3} f_{0a}(E_p) [1 + s_a f_{0a}(E_p)] (\psi_a - \psi_{*,a}) [\mathcal{L}(\psi - \psi_*)]_a \leq \frac{\eta}{2\beta}. \end{aligned}$$

- **Lower Bound:** nested trial spaces improve the maximum monotonically.
- **Current:** replace $p_x p_y / E_p$ by $q_a v_p^i$ for a fixed component.

From a Functional to a Matrix

- **K Real Basis Functions:** $\psi_a(\mathbf{p}) = \sum_{r=1}^K a_r \Phi_{r,a}(\mathbf{p})$.

$$M_{rs} = \sum_a \nu_a \int \frac{d^3p}{(2\pi)^3} f_{0a}(E_p) [1 + s_a f_{0a}(E_p)] \Phi_{r,a}(\mathbf{p}) (\mathcal{L}\Phi_s)_a(\mathbf{p})$$

$$b_r = \sum_a \nu_a \int \frac{d^3p}{(2\pi)^3} f_{0a}(E_p) [1 + s_a f_{0a}(E_p)] \Phi_{r,a}(\mathbf{p}) \frac{p_x p_y}{E_p}$$

Finite-Basis Solution

$$M\mathbf{a} = \mathbf{b}, \quad Q_{\max}^{(K)} = \frac{1}{2} \mathbf{b}^T M^{-1} \mathbf{b}$$

- **Basis:** choose the shear angular structure and appropriate radial behavior.
- **Matrix Elements:** elastic phase space and embedded LPM solutions.
- **Convergence:** refine quadrature and basis separately.

A Two-Component Variational Example

$$\begin{aligned} M &= \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}, & \mathbf{b} &= \begin{pmatrix} 1 \\ 0 \end{pmatrix}, & \mathcal{Q} &= a_1 - a_1^2 + a_1 a_2 - a_2^2 \\ a_2 = 0 : & a_1 &= \frac{1}{2}, & \mathcal{Q}_{\max}^{(1)} &= \frac{1}{4} \\ \mathbf{a}_* &= M^{-1} \mathbf{b} = \begin{pmatrix} 2/3 \\ 1/3 \end{pmatrix}, & \mathcal{Q}_{\max}^{(2)} &= \frac{1}{3} > \frac{1}{4} \end{aligned}$$

- **Mixing:** collisions populate the component absent from the source.

Expanding in the Inverse Logarithm

Recall The Same LO Operator at a Chosen Logarithm Scale; p. 50

$$\mathcal{L} = g^4 T [\Lambda \mathcal{A} + \mathcal{B}(\mu_{\log})], \quad \Lambda = \log(\mu_{\log}/m_D), \quad \mu_{\log} \sim T$$

$$\mathcal{B}(\mu_{\log}) = \mathcal{B} + \mathcal{A} \log(T/\mu_{\log})$$

$$\mathcal{L}^{-1} \sim \frac{1}{g^4 T} \left[\frac{\mathcal{A}^{-1}}{\Lambda} - \frac{\mathcal{A}^{-1} \mathcal{B}(\mu_{\log}) \mathcal{A}^{-1}}{\Lambda^2} + \frac{\mathcal{A}^{-1} \mathcal{B}(\mu_{\log}) \mathcal{A}^{-1} \mathcal{B}(\mu_{\log}) \mathcal{A}^{-1}}{\Lambda^3} + \dots \right]$$

Inverse-Log Expansions

$$D \sim \frac{1}{g^4 T} \sum_{n \geq 1} \frac{D_n(\mu_{\log})}{\log^n(\mu_{\log}/m_D)}, \quad \eta \sim \frac{T^3}{g^4} \sum_{n \geq 1} \frac{\eta_n(\mu_{\log})}{\log^n(\mu_{\log}/m_D)}$$

- **Operator Ordering:** Preserve the order of noncommuting operator factors.
- **Inverse-Log Expansion:** Asymptotic inverse-log expansion; zero radius of convergence.

- **Scale Choice:** μ_* cancels the second inverse-log coefficient.

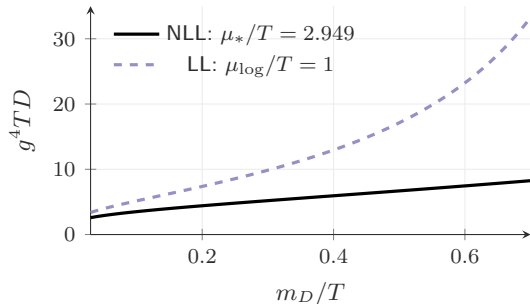
Next-to-Leading-Log Approximation

$$D_{\text{NLL}} = \frac{D_1}{g^4 T \log(\mu_{*,D}/m_D)}, \quad \eta_{\text{NLL}} = \frac{\eta_1 T^3}{g^4 \log(\mu_{*,\eta}/m_D)}$$

Group	N_f	D_1	$\mu_{*,D}/T$	η_1	$\mu_{*,\eta}/T$
SU(3)	0	16.060	2.699	27.126	2.765
SU(3)	2	12.999	2.887	86.47	2.954
SU(3)	3	11.869	2.949	106.66	2.957

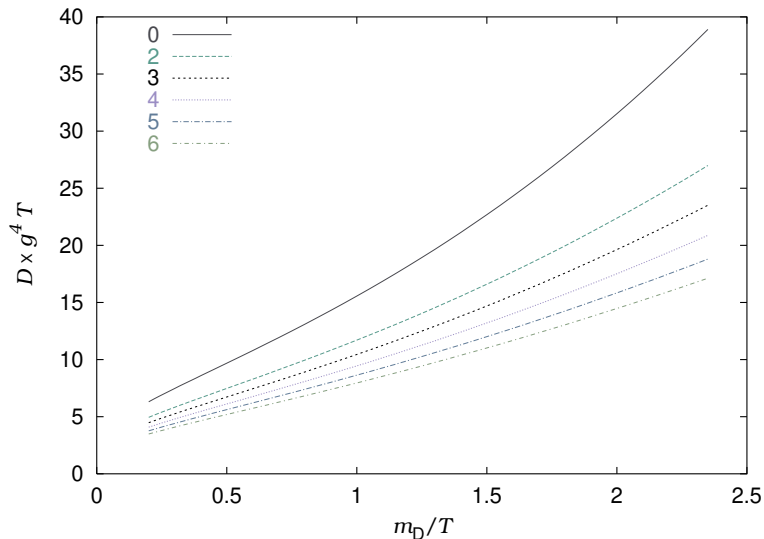
- $N_f = 0$: quark probe in a gluon bath.

The Constant Inside the Logarithm



- **Plasma:** $SU(3)$, $N_f = 3$.
- **Leading Log:** common asymptote.
- **Finite Coupling:** separated curves.
- **Check:** NLL against complete LO.

Complete LO Flavor Diffusion



- **Vertical:** $g^4 T D_q$.
- **Horizontal:** m_D/T .
- **Flavors:**
 $N_f = 0, 2, 3, 4, 5, 6$.
- **Collision Sectors:**
Elastic + LPM.
- **Validity:** $m_D/T \ll 1$ required.

- **Complete LO:** η , D , σ_Q from elastic and splitting sectors.
- **Reported Basis Errors:** $< 10^{-4}$ with four functions; about 1% with two.
- **Inverse-Log Series:** asymptotic, with zero convergence radius.
- **Uncertainties:** Numerical convergence and omitted higher orders in g : separate uncertainties.

VI

Retarded Response and Spectral Functions

How does Relaxation Determine the Response at Finite Frequency?

Which Response are We Calculating?

- **Different Observables:** color response versus stress relaxation.

Perturbation	Responding Quantity	Weak-Coupling Scale
Soft Color Field	Induced Color Current	gT
Shear Deformation	Stress T^{xy}	$g^4T \log(1/g)$

$$D_{\text{HTL}}^R \longrightarrow \text{Screened Scattering Rates} \longrightarrow \mathcal{L} \longrightarrow G_R^{xyxy}(\omega, \mathbf{k})$$

- **HTL Input:** screened rates inside $\mathcal{L}$.
- **Metric Source** h_{ij} : differentiate stress at $h_{ij} = 0$.
- **Response Probe:** no physical gravitational field required.

Retarded Response: Definition and Sign

- **Finite-Frequency Drive:** resolve relaxation times.

$$H_{\text{int}}(t) = - \int d^3x h(t, \mathbf{x}) B(t, \mathbf{x})$$

$$\delta \langle A(t) \rangle = i \int_{-\infty}^t dt' \langle [H_{\text{int}}(t'), A(t)] \rangle$$

$$G_R^{AB}(t, \mathbf{x}) = -i\theta(t) \langle [A(t, \mathbf{x}), B(0)] \rangle$$

$$G_R^{AB}(\omega, \mathbf{k}) = \int dt d^3x e^{i\omega t - i\mathbf{k} \cdot \mathbf{x}} G_R^{AB}(t, \mathbf{x})$$

Response and Spectral Convention

$$\delta \langle A \rangle = -G_R^{AB} h, \quad \rho^{AA} = -2 \text{Im} G_R^{AA}$$

- **Causality:** analytic for $\text{Im} \omega > 0$.
- **Decay at $k = 0$:** poles $\omega = -i\lambda$, $\lambda > 0$.

Why Spectral Weight Describes Absorption

- **Hermitian Observable:** $H|n\rangle = E_n|n\rangle$, $Z = \sum_n e^{-\beta E_n}$; $A(t) = e^{iHt} A e^{-iHt}$.

Lehmann Spectrum

$$\rho^{AA}(\omega) = \int dt e^{i\omega t} \langle [A(t), A(0)] \rangle = \frac{2\pi}{Z} \sum_{nm} (e^{-\beta E_n} - e^{-\beta E_m}) |\langle n|A|m\rangle|^2 \delta(\omega - E_m + E_n).$$

$$E_m - E_n = \omega > 0 : \quad \frac{e^{-\beta E_n} - e^{-\beta E_m}}{Z} = \frac{e^{-\beta E_n}}{Z} (1 - e^{-\beta \omega}) \geq 0$$

- **Absorption:** $n \rightarrow m$ absorbs ω ; reverse transitions subtract emission.
- **Parity:** exchange n, m to obtain $\rho^{AA}(-\omega) = -\rho^{AA}(\omega)$.
- **Channel:** the actual matrix element selects the observable.

- **Free Motion:** geodesics in $g_{\mu\nu}(x)$.

Mass Shell and Geodesic Equations

$$g_{\mu\nu}P^\mu P^\nu = -m_a^2, \quad \frac{dx^\mu}{d\lambda} = P^\mu, \quad \frac{dP^\mu}{d\lambda} = -\Gamma_{\alpha\beta}^\mu P^\alpha P^\beta$$

- **Connection:** $\Gamma_{\alpha\beta}^\mu$: gravitational momentum change.
- **Position-Dependent Mass:** an additional mean-field force is required.
- **Streaming:** use this phase-space flow in the kinetic equation.

From Geodesic Motion to Coordinate-Time Evolution

- **Input:** phase-space flow from p. 117.

Recall Flat-Space Collision Normalization; p. 26

$$\mathcal{C}_a^{\text{cov}} = E_p \mathcal{C}_a, \quad P^0 = E_p \quad \text{in the flat-space inertial frame}$$

Covariant Streaming and Coordinate Time

$$\left[P^\mu \frac{\partial}{\partial x^\mu} - \Gamma_{\alpha\beta}^i P^\alpha P^\beta \frac{\partial}{\partial p^i} \right] f_a = -\mathcal{C}_a^{\text{cov}}[f]$$
$$\frac{df_a}{dt} = \left[\partial_t + \frac{P^i}{P^0} \partial_i - \frac{\Gamma_{\alpha\beta}^i P^\alpha P^\beta}{P^0} \frac{\partial}{\partial p^i} \right] f_a = -\frac{\mathcal{C}_a^{\text{cov}}}{P^0}$$

- **Collisions:** evaluate in the local orthonormal frame.
- **Metric Perturbation:** weak h_{ij} supplies the streaming source.

