upper limit,exclusion&discovery

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9.3

abstract

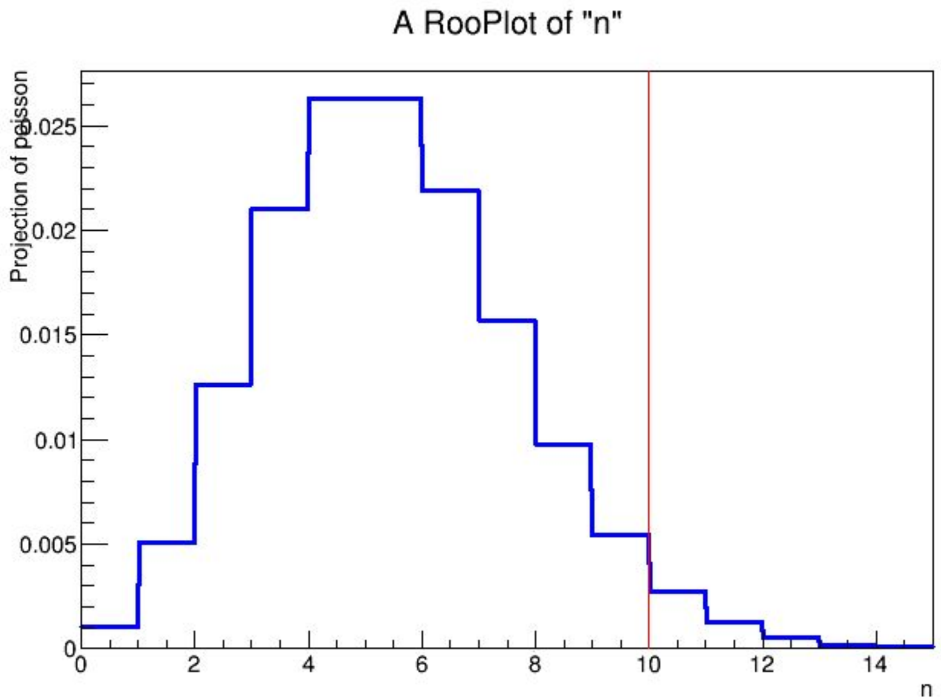
- upper limit and exclusion
  (what I understand before)
- discovery and their significance
  (what we talked about last week ,a little confused)
- I won't talk about any RooStat code here
- we can skip some basic knowledge about probability and hypothesis test here

upper limit

- upper limit means p(mu<mu_up)=0.95
- consider a poisson ditribution:b=4.5,n_obs=10,

what is the upper limit of s?

- we scan the value of mu and when the area on the right of the red line is 0.05,we can get the upper limit.



a simple method-PLR fit

Expand $\ln L(\theta)$ about its maximum:

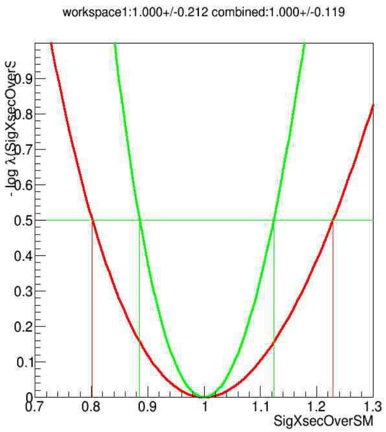
$$\ln L(\theta) = \ln L(\widehat{\theta}) + \left[\frac{\partial \ln L}{\partial \theta}\right]_{\theta=\widehat{\theta}} (\theta - \widehat{\theta}) + \frac{1}{2!}\left[\frac{\partial^2 \ln L}{\partial \theta^2}\right]_{\theta=\widehat{\theta}} (\theta - \widehat{\theta})^2 +$$

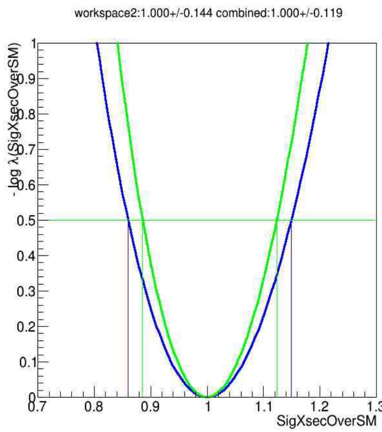
$$\ln L(\theta) \approx \ln L_{\max} - \frac{(\theta - \widehat{\theta})^2}{2\widehat{\sigma^2}_{\widehat{\theta}}}$$

$$\ln L(\widehat{\theta} \pm \widehat{\sigma}_{\widehat{\theta}}) \approx \ln L_{\max} - \frac{1}{2}$$

$$\ln L(\hat{\theta} \pm 2\hat{\sigma}) \approx \ln L_{\max} - 2$$

$$-Ln \frac{L(\hat{\theta} \pm 2\hat{\sigma})}{L_{\max}} \approx 2$$





- so when $-\ln\lambda(\mu)=2$, upper limit can be obtained

general method

- Null hypo:mu=0,alternate hypo:mu=1
- probability of x given θ:f(x|θ)
- test statistic: $q_{\mu} = -2Ln\frac{L(\mu,\hat{\hat{\theta}})}{L(\hat{\mu},\hat{\theta})}$ if mu^hat>0
- parameters with one hat mean the best fit,
- mu is a variable and parameters with two hat is the best fit for a certain given mu