How the Metric Drives a Distribution

- **Massless Limit:** $g_{ij} = \delta_{ij} + h_{ij}$, $g_{00} = -1$, $g_{0i} = 0$; use p^i .

$$E_h = \sqrt{g_{ij}p^ip^j}, \quad \frac{dP^\mu}{dt} = -\frac{\Gamma_{\alpha\beta}^\mu P^\alpha P^\beta}{E_h}$$

$$\Gamma_{\alpha\beta}^\mu = \frac{1}{2}g^{\mu\nu}(\partial_\alpha g_{\nu\beta} + \partial_\beta g_{\nu\alpha} - \partial_\nu g_{\alpha\beta}), \quad \Gamma_{ij}^0 = \frac{1}{2}\partial_t h_{ij}$$

$$D_h E_h \equiv \frac{dE_h}{dt} = -\frac{p^ip^j}{2E_p}\partial_t h_{ij}, \quad \partial_E f_{0a}(E) = -\beta f_{0a}(E)[1 + s_a f_{0a}(E)]$$

Metric-Induced Source to First Order

$$f_a = f_{0a}(E_h) + \delta f_a, \quad D_h f_{0a}(E_h) = \beta f_{0a}(E_p)[1 + s_a f_{0a}(E_p)] \frac{p^ip^j}{2E_p} \partial_t h_{ij}$$

- **Reference:** $\delta f_a = f_a - f_{0a}(E_h)$.
- **Shear:** $h_{xy} = h_{yx} = h$ gives $\beta f_{0a}(E_p)[1 + s_a f_{0a}(E_p)](p_x p_y / E_p) \partial_t h$.

A Metric Source for Shear Stress

- **Stress Source:** $H_{\text{int}} = -\frac{1}{2} \int d^3x T^{ij} h_{ij}$; $h_{xy} = h_{yx} = h(t, z)$.

$$\delta f_a = f_{0a}(E_p)[1 + s_a f_{0a}(E_p)]\phi_a$$

$$(\partial_t + v_z \partial_z) \delta f_a + \beta f_{0a}(E_p)[1 + s_a f_{0a}(E_p)] \frac{p_x p_y}{E_p} \partial_t h = -f_{0a}(E_p)[1 + s_a f_{0a}(E_p)] (\mathcal{L}\phi)_a.$$

Kinetic Response to Shear

$$(\mathcal{L} - i\omega + ikv_z)\phi = i\omega\beta \frac{p_x p_y}{E_p} h$$

$$\delta T^{xy} = -Ph + \sum_a \nu_a \int \frac{d^3p}{(2\pi)^3} f_{0a}(E_p)[1 + s_a f_{0a}(E_p)] \frac{p_x p_y}{E_p} \phi_a = -Ph - G_R^{xyxy} h$$

- **Contact:** $T_{\text{eq}}^{ij} = P g^{ij}$ gives $-Ph$; the integral is the dynamical part.

- **Retarded Boundary:** turn on the source from the remote past.

Retarded Inverse and Stress Response

$$\begin{aligned} R(\omega, k) &= [\mathcal{L} - i(\omega + i0^+) + ikv_z]^{-1} \\ G_R^{xyxy}(\omega, k) &= -i\omega\beta \sum_a \nu_a \int \frac{d^3p}{(2\pi)^3} f_{0a}(E_p) [1 + s_a f_{0a}(E_p)] \\ &\quad \times \frac{p_x p_y}{E_p} \left[R(\omega, k) \left(\frac{p_x p_y}{E_p} \right) \right]_a(\mathbf{p}). \end{aligned} \tag{9}$$

- **Static Limit:** $R(0, 0) = \mathcal{L}^{-1}$ on nonconserved modes gives viscosity.
- **Time and Space:** $-i\omega$ describes time dependence; ikv_z describes streaming.
- **Finite k :** angular mixing and conservation constraints.

Deriving the Transport Limit

- $k = 0$ **Shear**: no overlap with conserved modes; assume finite inverse moments.

$$(\mathcal{L} - i\omega)^{-1} = \mathcal{L}^{-1} + i\omega \mathcal{L}^{-2} + O(\omega^2)$$

$$\begin{aligned} G_R^{xyxy}(\omega, 0) &= -i\omega\beta \sum_a \nu_a \int \frac{d^3p}{(2\pi)^3} f_{0a}(E_p) [1 + s_a f_{0a}(E_p)] \frac{p_x p_y}{E_p} \left[\mathcal{L}^{-1} \left(\frac{p_x p_y}{E_p} \right) \right]_a + O(\omega^2) \\ &= -i\omega\eta + O(\omega^2). \end{aligned}$$

Kubo Limits

$$\rho^{xyxy} = -2 \operatorname{Im} G_R^{xyxy} \implies \eta = \lim_{\omega \rightarrow 0} \frac{\rho^{xyxy}(\omega, \mathbf{0})}{2\omega}$$

$$\sigma_Q = \lim_{\omega \rightarrow 0} \frac{\rho^{xx}(\omega, \mathbf{0})}{2\omega} \quad \text{for the flavor current } J_Q^x$$

- **Dissipation**: the viscosity in p. 102 is the spectral slope.
- **Real Contact Terms**: no spectral contribution.

What Imaginary-Time Correlations Measure

- **Connected, Relaxing Correlator:** imaginary-time weights.

Recall Thermal Weight and Spectral Parity; p. 43, p. 116

$$G^>(\omega) = \frac{\rho(\omega)}{1 - e^{-\beta\omega}}, \quad \rho(-\omega) = -\rho(\omega)$$

$$G_E(\tau) = \langle A(-i\tau)A(0) \rangle = \frac{1}{Z} \sum_{nm} e^{-\beta E_n - \tau(E_m - E_n)} |\langle n|A|m \rangle|^2 \quad G_E(\tau) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{\rho(\omega)e^{-\omega\tau}}{1 - e^{-\beta\omega}}$$

Imaginary-Time Spectral Kernel

$$G_E(\tau, \mathbf{k}) = \int_0^{\infty} \frac{d\omega}{2\pi} \rho(\omega, \mathbf{k}) \frac{\cosh[\omega(\tau - \beta/2)]}{\sinh(\beta\omega/2)}$$

- **Odd Spectrum:** combine positive and negative frequencies.
- $\omega \ll T$: kernel $2T/\omega$ averages over the transport peak.

Resolving the Relaxation Spectrum

- **Nonconserved Modes:** $\mathcal{L}e_n = \lambda_n e_n$, $\lambda_n > 0$ at $k = 0$.

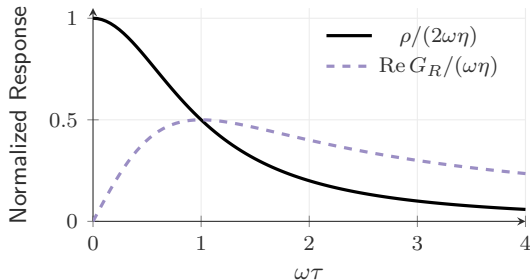
$$\sum_a \nu_a \int \frac{d^3p}{(2\pi)^3} f_{0a}(E_p) [1 + s_a f_{0a}(E_p)] e_{n,a}^*(\mathbf{p}) e_{m,a}(\mathbf{p}) = \delta_{nm}$$

Relaxation Spectrum with Explicit Overlaps

$$\frac{\rho^{xyxy}(\omega, 0)}{2\omega} = \beta \sum_n \frac{\lambda_n}{\lambda_n^2 + \omega^2} \left| \sum_a \nu_a \int \frac{d^3p}{(2\pi)^3} f_{0a}(E_p) [1 + s_a f_{0a}(E_p)] e_{n,a}^*(\mathbf{p}) \frac{p_x p_y}{E_p} \right|^2$$
$$\eta = \beta \sum_n \frac{1}{\lambda_n} \left| \sum_a \nu_a \int \frac{d^3p}{(2\pi)^3} f_{0a}(E_p) [1 + s_a f_{0a}(E_p)] e_{n,a}^*(\mathbf{p}) \frac{p_x p_y}{E_p} \right|^2$$

- **Width and Weight:** λ_n sets the width; the full squared integral sets its contribution.
- **Continuous Spectrum:** replace the mode sum by an integral.
- **Scope:** viscosity alone cannot determine the full spectrum.

A Single Relaxation Mode



$$\lambda = 1/\tau, \quad G_R = -\frac{i\omega\eta}{1 - i\omega\tau}$$

- **Width:** $\sim 1/\tau$.
- **Longer Lifetime:** narrower peak.
- **Finite Frequency:** phase delay.
- **Example:** one analytic mode.

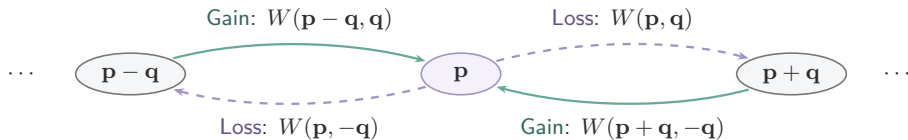
VII

Momentum Diffusion and QCD Spectra

What does the QCD Collision Operator Predict for Spectral Functions?

Gain and Loss between Momentum States

- **Probability Density:** $dP = W(\mathbf{p}, \mathbf{q}) d^3q dt$ for $\mathbf{p} \rightarrow \mathbf{p} + \mathbf{q}$.



Shown Transfers: one pair $\pm \mathbf{q}$; the integral includes all momentum transfers.

Momentum-Space Gain and Loss

$$\partial_t f(\mathbf{p}) = \int d^3q [W(\mathbf{p} - \mathbf{q}, \mathbf{q}) f(\mathbf{p} - \mathbf{q}) - W(\mathbf{p}, \mathbf{q}) f(\mathbf{p})]$$

- **Gain:** a populated source state enters the selected momentum state.
- **Loss:** a particle leaves the selected momentum state.

From Small Momentum Transfers to Diffusion

- **Expansion:** second order in momentum transfer; start from p. 127.

$$(Wf)(\mathbf{p} - \mathbf{q}, \mathbf{q}) = (Wf) - q_i \partial_{p_i} (Wf) + \frac{1}{2} q_i q_j \partial_{p_i} \partial_{p_j} (Wf) + \dots$$

$$\mathcal{A}_i = \int d^3q q_i W, \quad \mathcal{B}_{ij} = \int d^3q q_i q_j W$$

Momentum-Space Diffusion

$$\partial_t f = -\partial_{p_i} (\mathcal{A}_i f) + \frac{1}{2} \partial_{p_i} \partial_{p_j} (\mathcal{B}_{ij} f)$$

- **Moments:** first $\rightarrow$ drift; second $\rightarrow$ diffusion.
- **Whole Plasma:** perturbed partners supply the conserving bath response.

From Elastic Scattering to Soft Gradients

- **Pure Glue, Leading Log:** expand the event perturbation for soft $\mathbf{q}$.

$$\mathbf{p}' = \mathbf{p} - \mathbf{q}, \quad \mathbf{k}' = \mathbf{k} + \mathbf{q}$$

$$\phi_g(\mathbf{p}) + \phi_g(\mathbf{k}) - \phi_g(\mathbf{p}') - \phi_g(\mathbf{k}') = q_i [\partial_{p_i} \phi_g(\mathbf{p}) - \partial_{k_i} \phi_g(\mathbf{k})] + O(q^2)$$

$$\mu_g^{\text{drag}} = \frac{g^2 C_A m_D^2}{8\pi} \log(T/m_D), \quad \int \frac{d^3 p}{(2\pi)^3} f_{0g}(p) [1 + f_{0g}(p)] = \frac{T^3}{6}, \quad T\mu_g^{\text{drag}} = \frac{\hat{q}_g}{4} \quad (\text{LL})$$

Leading-Log Soft-Exchange Tensor

$$I^{ij}(\mathbf{p}, \mathbf{k}) = \frac{3\mu_g^{\text{drag}}}{T^2} [\hat{p}_i \hat{k}_j + \hat{k}_i \hat{p}_j + (1 - \hat{\mathbf{p}} \cdot \hat{\mathbf{k}}) \delta_{ij}]$$

- **Exchange Tensor:** I^{ij} is the soft-exchange integral weighted by $q_i q_j$.
- **Coulomb Logarithm:** screened exchange supplies $\int_{m_D}^T dq/q$.
- **Event Square:** local loss and correlated bath response come from the same square.

Why Only Two Bath Moments Remain

- **Isotropic Equilibrium:** $f_{0g}(k)$ depends only on $k = |\mathbf{k}|$.

$$\int \frac{d^3k}{(2\pi)^3} f_{0g}(k) [1 + f_{0g}(k)] [\hat{p}_i \hat{k}_j + \hat{k}_i \hat{p}_j + (1 - \hat{\mathbf{p}} \cdot \hat{\mathbf{k}}) \delta_{ij}] = \frac{T^3}{6} \delta_{ij}.$$

The Antisymmetric Moment Vanishes

$$\begin{aligned} & \int \frac{d^3p}{(2\pi)^3} f_{0g}(p) [1 + f_{0g}(p)] [\hat{p}_i \partial_{p_j} \phi_g - \hat{p}_j \partial_{p_i} \phi_g] \\ &= - \int \frac{d^3p}{(2\pi)^3} \phi_g \left\{ \partial_{p_j} [f_{0g}(p) [1 + f_{0g}(p)] \hat{p}_i] - \partial_{p_i} [f_{0g}(p) [1 + f_{0g}(p)] \hat{p}_j] \right\} = 0. \end{aligned}$$

- **Integration by Parts:** surface terms vanish at the physical boundaries.
- **Radial Weight:** its derivatives are parallel to $\hat{\mathbf{p}}$; the antisymmetric part is zero.

The Correlated Bath Response Leaves Two Integrals

- **Contraction:** use the vanishing antisymmetric moment from p. 130.

Scalar and Vector Bath Contributions

$$\begin{aligned} & \int \frac{d^3p d^3k}{(2\pi)^6} f_{0g}(p)[1 + f_{0g}(p)]f_{0g}(k)[1 + f_{0g}(k)] \times [\partial_{p_i} \phi_g(\mathbf{p})]^* [\hat{p}_i \hat{k}_j + \hat{k}_i \hat{p}_j + (1 - \hat{\mathbf{p}} \cdot \hat{\mathbf{k}}) \delta_{ij}] \partial_{k_j} \phi_g(\mathbf{k}) \\ &= \left| \int \frac{d^3p}{(2\pi)^3} f_{0g}(p)[1 + f_{0g}(p)] \hat{\mathbf{p}} \cdot \nabla_p \phi_g(\mathbf{p}) \right|^2 + \left| \int \frac{d^3p}{(2\pi)^3} f_{0g}(p)[1 + f_{0g}(p)] \nabla_p \phi_g(\mathbf{p}) \right|^2. \end{aligned}$$

- **Scalar Integral:** energy transferred to the bath.
- **Vector Integral:** momentum transferred to the bath.
- **Conservation:** both integrals are needed in the conserving collision operator.

The Conserving Quadratic Functional

- **Per Gluon State:** local gradients and the two explicit bath integrals.

Conserving Collision Functional

$$\mathcal{I}[\phi_g] = \frac{T\mu_g^{\text{drag}}}{2} \int \frac{d^3p}{(2\pi)^3} f_{0g}(p)[1 + f_{0g}(p)] |\nabla_p \phi_g|^2$$
$$- \frac{3\mu_g^{\text{drag}}}{T^2} \left| \int \frac{d^3p}{(2\pi)^3} f_{0g}(p)[1 + f_{0g}(p)] \hat{\mathbf{p}} \cdot \nabla_p \phi_g \right|^2 - \frac{3\mu_g^{\text{drag}}}{T^2} \left| \int \frac{d^3p}{(2\pi)^3} f_{0g}(p)[1 + f_{0g}(p)] \nabla_p \phi_g \right|^2.$$

$$\delta \mathcal{I} = \text{Re} \int \frac{d^3p}{(2\pi)^3} \delta \phi_g^* f_{0g}(p)[1 + f_{0g}(p)] \mathcal{L}_{\text{LL}} \phi_g$$

$$\partial_t \delta f_g = -f_{0g}(p)[1 + f_{0g}(p)] \mathcal{L}_{\text{LL}} \phi_g$$

- **Variation:** integration by parts gives local diffusion and conserving gain.

- **Local Term:** vary the first integral in p. 132.

Fokker–Planck Equation

$$\begin{aligned}(\partial_t + \mathbf{v} \cdot \nabla_x) \delta f_g &= T \mu_g^{\text{drag}} \nabla_p \cdot \{f_{0g}(p)[1 + f_{0g}(p)] \nabla_p \phi_g\} + \text{Gain}, \\ \delta f_g &= f_{0g}(p)[1 + f_{0g}(p)] \phi_g.\end{aligned}\tag{10}$$

$$\mathbf{j}_p = -T \mu_g^{\text{drag}} f_{0g}(p)[1 + f_{0g}(p)] \nabla_p \phi_g$$

- **Drag and Diffusion:** μ_g^{drag} is the drag scale; $T \mu_g^{\text{drag}}$ sets diffusion.
- **Loss:** $-\nabla_p \cdot \mathbf{j}_p$ redistributes particles in momentum space.
- **Positive Form:** $T \mu_g^{\text{drag}} \int \frac{d^3 p}{(2\pi)^3} f_{0g}(p)[1 + f_{0g}(p)] |\nabla_p \phi_g|^2 \geq 0.$

- **Bath Response:** the two nonlocal terms follow by varying the last two squares in p. 132.

Conserving Gain with Explicit Statistical Factors

$$\begin{aligned} \text{Gain} = & -\frac{6\mu_g^{\text{drag}}}{T^2 p^2} \partial_p \{p^2 f_{0g}(p)[1 + f_{0g}(p)]\} \int \frac{d^3 k}{(2\pi)^3} f_{0g}(k)[1 + f_{0g}(k)] \hat{\mathbf{k}} \cdot \nabla_k \phi_g(\mathbf{k}) \\ & - \frac{6\mu_g^{\text{drag}}}{T^2} \nabla_p \{f_{0g}(p)[1 + f_{0g}(p)]\} \cdot \int \frac{d^3 k}{(2\pi)^3} f_{0g}(k)[1 + f_{0g}(k)] \nabla_k \phi_g(\mathbf{k}). \end{aligned} \quad (11)$$

$$\int \frac{d^3 p}{(2\pi)^3} p \text{Gain} = - \int \frac{d^3 p}{(2\pi)^3} \hat{\mathbf{p}} \cdot \mathbf{j}_p, \quad \int \frac{d^3 p}{(2\pi)^3} p_i \text{Gain} = - \int \frac{d^3 p}{(2\pi)^3} j_{p,i}$$

- **Cancellation:** these moments cancel those of $-\nabla_p \cdot \mathbf{j}_p$.
- **Normalization:** $\int \frac{d^3 p}{(2\pi)^3} f_{0g}(p)[1 + f_{0g}(p)] = T^3/6$ fixes the coefficient.

- **Temperature Mode:** $\phi_g = p/T$.

$$\mathbf{j}_p = -\mu_g^{\text{drag}} f_{0g}(p)[1 + f_{0g}(p)]\hat{\mathbf{p}}, \quad \int \frac{d^3p}{(2\pi)^3} \hat{\mathbf{p}} \cdot \mathbf{j}_p = -\frac{\mu_g^{\text{drag}} T^3}{6}$$

$$\text{Loss} = \frac{\mu_g^{\text{drag}}}{p^2} \partial_p \{p^2 f_{0g}(p)[1 + f_{0g}(p)]\}, \quad \text{Gain} = -\frac{\mu_g^{\text{drag}}}{p^2} \partial_p \{p^2 f_{0g}(p)[1 + f_{0g}(p)]\}$$

$$\phi_g = p_z/T : \quad \text{Loss} = \mu_g^{\text{drag}} \partial_{p_z} \{f_{0g}(p)[1 + f_{0g}(p)]\}, \quad \text{Gain} = -\text{Loss}$$

- **Conserved Modes:** loss and gain cancel for energy and momentum shifts.
- **Gain Sectors:** $\ell = 0, 1$; no gain for $\ell = 2$ shear at $k = 0$.
- **Streaming:** preserves m and couples adjacent ℓ .

The Soft Boundary Condition

- **Soft Gluons:** $f_{0g}(p)[1 + f_{0g}(p)] \simeq T^2/p^2$; density per internal state.

$$\frac{\delta n_{g, < \Delta p}^{\text{num}}}{\nu_g} \simeq \frac{T^2}{2\pi^2} \phi_g(0) \Delta p$$

$$\Gamma_{\text{soft number}} \sim g^3 T \gg \Gamma_{\text{tr}} \sim g^4 T \log(1/g)$$

Soft Boundary Condition

$$\phi_g(p) \longrightarrow 0 \quad (p \rightarrow 0) \quad (12)$$

- **Soft Equilibrium:** Fast emission and absorption pin the soft population to equilibrium.
- **Gluon Number:** not a conserved zero mode.
- **Boundary Physics:** Soft number-changing physics survives in this LL boundary condition.

- **Angular Mode:** $\phi_g(\mathbf{p}) = \sum_{\ell m} \phi_{g,\ell m}(p) Y_{\ell m}(\hat{\mathbf{p}})$.

$$\mathcal{L}^{\text{loss}} \phi_g = - \frac{T \mu_g^{\text{drag}}}{f_{0g}(p)[1 + f_{0g}(p)]} \nabla_p \cdot \{f_{0g}(p)[1 + f_{0g}(p)] \nabla_p \phi_g\}$$

Radial Loss Operator

$$\mathcal{L}_\ell^{\text{loss}} = - \frac{T \mu_g^{\text{drag}}}{p^2 f_{0g}(p)[1 + f_{0g}(p)]} \partial_p \{p^2 f_{0g}(p)[1 + f_{0g}(p)] \partial_p\} + \frac{T \mu_g^{\text{drag}} \ell(\ell + 1)}{p^2}$$

$$\cos \theta Y_{\ell m} = a_{\ell+1,m} Y_{\ell+1,m} + a_{\ell m} Y_{\ell-1,m}, \quad a_{\ell m} = \sqrt{\frac{\ell^2 - m^2}{(2\ell - 1)(2\ell + 1)}}$$

- **Gain:** allowed $\ell = 0, 1$ sectors; $\phi_g \rightarrow 0$ at $p = 0$.
- **Large p :** no exponential growth; $f_{0g}(p)[1 + f_{0g}(p)] \phi_g$ decays.

Solving at Fixed Frequency and Momentum

Recall Shear Equation and Full Measured Momentum; p. 120, p. 121

$$(\mathcal{L} - i\omega + ikv_z)\phi = i\omega\beta\frac{p_x p_y}{E_p}h$$

$$G_R^{xyxy} = -\frac{1}{h} \sum_a \nu_a \int \frac{d^3p}{(2\pi)^3} f_{0a}(E_p) [1 + s_a f_{0a}(E_p)] \frac{p_x p_y}{E_p} \phi_a(\mathbf{p})$$

1. **Source:** choose the metric or flavor-current source.
2. **Basis:** represent $\mathcal{L} - i\omega + ikv_z$ in radial and angular bases.
3. **Constraints:** impose boundaries and conservation constraints.
4. **Response:** solve for ϕ and evaluate the full stress or current integral.
5. **Convergence:** refine radial spacing, momentum range and angular cutoff.
 - **Checks:** zero modes, Kubo limits, hydrodynamic poles and free streaming.

Stress Channels at Finite Momentum

- **Wave Vector:** $\mathbf{k} = k\hat{\mathbf{z}}$.

Channel	Probe	Long-Wavelength Content
Shear	T^{zx}	Transverse Momentum Diffusion
Sound	T^{zz}	Energy and Longitudinal Momentum
Tensor	T^{xy}	Stress; No Conserved-Density Pole
Bulk	$T^\mu{}_\mu$	Scale Breaking and Sound

- $k = 0$: rotationally equivalent stress components.
- **Finite k** : distinct channels.

Taking Moments Gives Fluid Conservation

- **Collision Moments:** weights E_p, p_i ; sum all species.

$$\sum_a \nu_a \int \frac{d^3p}{(2\pi)^3} (E_p, p_i) \mathcal{C}_a[f] = 0 \quad \implies \quad \partial_t T^{00} + \partial_j T^{j0} = 0, \quad \partial_t T^{0i} + \partial_j T^{ji} = 0$$

$$\delta T^{00} = \delta e, \quad T^{0i} = (e + P)u_i$$

$$\delta T^{ij} = c_s^2 \delta e \delta_{ij} - \eta \Sigma^{ij} - \zeta \delta_{ij} \nabla \cdot \mathbf{u}, \quad c_s^2 = \frac{\partial P}{\partial e}$$

- **Hydrodynamic Closure:** $\omega, k \ll \Gamma_{\text{tr}}$.

Hydrodynamic Poles from Conservation

- **Conserved Disturbances:** transverse flow and longitudinal density.
- **Attenuation Coefficients:** D_η, D_s have dimension E^{-1} .

$$(e + P)\partial_t u_x - \eta\partial_z^2 u_x = 0 \quad \implies \quad \omega = -iD_\eta k^2, \quad D_\eta = \frac{\eta}{e + P}$$

$$\partial_t \delta e + (e + P)\partial_z u_z = 0, \quad (e + P)\partial_t u_z + c_s^2 \partial_z \delta e - \left(\frac{4}{3}\eta + \zeta\right)\partial_z^2 u_z = 0$$

Sound Attenuation and Poles

$$\omega^2 - c_s^2 k^2 + iD_s k^2 \omega = 0, \quad D_s = \frac{4\eta/3 + \zeta}{e + P}$$

$$\omega \simeq \pm c_s k - \frac{i}{2} D_s k^2, \quad c_s^2 = \frac{\partial P}{\partial e}$$

A Diffusion Pole and a Stress-Spectrum Dip

$$(-i\omega(e + P) + \eta k^2)u_x = \eta k\omega h_{zx}, \quad \delta T_{\text{dyn}}^{zx} = -i\eta k u_x + i\eta\omega h_{zx}$$

Diffusion Pole in the Stress Response

$$G_R^{zxzx} = -\frac{\eta\omega^2}{-i\omega + D_\eta k^2}, \quad \frac{\rho^{zxzx}}{2\omega} = \eta \frac{\omega^2}{\omega^2 + (D_\eta k^2)^2}$$

- **Stress Response:** Eliminate u_x and use $\delta T_{\text{dyn}}^{zx} = -G_R^{zxzx} h_{zx}$.
- **Diffusion Dip:** The diffusion denominator and stress numerator give a finite- k dip.
- **Kubo Limit:** first $k = 0$, then $\omega \rightarrow 0$.
- **Tensor Channel, First Order:** $G_R^{xyxy} = -i\eta\omega$.

The Free-Streaming Limit

- **Streaming Limit:** neglect collisions for $\omega, k \gg \Gamma_{\text{tr}}$, within kinetic accuracy.

$$G_R^{xyxy} = -i\omega\beta \sum_a \nu_a \int \frac{d^3p}{(2\pi)^3} f_{0a}(E_p) [1 + s_a f_{0a}(E_p)] \frac{p_x p_y}{E_p} \left[R(\omega, k) \left(\frac{p_x p_y}{E_p} \right) \right]_a$$

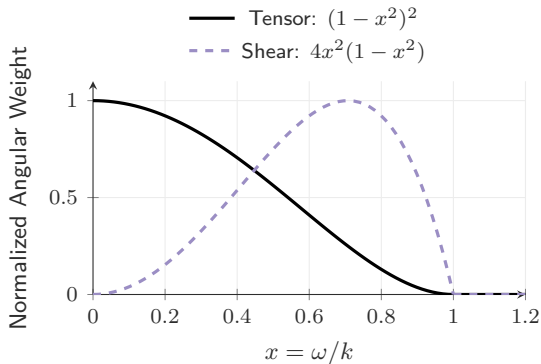
$$R(\omega, k) \simeq \frac{i}{\omega - kv_z + i0^+}, \quad \text{Im} \frac{1}{x + i0^+} = -\pi\delta(x)$$

Resonant Streaming Spectrum

$$\frac{\rho^{xyxy}}{2\omega} = \pi\beta \sum_a \nu_a \int \frac{d^3p}{(2\pi)^3} f_{0a}(E_p) [1 + s_a f_{0a}(E_p)] \left(\frac{p_x p_y}{E_p} \right)^2 \delta(\omega - kv_z)$$

- **Resonance:** particle velocity $v_z = \omega/k$.
- **Massless Particles:** support within $|\omega| < k$.

Angular Structure of Free Streaming



Recall Massless Shear Weight; p. 94, p. 8

$$\frac{p_x p_y}{E_p} = p \hat{p}_x \hat{p}_y \quad (E_p = p)$$

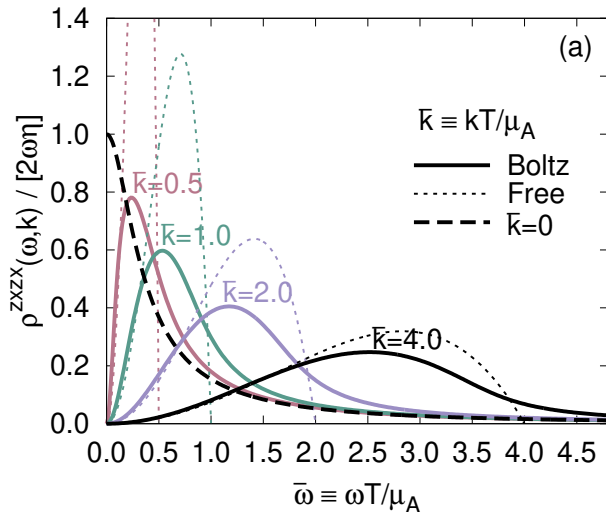
$$\langle \hat{p}_x^2 \hat{p}_y^2 \rangle_\varphi = \frac{(1 - x^2)^2}{8}$$

$$\frac{\rho^{xyxy}}{2\omega} = \frac{\nu_g \pi^3 T^4}{120k} (1 - x^2)^2 \theta(1 - |x|)$$

$$\int_0^\infty dp p^4 f_{0g}(p) [1 + f_{0g}(p)] = \frac{4\pi^4 T^5}{15}$$

- **Tensor:** $(1 - x^2)^2$.
- **Shear:** $4x^2(1 - x^2)$.

From Hydrodynamics to Streaming



Recall Drag Scale; p.129

$$\mu_g^{\text{drag}} = \frac{g^2 C_A m_D^2}{8\pi} \log(T/m_D)$$

$$\frac{\rho^{zzzz}}{2\omega\eta}, \quad \bar{\omega} = \frac{\omega T}{\mu_g^{\text{drag}}}, \quad \bar{k} = \frac{kT}{\mu_g^{\text{drag}}}$$

- **Solid:** kinetic solution.
- **Dotted:** free streaming.
- **Dashed Black:** $k = 0$.
- **Finite k :** diffusion dip.
- **Larger k :** approach to streaming.
- **Published Axes:** $\mu_A = \mu_g^{\text{drag}}$; pure-gluon LL.

A Second Inverse Moment

- $k = 0$: expand the retarded inverse on the nonconserved shear sector.

$$R(\omega, 0) = \mathcal{L}^{-1} + i\omega\mathcal{L}^{-2} + O(\omega^2)$$
$$G_R^{xyxy}(\omega, 0) = -i\eta\omega + \eta\tau_\pi\omega^2 + \dots$$

Second Inverse Moment with the Full Measure

$$\tau_\pi = \frac{\sum_a \nu_a \int \frac{d^3p}{(2\pi)^3} f_{0a}(E_p) [1 + s_a f_{0a}(E_p)] \frac{p_x p_y}{E_p} \left[\mathcal{L}^{-2} \left(\frac{p_x p_y}{E_p} \right) \right]_a}{\sum_a \nu_a \int \frac{d^3p}{(2\pi)^3} f_{0a}(E_p) [1 + s_a f_{0a}(E_p)] \frac{p_x p_y}{E_p} \left[\mathcal{L}^{-1} \left(\frac{p_x p_y}{E_p} \right) \right]_a}$$

- **Meaning:** a weighted inverse moment of the relaxation spectrum.
- **Curvature Response:** κ adds $-\kappa(\omega^2 + k^2)/2$.
- **HT Order:** $\kappa = 0$ at enhanced kinetic order; static $O(T^2)$ terms may remain.

Two Modes and an Effective Relaxation Time

- **Two Modes:** $\lambda_1 = \lambda_0$, $\lambda_2 = 4\lambda_0$; equal nonzero squared overlaps:

$$\left| \sum_a \nu_a \int \frac{d^3 p}{(2\pi)^3} f_{0a}(E_p) [1 + s_a f_{0a}(E_p)] e_{1,a}^*(\mathbf{p}) \frac{p_x p_y}{E_p} \right|^2$$

$$= \left| \sum_a \nu_a \int \frac{d^3 p}{(2\pi)^3} f_{0a}(E_p) [1 + s_a f_{0a}(E_p)] e_{2,a}^*(\mathbf{p}) \frac{p_x p_y}{E_p} \right|^2 \neq 0.$$

Cancel the Equal Overlap Integrals

$$\tau_\pi = \frac{\lambda_1^{-2} + \lambda_2^{-2}}{\lambda_1^{-1} + \lambda_2^{-1}} = \frac{1 + 1/16}{1 + 1/4} \frac{1}{\lambda_0} = \frac{17}{20\lambda_0}$$

$$\tau_{\text{slow}} = \frac{1}{\lambda_0}, \quad \tau_\pi < \tau_{\text{slow}}$$

$$\frac{\eta}{e + P} = \frac{\tau}{5} \quad (\text{Massless Gas, Constant Relaxation Time})$$

- **One Mode:** $\tau_\pi = \tau$ and $\tau_\pi / [\eta / (e + P)] = 5$.
- **Full Spectrum:** changes the ratio between gradient coefficients and the slowest time.

- **Published Results:** SU(3), leading-log accuracy.

N_f	0	2	3
$\tau_\pi/[\eta/(sT)]$	6.32	6.65	6.46
τ_J/D	3.776	3.756	3.748

$$\frac{\eta}{e+P} = 0.4613 \frac{T}{\mu_g^{\text{drag}}} \quad (\text{Pure Glue}), \quad e+P = sT \quad (\mu = 0)$$

- **Notation:** entropy density s ; current relaxation coefficient τ_J (p. 153).

Scale Breaking Drives the Bulk Response

- **Scale Breaking:** running $g(T)$; fixed m/T preserves scaling.

$$g_{\mu\nu} = (1 + H)\eta_{\mu\nu}, \quad T_H \sqrt{-g_{00}} = T, \quad T_H = T(1 - H/2) + O(H^2)$$

$$m_\infty^2 = \frac{C_A}{6} g^2(T) T^2, \quad b(g) = \frac{dg^2}{d \log T^2}$$

$$\delta m_\infty^2 = -T^2 \frac{\partial m_\infty^2}{\partial T^2} H$$

Scale-Breaking Mass Combination

$$\tilde{m}^2 = m_\infty^2 - T^2 \frac{\partial m_\infty^2}{\partial T^2} = -\frac{C_A T^2}{6} b(g)$$

- **Source:** gravitational redshift plus thermal-mass variation.

Deriving the Bulk Driving Term

- **Canonical Momentum:** $\kappa_i = P_i = (1 + H)p^i$; $\boldsymbol{\kappa} = \mathbf{p}$ at $H = 0$.
- **Reference Energy:** $E_\kappa^2 = \kappa^2 + m_\infty^2(T)$.

$$\mathcal{H} = -P_0 = \sqrt{\kappa^2 + (1 + H)m_\infty^2(T_H)} = E_\kappa + \frac{H\tilde{m}^2}{2E_\kappa} + O(H^2)$$

$$f_{0g}^H = \frac{1}{\exp(\mathcal{H}/T) - 1}, \quad D_H = \partial_t + \nabla_\kappa \mathcal{H} \cdot \nabla_x - \nabla_x \mathcal{H} \cdot \nabla_\kappa$$

$$D_H \mathcal{H} = \partial_t \mathcal{H}, \quad D_H f_{0g}^H = -\frac{f_{0g}^H(1 + f_{0g}^H)}{T} \partial_t \mathcal{H} = -f_{0g}(E_p)[1 + f_{0g}(E_p)] \frac{\tilde{m}^2}{2E_p T} \partial_t H$$

Bulk Driving Term to First Order

$$f_g = f_{0g}^H + \delta f_g$$

$$(\partial_t + \mathbf{v} \cdot \nabla_x) \delta f_g - f_{0g}(E_p)[1 + f_{0g}(E_p)] \frac{\tilde{m}^2}{2E_p T} \partial_t H = -f_{0g}(E_p)[1 + f_{0g}(E_p)] \mathcal{L}_{LL} \phi_g.$$

- **Fixed Coupling:** $\tilde{m}^2 = 0$; the bulk drive vanishes.

Measuring the Bulk Response

Recall Retarded and Spectral Convention; p. 115

$$\delta\langle A \rangle = -G_R^{AB} h, \quad \rho^{AA} = -2 \operatorname{Im} G_R^{AA}$$

$$\delta T^\mu{}_\mu|_{\text{kinetic}} = -\nu_g \int \frac{d^3 p}{(2\pi)^3} \frac{\tilde{m}^2}{E_p} \delta f_g$$

Bulk Kubo Relation

$$\mathcal{P} = \frac{1}{3} T^{ii} - c_s^2 T^{00}, \quad \zeta = \lim_{\omega \rightarrow 0} \frac{\rho^{\mathcal{P}\mathcal{P}}(\omega, 0)}{2\omega}$$

- **Scale Breaking:** Same scale-breaking factor in source and measured moment.
- **Energy Projection:** Subtract the conserved-energy contribution in the Kubo limit.
- **Finite Momentum:** trace response also contains sound.
- **Streaming Limit:** Naive free streaming can violate the soft boundary and diverge.

Recall Bath Screening and Color Factors; p.13, p.9

$$m_D^2 = g^2 T^2 \left(\frac{N_c}{3} + \frac{N_f}{6} \right), \quad C_q = C_F = \frac{N_c^2 - 1}{2N_c}, \quad C_g = N_c$$

Species-Dependent Drag

$$\mu_a^{\text{drag}} = \frac{g^2 C_a m_D^2}{8\pi} \log(T/m_D)$$

- **Soft Gluon Exchange:** species-dependent momentum diffusion.
- **Energy/Momentum Gain:** sum currents of all species.
- **Soft Quark Exchange:** additional LL conversion terms.
- **Flavor Source:** $\delta f_q - \delta f_{\bar{q}}$.
- **Flavor Difference:** bath momentum gain cancels; flavor-conserving conversion gain remains.

Flavor-Current Response

- **One Massless Dirac Flavor:** unit charge; $\chi_Q = N_c T^2/3$, $\sigma_Q = \chi_Q D$.

$$\tau_J = \frac{\sum_a \nu_a \int \frac{d^3 p}{(2\pi)^3} f_{0a}(E_p) [1 + s_a f_{0a}(E_p)] q_a v_p^x [\mathcal{L}^{-2}(q_b v_k^x)]_a(\mathbf{p})}{\sum_a \nu_a \int \frac{d^3 p}{(2\pi)^3} f_{0a}(E_p) [1 + s_a f_{0a}(E_p)] q_a v_p^x [\mathcal{L}^{-1}(q_b v_k^x)]_a(\mathbf{p})}$$

Current Response through Second Order

$$G_R^{zz} = -\frac{\sigma_Q \omega^2 + i\sigma_Q \tau_J \omega^3}{-i\omega + Dk^2}, \quad G_R^{xx} = -i\sigma_Q \omega + \sigma_Q \tau_J \omega^2 - \kappa_B k^2$$

$$\tau_J \partial_t \mathbf{J}_Q + \mathbf{J}_Q = -D \nabla n_Q + \sigma_Q \mathbf{E}$$

- **Operator Input:** $q_b v_k^x$ carries input species b and momentum $\mathbf{k}$.
- **Static Coefficient:** $\kappa_B = 0$ at retained order.
- **Relaxation Equation:** a low-frequency match; an extra pole requires a model assumption.

What Finite Frequency and Momentum Reveal

- **Low Frequency:** transport coefficients and hydrodynamic poles.
- **Finite Frequency:** microscopic relaxation spectrum.
- **Finite Momentum:** collective motion to resonant streaming.
- **LL Closure:** Conserving gain terms and the soft boundary complete the LL problem.

Perturbative Accuracy

- **AMY Transport:** complete LO.
- **Hong–Teaney Spectra:** leading log.

VIII

Microscopic Spin Dynamics

How do QCD Collisions Act on a Spin Density Matrix?

A Spin Density Matrix at Each Momentum

- **Spin Coherence:** matrix occupation in a fixed Pauli basis.

Spin Density Matrix

$$\varrho(\mathbf{p}) = \frac{f_{\text{tr}}(\mathbf{p})}{2} \mathbf{1} + \mathbf{S}(\mathbf{p}) \cdot \boldsymbol{\sigma}, \quad f_{\text{tr}} = \text{Tr } \varrho, \quad S_i = \frac{1}{2} \text{Tr}(\sigma_i \varrho) \quad (13)$$

$$\mathbf{P} = \frac{2\mathbf{S}}{f_{\text{tr}}} \quad (f_{\text{tr}} > 0), \quad \frac{f_{\text{tr}}}{2} \pm |\mathbf{S}| \geq 0 \quad \implies \quad |\mathbf{P}| \leq 1$$

- **Massive Species Q :** unpolarized $f_{\text{tr}} = 2f_Q$ per color.
- **Spin Observables:** $\mathbf{S}$: spin density; $\mathbf{P}$: polarization per particle.
- **Integrated Densities:** $n_{\text{tr}} = \int_{\mathbf{p}} f_{\text{tr}}$, $\mathbf{S}_{\text{tot}} = \int_{\mathbf{p}} \mathbf{S}$; both per color.

$$\varrho_{+x} = \frac{f_{\text{tr}}}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad \varrho_{\text{mix}} = \frac{f_{\text{tr}}}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$
$$\mathbf{S}_{+x} = \frac{f_{\text{tr}}}{2} \hat{\mathbf{x}}, \quad \mathbf{S}_{\text{mix}} = 0$$

- **Diagonal Populations:** Identical diagonal populations in the σ_z basis.
- **Off-Diagonal Entries:** distinguish polarization from a mixture.
- **Spin Coherence:** Degenerate spin states retain their relative phase under free evolution.

- **Helicity Spinors:** $(\hat{\mathbf{p}} \cdot \boldsymbol{\sigma})\xi_r = r\xi_r$, $r = \pm 1$.

$$\varrho(\mathbf{p}) = \sum_{rr'} \xi_r(\mathbf{p}) \varrho_{rr'}(\mathbf{p}) \xi_{r'}^\dagger(\mathbf{p})$$

$$\xi_r \rightarrow e^{i\alpha_r} \xi_r, \quad \varrho_{rr'} \rightarrow e^{-i(\alpha_r - \alpha_{r'})} \varrho_{rr'}$$

- **Basis Phases:** Phase factors cancel in the physical matrix ϱ .
- **Momentum Derivatives:** must include the changing spinor basis.
- **Observable Choice:** Other relativistic spin observables require an explicit basis map.

- **Mass Hierarchy:** $m \gg m_D \sim gT$; retain the full dependence on m/T .
- **Dilute Tracked Quarks:** negligible Pauli blocking and mutual collisions.
- **Bath Symmetries:** Isotropic, unpolarized, parity-even equilibrium bath.
- **Homogeneous Kernel:** zero vorticity and magnetic field.

Recall Light-Bath Screening; p. 13

$$m_D^2 = g^2 T^2 \left(\frac{N_c}{3} + \frac{N_f}{6} \right), \quad N_f = 2 \text{ in LY's setup}$$

- N_f : light bath flavors.

- **Reduced State:** thermal average of unitary evolution.

$$\varrho(t) = \langle U_1(t, 0) \varrho(0) U_2^\dagger(t, 0) \rangle_A, \quad U = \mathcal{T} \exp \left[-i \int dt H_I(t) \right]$$

- **Branch 1:** ket; branch 2: bra.
- **Zero Mean Field:** $\langle A_\mu \rangle = 0$: first nonzero correction is quadratic in H_I .
- **Coarse Graining:** $\tau_{\text{bath}} \ll \Delta t \ll \tau_{\text{relax}}$.
- **Secular Term** $\propto \Delta t$: local evolution with spin coherence.

Why There is One Gain and Two Losses

- **Second Order:** expand ket and bra evolution.

$$U = 1 + U^{(1)} + U^{(2)} + \dots, \quad U^{(1)} = -i \int dt H_I(t)$$

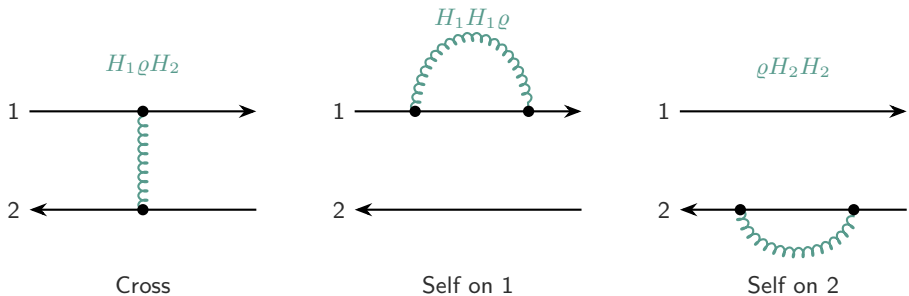
$$\Delta \varrho^{(2)} = \left\langle \underbrace{U^{(1)} \varrho U^{(1)\dagger}}_{\text{One Vertex on Each Branch}} + \underbrace{U^{(2)} \varrho + \varrho U^{(2)\dagger}}_{\text{Two Vertices on One Branch}} \right\rangle_A$$

$$U^\dagger U = 1 \quad \Rightarrow \quad U^{(2)} + U^{(2)\dagger} = -U^{(1)\dagger} U^{(1)}$$

$$\text{Tr } \Delta \varrho^{(2)} = 0 \quad \text{after summing all final momentum and spin states}$$

- **Cross Term:** probability gain.
- **Same-Branch Terms:** anticommutator loss on ket and bra.

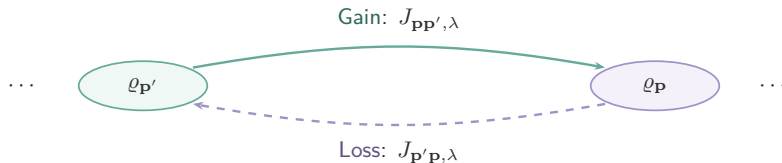
Cross and Self-Energy Contractions



- **Cross:** one vertex on each branch; gain from other momentum states.
- **Self-Energies:** probability loss on the ket and bra.
- **Probability Conservation:** Their sum conserves total quark probability.

Matrix Gain and Loss

- **Jump Vertex** $J_{\mathbf{p}\mathbf{p}',\lambda}$: spin transfer $\mathbf{p}' \rightarrow \mathbf{p}$; bath channel λ .



Matrix Gain and Loss

$$\partial_t \varrho_{\mathbf{p}} = \int_{\mathbf{p}'} \sum_{\lambda} \left[J_{\mathbf{p}\mathbf{p}',\lambda} \varrho_{\mathbf{p}'} J_{\mathbf{p}\mathbf{p}',\lambda}^{\dagger} - \frac{1}{2} \left\{ J_{\mathbf{p}'\mathbf{p},\lambda}^{\dagger} J_{\mathbf{p}'\mathbf{p},\lambda}, \varrho_{\mathbf{p}} \right\} \right]$$

$$\int_{\mathbf{p},\mathbf{p}'} \text{Tr}(J_{\mathbf{p}\mathbf{p}'} \varrho_{\mathbf{p}'} J_{\mathbf{p}\mathbf{p}'}^{\dagger}) = \int_{\mathbf{p},\mathbf{p}'} \text{Tr}(J_{\mathbf{p}'\mathbf{p}}^{\dagger} J_{\mathbf{p}'\mathbf{p}} \varrho_{\mathbf{p}})$$

- **Trace Conservation:** cyclicity and $\mathbf{p} \leftrightarrow \mathbf{p}'$.
- **Matrix Gain:** transfers and rotates a full spin matrix.
- **Microscopic Coefficients:** QCD coefficients follow from spin vertices and HTL correlators.

Momentum-State Gain: Retained

- **Inflow:** $\mathbf{p}' \rightarrow \mathbf{p}$; number conservation and detailed balance.

Bath Backreaction: Neglected

- **Fixed Reservoir:** absorbed spin/momentum; no evolved backreaction.
- **HT:** perturb the whole plasma; conserve total energy and momentum.
- **LY:** follow a dilute subsystem; conserve its quark number.

- **Bath Bose Factor:** signed frequency $\omega = E_p - E_{p'}$.

Recall KMS and Bath Spectra; p. 43

$$G_{\mu\nu}^< = f_{0g}(\omega)\rho_{\mu\nu}, \quad G_{\mu\nu}^> = [1 + f_{0g}(\omega)]\rho_{\mu\nu} = e^{\beta\omega} G_{\mu\nu}^<$$

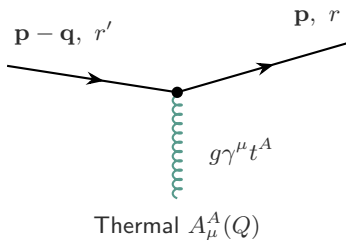
Thermal Detailed Balance

$$G^<(E_p - E_{p'})e^{-\beta E_{p'}} = G^>(E_p - E_{p'})e^{-\beta E_p}$$

$$\varrho_0 = f_{0Q}\mathbf{1}, \quad f_{0Q} \simeq z_Q e^{-\beta E_p}, \quad f_{0,\text{tr}} = 2f_{0Q}, \quad \mathbf{S}_0 = 0$$

- **Fugacity:** quark number fixes z_Q ; LY convention $z_{\text{LY}} = 2z_Q$.

Keeping the Spin Information in a QCD Vertex



- **Open Spin Indices:** retain incoming and outgoing polarization.

$$\mathbf{p}' = \mathbf{p} - \mathbf{q}, \quad \omega = E_p - E_{p'}$$
$$V_{rr'}^\mu(\mathbf{p}, \mathbf{p}') = \frac{\bar{u}(\mathbf{p}, r) \gamma^\mu u(\mathbf{p}', r')}{\sqrt{2E_p} \sqrt{2E_{p'}}}$$

- **External States:** $1/\sqrt{2E}$ each.
- **Color Vertex:** gt^A .
- **Spin Transfer:** two vertices contracted with $\langle A_\mu^A A_\nu^B \rangle$.

- **Conventions:** metric $(-, +, +, +)$; $\bar{u} = -u^\dagger \gamma^0$; $r = \pm 1$.

$$\gamma^0 = \begin{pmatrix} 0 & i\mathbf{1} \\ i\mathbf{1} & 0 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & i\sigma_i \\ -i\sigma_i & 0 \end{pmatrix}, \quad u(\mathbf{p}, r) = \begin{pmatrix} \sqrt{E_p - rp} \xi_r \\ \sqrt{E_p + rp} \xi_r \end{pmatrix}$$
$$\frac{\bar{u}(\mathbf{p}, r) \gamma^\mu u(\mathbf{p}', r')}{\sqrt{4E_p E_{p'}}}, \quad \xi_r \xi_r^\dagger = \frac{1}{2}(\mathbf{1} + r \hat{\mathbf{p}} \cdot \boldsymbol{\sigma})$$

- **Bath Contraction:** Contract two vertices with the bath correlator.
- **Spin Matrix:** Reconstruct the fixed-basis spin matrix.
- **Pauli Traces:** Take scalar and vector Pauli traces.

Writing the Vertex as a Two-Spin Matrix

- **Pauli Reduction:** spinor square roots.

$$\sqrt{E_p \pm \mathbf{p} \cdot \boldsymbol{\sigma}} = \frac{E_p + m \pm \mathbf{p} \cdot \boldsymbol{\sigma}}{\sqrt{2(E_p + m)}}$$

$$d_{pp'} = \sqrt{(E_p + m)(E_{p'} + m)}, \quad r_{pp'} = \sqrt{\frac{E_p + m}{E_{p'} + m}}$$

Pauli-Basis Vertices

$$V^0 = \frac{d_{pp'} \mathbf{1} + (\mathbf{p} \cdot \boldsymbol{\sigma})(\mathbf{p}' \cdot \boldsymbol{\sigma})/d_{pp'}}{2\sqrt{E_p E_{p'}}} \quad V^i = \frac{r_{pp'} \sigma_i (\mathbf{p}' \cdot \boldsymbol{\sigma}) + r_{pp'}^{-1} (\mathbf{p} \cdot \boldsymbol{\sigma}) \sigma_i}{2\sqrt{E_p E_{p'}}}$$

- **Forward Limit:** $\mathbf{p}' = \mathbf{p}$ gives $V^0 = 1$, $V^i = v_p^i \mathbf{1}$.
- **Unresolved Kick:** spin independent.

Evaluating a Spin Transfer Explicitly

- **Real Polarization:** $V = a\mathbf{1} + i\mathbf{c} \cdot \boldsymbol{\sigma}$; real $a, \mathbf{c}$.

Recall Spin Matrix of the Incoming State; p. 156

$$\varrho' = \frac{1}{2}f'_{\text{tr}}\mathbf{1} + \mathbf{S}' \cdot \boldsymbol{\sigma}, \quad S'_i = \frac{1}{2} \text{Tr}(\sigma_i \varrho')$$

$$(\mathbf{a} \cdot \boldsymbol{\sigma})(\mathbf{b} \cdot \boldsymbol{\sigma}) = (\mathbf{a} \cdot \mathbf{b})\mathbf{1} + i(\mathbf{a} \times \mathbf{b}) \cdot \boldsymbol{\sigma}$$

Scalar and Spin Transfer

$$\text{Tr}(V \varrho' V^\dagger) = (a^2 + c^2)f'_{\text{tr}} \quad \frac{1}{2} \text{Tr}(\boldsymbol{\sigma} V \varrho' V^\dagger) = (a^2 - c^2)\mathbf{S}' + 2\mathbf{c}(\mathbf{c} \cdot \mathbf{S}') - 2a\mathbf{c} \times \mathbf{S}'$$

$$a_L = \frac{d_{pp'} + (\mathbf{p} \cdot \mathbf{p}')/d_{pp'}}{2\sqrt{E_p E_{p'}}}, \quad \mathbf{c}_L = \frac{\mathbf{p} \times \mathbf{p}'}{2d_{pp'}\sqrt{E_p E_{p'}}}$$

- **Transverse Modes:** $V_\lambda = \epsilon_{\lambda i} V^i$; $\epsilon_\lambda \cdot \mathbf{q} = 0$.

Combining the Vertex with the Thermal Bath

- **Coulomb Components:** $\lambda = L, T_1, T_2$; $V_L = V^0$.
- **Pauli Vertex:** $V_\lambda = a_\lambda \mathbf{1} + i c_\lambda \cdot \boldsymbol{\sigma}$.

$$\rho_{\mu\nu} = \delta_{\mu 0} \delta_{\nu 0} \rho_L + \delta_{\mu i} \delta_{\nu j} (\delta_{ij} - \hat{q}_i \hat{q}_j) \rho_T, \quad \rho_L = -2 \text{Im } D_{00}^R, \quad \rho_T = -2 \text{Im } D_T^R$$

Microscopic Spin Evolution

$$\partial_t \varrho_{\mathbf{p}} = g^2 C_F \int \frac{d^3 q}{(2\pi)^3} \sum_{\lambda} \rho_{\lambda}(\omega, q) \left[f_{0g}(\omega) V_{\lambda} \varrho_{\mathbf{p}-\mathbf{q}} V_{\lambda}^{\dagger} - (1 + f_{0g}(\omega)) (a_{\lambda}^2 + c_{\lambda}^2) \varrho_{\mathbf{p}} \right]$$

$$\omega = E_p - E_{\mathbf{p}-\mathbf{q}}, \quad \partial_t f_{\text{tr}} = \text{Tr}(\partial_t \varrho), \quad \partial_t S_i = \frac{1}{2} \text{Tr}(\sigma_i \partial_t \varrho)$$

- **Vertex Normalization:** The normalized vertices already contain $1/\sqrt{4E_p E_{p'}}$.
- **Pauli Decomposition:** scalar and vector kernels before the soft expansion.
- **Unresolved Kicks:** $f_{0g} \sim T/\omega$, $\mathbf{q} \rightarrow 0$: identical gain and loss cancel.

- **Soft Transfer:** $\mathbf{p}' = \mathbf{p} - \mathbf{q}$.

$$\varrho(\mathbf{p} - \mathbf{q}) = \varrho - q_i \partial_{p_i} \varrho + \frac{1}{2} q_i q_j \partial_{p_i} \partial_{p_j} \varrho + \dots$$

$$f_{0g}(\omega) = \frac{T}{\omega} - \frac{1}{2} + \dots$$

- **Consistent Expansion:** spinor vertices, occupations and the energy constraint.
- **Gain-Loss Cancellation:** Leading unresolved kicks cancel between gain and self-energies.
- **Transport Logarithm:** Retained transport terms sample $m_D \ll q \ll T$: $\int dq/q$.

Mass Shifts the Energy-Transfer Endpoints

- **On-Shell Energy Transfer:** $\omega = E_p - E_{\mathbf{p}-\mathbf{q}}$.

$$\hat{\mathbf{p}} \cdot \mathbf{q} = \frac{E_p}{p} \omega + \frac{q^2 - (\omega)^2}{2p}$$

Shifted Transfer Endpoints

$$\omega_{\max/\min} = E_p - \sqrt{(p \mp q)^2 + m^2} = \pm vq - \frac{m^2 q^2}{2E_p^3} + O(q^3), \quad v = \frac{p}{E_p} \quad (14)$$

- **Endpoint Shift:** Both endpoints shift downward by the same amount.
- **Odd Integrands:** acquire nonzero contributions at the retained LL order.

The Angular Average and Its Jacobian

- **Longitudinal Transfer:** $q_L = \hat{\mathbf{p}} \cdot \mathbf{q}$; azimuth around $\hat{\mathbf{p}}$.

$$\langle q_i \rangle_\varphi = q_L \hat{p}_i, \quad \langle q_i q_j \rangle_\varphi = q_L^2 \hat{p}_i \hat{p}_j + \frac{1}{2}(q^2 - q_L^2)(\delta_{ij} - \hat{p}_i \hat{p}_j)$$

Energy-Transfer Measure

$$\frac{d\omega}{d \cos \theta_{pq}} = \frac{pq}{E_{p'}}, \quad \int \frac{d^3 q}{(2\pi)^3} \frac{\mathcal{N}}{4E_p E_{p'}} = \frac{1}{4pE_p} \int \frac{q dq}{2\pi} \int_{\omega_{\min}}^{\omega_{\max}} \frac{d\omega}{2\pi} \langle \mathcal{N} \rangle_\varphi$$

- **Integrand:** $\mathcal{N}$: unnormalized vertex contractions, occupations and spectral density.
- **Longitudinal Transfer:** Replace q_L using the energy constraint on the preceding slide.
- **Spectral Moments:** The remaining energy integrals are the spectral moments $J_n^{L/T}$.

The Landau Cut Fixes the Imaginary Self-Energies

- **LL Window:** $m_D \ll q \ll T$; $x = \omega/q$, $|x| \lesssim v < 1$.

Recall Full HTL Self-Energies before Expansion; p. 40, p. 41

$$L(x) = \log \frac{x+1+i0^+}{x-1+i0^+}, \quad \Pi_{\text{el}} = m_D^2 \left[1 - \frac{x}{2} L(x) \right]$$

$$\Pi_T = \frac{m_D^2}{2} \left[x^2 + \frac{x(1-x^2)}{2} L(x) \right], \quad x = \omega/q$$

Imaginary Parts on the Landau Cut

$$\log \frac{\omega+q+i0}{\omega-q+i0} = \log \left| \frac{\omega+q}{\omega-q} \right| - i\pi, \quad |\omega| < q$$

$$\text{Im } \Pi_{\text{el}} = \frac{\pi m_D^2 \omega}{2q}, \quad \text{Im } \Pi_T = -\frac{\pi m_D^2 \omega}{4q} (1-x^2)$$

Obtaining the Bath Spectra from HTL

- **Input:** Landau-cut imaginary parts from p.174.
- **Expansion:** momentum transfer above screening; fixed subluminal velocity.

Longitudinal and Transverse Bath Spectra

$$D_{00}^R = \frac{1}{q^2 + \Pi_{\text{el}}} \implies \rho_L = -2 \text{Im } D_{00}^R \simeq \frac{2 \text{Im } \Pi_{\text{el}}}{q^4} = \frac{\pi m_D^2 \omega}{q^5}$$

$$D_T^R = \frac{1}{(\omega + i0)^2 - q^2 - \Pi_T} \implies \rho_T \simeq \frac{-2 \text{Im } \Pi_T}{q^4(1 - x^2)^2} = \frac{\pi m_D^2 \omega}{2q^5(1 - x^2)}$$

- **Fixed** $v < 1$: spectra odd in ω ; positive for $\omega > 0$.
- **Spectral Moments:** integrate over the allowed energy-transfer interval.

Spectral Moments and the Shifted Interval

$$J_n^{L/T} = \int_{\omega_{\min}}^{\omega_{\max}} \frac{d\omega}{2\pi} (\omega)^{2n-1} \rho_{L/T}, \quad \rho_L \simeq \frac{\pi m_D^2 \omega}{q^5}$$
$$J_1^L \simeq \frac{m_D^2}{2q^5} \int_{-vq}^{vq} d\omega (\omega)^2 = \frac{m_D^2 v^3}{3q^2}$$

Odd Moment from Shifted Endpoints

$$\delta = \frac{m^2 q^2}{2E_p^3}, \quad J_{1/2}^L \simeq \frac{m_D^2}{4q^5} [(vq - \delta)^2 - (-vq - \delta)^2] = -\frac{m_D^2 m^2 v}{2E_p^3 q^2}$$

- **Endpoint Shift:** essential for nonzero half-integer moments.

Where the Rapidity Dependence Comes from

- **Dimensionless Moments:** factor out m_D, q, E_p .

$$J_n^{L/T} = \begin{cases} m_D^2 q^{2n-4} j_n^{L/T}, & n \text{ integer}, \\ m_D^2 m^2 E_p^{-3} q^{2n-3} j_n^{L/T}, & n \text{ half integer} \end{cases}$$

$$\rho_T \simeq \frac{\pi m_D^2 \omega}{2q^5(1-x^2)}, \quad x = \omega/q, \quad y_p = \operatorname{arctanh} v$$

$$j_0^T = \frac{1}{4} \int_{-v}^v \frac{dx}{1-x^2} = \frac{y_p}{2}, \quad j_1^T = \frac{1}{4} \int_{-v}^v \frac{x^2 dx}{1-x^2} = \frac{y_p - v}{2}$$

$$j_2^T = \frac{1}{4} \int_{-v}^v \frac{x^4 dx}{1-x^2} = \frac{y_p - v}{2} - \frac{v^3}{6}, \quad j_{1/2}^T = -\frac{v}{4(1-v^2)} = -\frac{pE_p}{4m^2}$$

- **LL Window:** $m_D \ll q \ll T$; shifted endpoints for half-integer moments.

- **Dimensionless Coefficients:** $y_p = \operatorname{arctanh} v = \operatorname{arsinh}(p/m)$.

n	j_n^L	j_n^T
0	v	$y_p/2$
1	$v^3/3$	$(y_p - v)/2$
2	$v^5/5$	$(y_p - v)/2 - v^3/6$
1/2	$-v/2$	$-pE_p/(4m^2)$
3/2	$-v^3/2$	$-p^3/(4m^2E_p)$

IX

Spin Relaxation

How do Microscopic Spin Transitions Become Polarization Decay?

Organizing the Scalar and Spin Operators

- **Radial Exchange Integral:** $\int_{m_D}^T dq/q \simeq \log(1/g)$.

$$K(p) = \frac{C_F g^2 m_D^2 \log(1/g)}{8\pi p E_p}, \quad \mathcal{D}_p = \mathbf{p} \cdot \nabla_p$$

$$\mathcal{G}_f = c_0 + c_\Delta \nabla_p^2 + c_2 \mathcal{D}_p^2 + c_1 \mathcal{D}_p, \quad \partial_t f_{\text{tr}} = K \mathcal{G}_f f_{\text{tr}}$$

$$\mathcal{W}_i[\mathbf{S}] = p_i \nabla_p \cdot \mathbf{S} - \partial_{p_i}(\mathbf{p} \cdot \mathbf{S})$$

Spin Collision Operator

$$(\mathcal{G}_S \mathbf{S})_i = (\mathcal{G}_f + c_{\text{spin}}) S_i + c_w \mathcal{W}_i[\mathbf{S}] + c_p p_i (\mathbf{p} \cdot \mathbf{S}), \quad \partial_t \mathbf{S} = K \mathcal{G}_S \mathbf{S}$$

- **Coefficient Matching:** collect derivative structures in the Pauli traces.

From Bath Moments to Momentum Diffusion

- **Azimuthal Average:** collect $\partial_{p_i} \partial_{p_j}$.

Recall Bath Moments Used Below; p. 178

$$j_0^L = v, \quad j_1^L = \frac{v^3}{3}, \quad j_0^T = \frac{y_p}{2}, \quad j_1^T = \frac{y_p - v}{2}, \quad j_2^T = \frac{y_p - v}{2} - \frac{v^3}{6}$$

$$v = p/E_p, \quad y_p = \text{arctanh } v$$

$$c_\Delta = TE_p^2 \left(j_0^L - \frac{E_p^2}{p^2} j_1^L \right) + Tp^2 j_0^T - 2TE_p^2 j_1^T + \frac{TE_p^4}{p^2} j_2^T$$

$$c_2 = -\frac{TE_p^2}{p^2} \left(j_0^L - \frac{3E_p^2}{p^2} j_1^L \right) - Tj_0^T + \frac{4TE_p^2}{p^2} j_1^T - \frac{3TE_p^4}{p^4} j_2^T$$

$$\mathcal{D}_p^2 = p_i p_j \partial_{p_i} \partial_{p_j} + \mathcal{D}_p$$

- **Cancellation:** $j_0^L - 3E_p^2 j_1^L / p^2 = v - v = 0$.
- **Transverse Moments:** Next: insert j_n^T and $E_p^2 - p^2 = m^2$.
- **Shared Diffusion:** Identical second derivatives for scalar and spin occupations.

- **Kinematics:** $E_p = \sqrt{p^2 + m^2}$; $y_p = \text{arsinh}(p/m)$.

Diffusion and Drift Coefficients

$$c_0 = 2p, \quad c_\Delta = TpE_p - \frac{Tm^2E_p}{2p} + \frac{y_pTm^4}{2p^2}$$

$$c_2 = \frac{Tm^2}{2p^2} \left[y_p \left(1 - \frac{3E_p^2}{p^2} \right) + \frac{3E_p}{p} \right]$$

$$c_1 = \frac{1}{p^2} \left[pE_p^2 - \frac{3Tm^2E_p}{2p} + y_pm^2 \left(-E_p - \frac{T}{2} + \frac{3TE_p^2}{2p^2} \right) \right]$$

- **Diffusion:** c_Δ, c_2 ; drift fixed by probability balance and equilibrium.

Recovering Drift from a Conserved Probability Flux

- **Radial Diffusion:** $a_{\parallel} = K(c_{\Delta} + p^2 c_2)$.

Recall Moments for the Scalar Drift; p. 178

$$j_{1/2}^L = -\frac{v}{2}, \quad j_{1/2}^T = -\frac{pE_p}{4m^2}, \quad j_{3/2}^T = -\frac{p^3}{4m^2E_p}, \quad j_1^L = \frac{v^3}{3}, \quad j_1^T - j_2^T = \frac{v^3}{6}$$

Conserved Probability Flux

$$\partial_t f_{\text{tr}} = \frac{1}{p^2} \partial_p \left[p^2 a_{\parallel} \left(\partial_p f_{\text{tr}} + \frac{p}{E_p T} f_{\text{tr}} \right) \right]$$

$$K \left(\frac{2c_{\Delta}}{p} + pc_2 + pc_1 \right) = a'_{\parallel} + \frac{2a_{\parallel}}{p} + \frac{pa_{\parallel}}{E_p T}, \quad Kc_0 = \frac{1}{p^2} \partial_p \left(\frac{p^3 a_{\parallel}}{E_p T} \right)$$

$$c_0 = 4 \left[-\frac{m^2}{E_p} j_{1/2}^L + E_p j_1^L - \frac{m^2}{E_p^3} (p^2 j_{1/2}^T - E_p^2 j_{3/2}^T) + E_p (j_1^T - j_2^T) \right] = 2p$$

- **Zero Boundary Flux:** conserved n_{tr} .
- **Equilibrium:** $f_{\text{tr}} \propto e^{-E_p/T}$ gives zero flux.

From Pauli Rotation to Spin Coefficients

- **Spin Transfer:** local term plus gradient $\mathcal{W}_i$.

Recall Spin-Transfer Trace; p. 169

$$\frac{1}{2} \text{Tr}(\boldsymbol{\sigma} V \boldsymbol{\varrho}' V^\dagger) = (a^2 - c^2) \mathbf{S}' + 2\mathbf{c}(\mathbf{c} \cdot \mathbf{S}') - 2a\mathbf{c} \times \mathbf{S}'$$

$$c_{\text{spin}} = T \left[j_0^L - \frac{E_p^2}{p^2} j_1^L + j_0^T + \left(1 - \frac{3E_p^2}{p^2} \right) j_1^T + \frac{E_p^2}{p^2} j_2^T \right]$$

$$c_w = 2T \left[\frac{E_p}{E_p + m} \left(j_0^L - \frac{E_p^2}{p^2} j_1^L \right) + j_0^T - \frac{E_p(2E_p - m)}{p^2} j_1^T + \frac{E_p^3}{p^2(E_p + m)} j_2^T \right]$$

For Example:
$$\frac{c_{\text{spin}}}{T} = \frac{2v}{3} + \frac{y_p}{2} + \frac{1 - 3/v^2}{2} (y_p - v) + \frac{1}{v^2} \left(\frac{y_p - v}{2} - \frac{v^3}{6} \right)$$

Local Spin Coefficient

$$\frac{c_{\text{spin}}}{T} = \frac{1}{v} - \frac{1 - v^2}{v^2} y_p$$

- **Scalar/Spin Sectors:** common diffusion; distinct spin-transfer terms.

The Longitudinal Spin Contribution

- **Longitudinal Projection:** from $2\mathbf{c}(\mathbf{c} \cdot \mathbf{S}')$.

$$c_p = -\frac{T}{p^2} \left[\frac{E_p - m}{E_p + m} \left(j_0^L - \frac{E_p^2}{p^2} j_1^L \right) + j_0^T \right. \\ \left. - \left(1 + \frac{3E_p^2}{p^2} - \frac{2E_p}{E_p + m} \right) j_1^T + \frac{E_p^2}{p^2} \left(3 + \frac{4p^2}{(E_p + m)^2} - \frac{6E_p}{E_p + m} \right) j_2^T \right]$$

Longitudinal Spin Coefficient

$$c_p = -\frac{T}{p^2} \left[\frac{E_p(E_p + 2m)}{p(E_p + m)} + \frac{y_p m E_p}{E_p + m} \left(-\frac{3E_p}{p^2} + \frac{1}{E_p + m} \right) \right]$$

- **Reduction:** insert $j_n^{L/T}$; collect y_p ; use $E_p^2 = p^2 + m^2$.

The Resulting Spin-Dependent Terms

Recall Scalar Operator and Spin Gradient; p. 180

$$\mathcal{G}_f = c_0 + c_\Delta \nabla_p^2 + c_2 \mathcal{D}_p^2 + c_1 \mathcal{D}_p$$

$$\mathcal{W}_i[\mathbf{S}] = p_i \nabla_p \cdot \mathbf{S} - \partial_{p_i}(\mathbf{p} \cdot \mathbf{S})$$

Spin-Dependent Operator Terms

$$c_{\text{spin}} = \frac{TE_p}{p} - \frac{y_p m^2 T}{p^2} \quad c_w = \frac{T}{p^2(E_p + m)} [p(2p^2 + mE_p) - y_p m^3]$$

$$(\mathcal{G}_S \mathbf{S})_i = \mathcal{G}_f S_i + c_{\text{spin}} S_i + c_w \mathcal{W}_i[\mathbf{S}] + c_p p_i (\mathbf{p} \cdot \mathbf{S})$$

- **Spin Coefficients:** c_w : spin-momentum gradients; c_p : longitudinal spin projection.
- **Isotropic, Parity-Even Bath:** scalar and spin sectors decouple.
- **Small-Momentum Limit:** combine terms before $p \rightarrow 0$; singular factors cancel.

Why Polarization Needs Two Radial Functions

- **Small Polarization:** fixed axial vector $\mathbf{e}$.
- **Bath Symmetry:** isotropic and parity even.

$$S_i(\mathbf{p}) = [a(p)\delta_{ij} + b(p)\hat{p}_i\hat{p}_j] e_j$$

Two Radial Spin Functions

$$\hat{p}_i\hat{p}_j = \frac{\delta_{ij}}{3} + \left(\hat{p}_i\hat{p}_j - \frac{\delta_{ij}}{3} \right) \implies \mathbf{S} = \underbrace{(a + b/3)}_{s_0} \mathbf{e} + \underbrace{b}_{s_2} \left[\hat{\mathbf{p}}(\hat{\mathbf{p}} \cdot \mathbf{e}) - \frac{\mathbf{e}}{3} \right]$$

$$\langle \hat{p}_i\hat{p}_j \rangle_{\Omega} = \frac{\delta_{ij}}{3}, \quad \langle \mathbf{S} \rangle_{\Omega} = s_0(p)\mathbf{e}$$

- **Allowed Orbital Sectors:** $\ell = 0, 2$; mixed by collisions.
- **Excluded:** parity-odd $\hat{\mathbf{p}} \times \mathbf{e}$.

Angular Structure of a Polarized State

- **Parity-Even** $J = 1$: orbital $\ell = 0, 2$.

Recall Radial Dilation Operator; p. 180

$$\mathcal{D}_p = \mathbf{p} \cdot \nabla_p, \quad \mathcal{D}_p s(p) = p s'(p)$$

$$\mathbf{S} = s_0(p)\mathbf{e} + s_2(p) \left[\hat{\mathbf{p}}(\hat{\mathbf{p}} \cdot \mathbf{e}) - \frac{\mathbf{e}}{3} \right], \quad \partial_{p_i} \hat{p}_j = \frac{\delta_{ij} - \hat{p}_i \hat{p}_j}{p}$$

$$\nabla_p^2 \mathbf{S} = \left(s_0'' + \frac{2}{p} s_0' \right) \mathbf{e} + \left(s_2'' + \frac{2}{p} s_2' - \frac{6}{p^2} s_2 \right) \left[\hat{\mathbf{p}}(\hat{\mathbf{p}} \cdot \mathbf{e}) - \frac{\mathbf{e}}{3} \right]$$

$$\mathcal{D}_p \mathbf{S} = p s_0' \mathbf{e} + p s_2' \left[\hat{\mathbf{p}}(\hat{\mathbf{p}} \cdot \mathbf{e}) - \frac{\mathbf{e}}{3} \right], \quad \mathcal{D}_p^2 s_\ell = p^2 s_\ell'' + p s_\ell'$$

- **Axis:** $\mathbf{e}$; traceless term: quadrupole distortion.

Cartesian Identities for the Spin Gradients

- **Longitudinal Projection:** $s_0 + 2s_2/3$; coupled radial components.

Recall Spin–Momentum Gradient Operator; p. 180

$$\mathcal{W}_i[\mathbf{S}] = p_i \nabla_p \cdot \mathbf{S} - \partial_{p_i}(\mathbf{p} \cdot \mathbf{S})$$

$$\mathbf{p} \cdot \mathbf{S} = p(s_0 + 2s_2/3)(\hat{\mathbf{p}} \cdot \mathbf{e}), \quad \nabla_p \cdot \mathbf{S} = \left(s'_0 + \frac{2}{3}s'_2 + \frac{2s_2}{p} \right) (\hat{\mathbf{p}} \cdot \mathbf{e})$$

$$\nabla_p(\mathbf{p} \cdot \mathbf{S}) = (s_0 + 2s_2/3)\mathbf{e} + p(s'_0 + 2s'_2/3)(\hat{\mathbf{p}} \cdot \mathbf{e})\hat{\mathbf{p}}$$

- **Substitution:** insert the divergence and gradient into the spin–momentum operator.

Reducing the Spin–Momentum Gradients

- **Input:** Cartesian identities from p. 189.

Cancellation of Radial Derivatives

$$\mathcal{W}[\mathbf{S}] = 2s_2(\hat{\mathbf{p}} \cdot \mathbf{e})\hat{\mathbf{p}} - (s_0 + 2s_2/3)\mathbf{e} = -s_0\mathbf{e} + 2s_2 \left[\hat{\mathbf{p}}(\hat{\mathbf{p}} \cdot \mathbf{e}) - \frac{\mathbf{e}}{3} \right]$$

$$\mathbf{p}(\mathbf{p} \cdot \mathbf{S}) = p^2(s_0 + 2s_2/3) \left(\left[\hat{\mathbf{p}}(\hat{\mathbf{p}} \cdot \mathbf{e}) - \frac{\mathbf{e}}{3} \right] + \frac{1}{3}\mathbf{e} \right)$$

- **Derivative Cancellation:** only the two angular structures remain.
- **Radial Closure:** insert both structures into the coupled spin equations.

Coupled Radial Spin Equations

Recall Collision Prefactor; p. 180

$$K(p) = \frac{C_F g^2 m_D^2 \log(1/g)}{8\pi p E_p}, \quad E_p = \sqrt{p^2 + m^2}$$

$$\mathcal{H}_\ell = 2p + c_\Delta \left(\partial_p^2 + \frac{2}{p} \partial_p - \frac{\ell(\ell+1)}{p^2} \right) + c_2 (p \partial_p)^2 + c_1 p \partial_p$$

Coupled Radial Evolution

$$\partial_t s_0 = K \left[(\mathcal{H}_0 + c_{\text{spin}} - c_w) s_0 + \frac{c_p p^2}{3} (s_0 + 2s_2/3) \right]$$

$$\partial_t s_2 = K \left[(\mathcal{H}_2 + c_{\text{spin}} + 2c_w) s_2 + c_p p^2 (s_0 + 2s_2/3) \right]$$

- **Measured Spin:** s_0 ; dynamical feedback: s_2 .

- **Initial State:** $s_2(p, 0) = 0$, $s_0(p, 0) \neq 0$.

$$\partial_t s_2|_{t=0} = K(p)c_p(p)p^2 s_0(p, 0) \neq 0 \quad \text{generically}$$

$$\langle [\hat{\mathbf{p}}(\hat{\mathbf{p}} \cdot \mathbf{e}) - \frac{\mathbf{e}}{3}] \rangle_\Omega = 0, \quad \mathbf{S}_{\text{tot}} = \frac{\mathbf{e}}{2\pi^2} \int_0^\infty dp p^2 s_0(p, t)$$

Angular Feedback

$$\partial_t s_0|_{s_2 \text{ contribution}} = \frac{2}{9} K c_p p^2 s_2$$

- **Angular Feedback:** zero-average s_2 still changes total spin.

Checking the Scalar Equilibrium State

- **Dilute Equilibrium:** $f_{0,\text{tr}} = 2z_Q e^{-E_p/T}$.

Recall Conserved Radial Flux; p. 183

$$K\mathcal{G}_f f_{\text{tr}} = \frac{1}{p^2} \partial_p \left[p^2 a_{\parallel} \left(\partial_p + \frac{p}{E_p T} \right) f_{\text{tr}} \right], \quad a_{\parallel} = K(c_{\Delta} + p^2 c_2)$$

$$\frac{\mathcal{D}_p f_{0,\text{tr}}}{f_{0,\text{tr}}} = -\frac{p^2}{E_p T}, \quad \frac{\nabla_p^2 f_{0,\text{tr}}}{f_{0,\text{tr}}} = \frac{p^2}{E_p^2 T^2} - \frac{3}{E_p T} + \frac{p^2}{E_p^3 T}, \quad \frac{\mathcal{D}_p^2 f_{0,\text{tr}}}{f_{0,\text{tr}}} = \frac{p^4}{E_p^2 T^2} - \frac{2p^2}{E_p T} + \frac{p^4}{E_p^3 T}$$

Equilibrium Cancellation

$$(2p + c_{\Delta} \nabla_p^2 + c_2 \mathcal{D}_p^2 + c_1 \mathcal{D}_p) f_{0,\text{tr}} = 0$$

- **Detailed Balance:** drift, diffusion and rapidity terms cancel.

A Formal Helicity Consistency Check

- **Massless Vector Interaction:** helicity conservation.

Recall Spin-Trace Normalization; p. 156

$$f_{\text{tr}} = \text{Tr } \varrho, \quad S_i = \frac{1}{2} \text{Tr}(\sigma_i \varrho)$$

$$\varrho = f_+ P_+ + f_- P_-, \quad P_{\pm} = \frac{1}{2}(\mathbf{1} \pm \hat{\mathbf{p}} \cdot \boldsymbol{\sigma})$$

Helicity Decomposition

$$f_{\text{tr}} = f_+ + f_-, \quad \mathbf{S} = \frac{1}{2}(f_+ - f_-)\hat{\mathbf{p}}$$

- **Elastic Kernel:** identical scalar evolution for the helicity difference.
- **LY Check:** uses the unspecialized spectral-moment operator.
- **Massless Theory:** Physical massless theory also needs conversion and revised soft integrals.
- **Scope:** algebraic check; no extension of the massive approximation to $m = 0$.

The Scale of Spin Relaxation

- **Scaling Limit:** fixed $p/T, m/T$; $m \gg gT$.

Recall Prefactor and Thermal Screening; p. 180, p. 13

$$K(p) = \frac{C_F g^2 m_D^2 \log(1/g)}{8\pi p E_p}, \quad m_D^2 \sim g^2 T^2, \quad p, E_p \sim T$$

Parametric Relaxation Scale

$$K\mathcal{G}_f, K\mathcal{G}_S \sim g^4 T \log(1/g) \sim \alpha_s^2 T \log(1/\alpha_s) \quad \text{in coupling order}$$

- **Operator Dependence:** Mass, momentum and angular dependence remain in the operator.
- **Initial State:** generally excites several modes.
- **Late-Time Exponential:** an isolated slow mode with nonzero overlap.
- **Numerical Rate:** requires a specified spectral problem for the LY kernel.

- **Computed Equilibrium:** $S = 0$.
- **Vorticity or Magnetic Field:** match both source and collision kernel.
- **Orbital-to-Spin Transfer:** spatial gradients and angular momentum balance.
- **Light-Quark/Gluon Spin:** coupled distributions and conversion.
- **LY Scope:** relaxation of existing dilute massive-quark polarization.

From Microscopic Collisions to Response

Work	Evolving Object	Main Construction
AMY-EKT	Scalar f_a	Complete LO Elastic and LPM Dynamics
AMY-T	δf_a	Variational Collision-Operator Inverse
Hong–Teaney	$\delta f_a(\omega, k)$	Conserving LL Spectral Response
Li–Yee	Spin Matrix $\varrho(\mathbf{p})$	Dilute Massive-Quark LL Evolution

State and Source $\longrightarrow$ Collision Operator $\longrightarrow$ Observable Moment