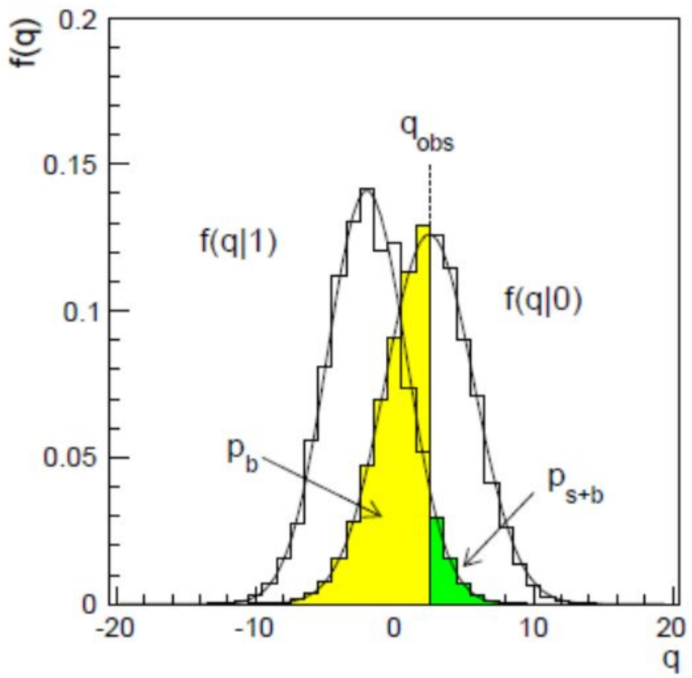
upper limit

- q_{obs} is calculated from data,q_{expect} is the median of background-only hypothesis

- I put Xiaohu's slides here......

Test statistic → exclusion or upper limit

- Calculate the test statistic of S+B hypo based on its p-value
- The signal model is regarded as excluded at a confidence level of 1-alpha = 95% if one finds



$$p_{s+b} < \alpha$$

$$1 - \alpha = 95\%$$

$$\text{confidence level CL} = 1 - \alpha$$

NOTE: in another word, the signal rate s under which the 5% p_{s+b} reaches, can be regarded as a upper limit s_{up}, for which signal is not excluded.

So the interval $[0,s_{up}]$ covers s with a probability of at least 95%

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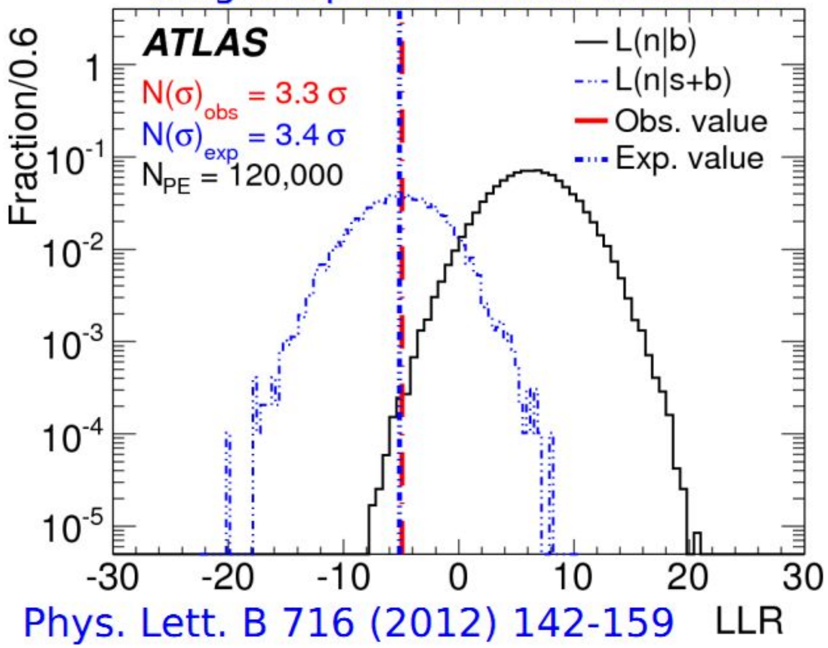
Test statistic → discovery

- Calculate the test statistic of B hypo based on its p-value: p_b
- Convert p-value into standard deviation (XX sigma)
- 3 sigma → evidence; 5 sigma → discovery
- The background-only model is regarded as rejected if 5 sigma

Single top Wt channel evidence



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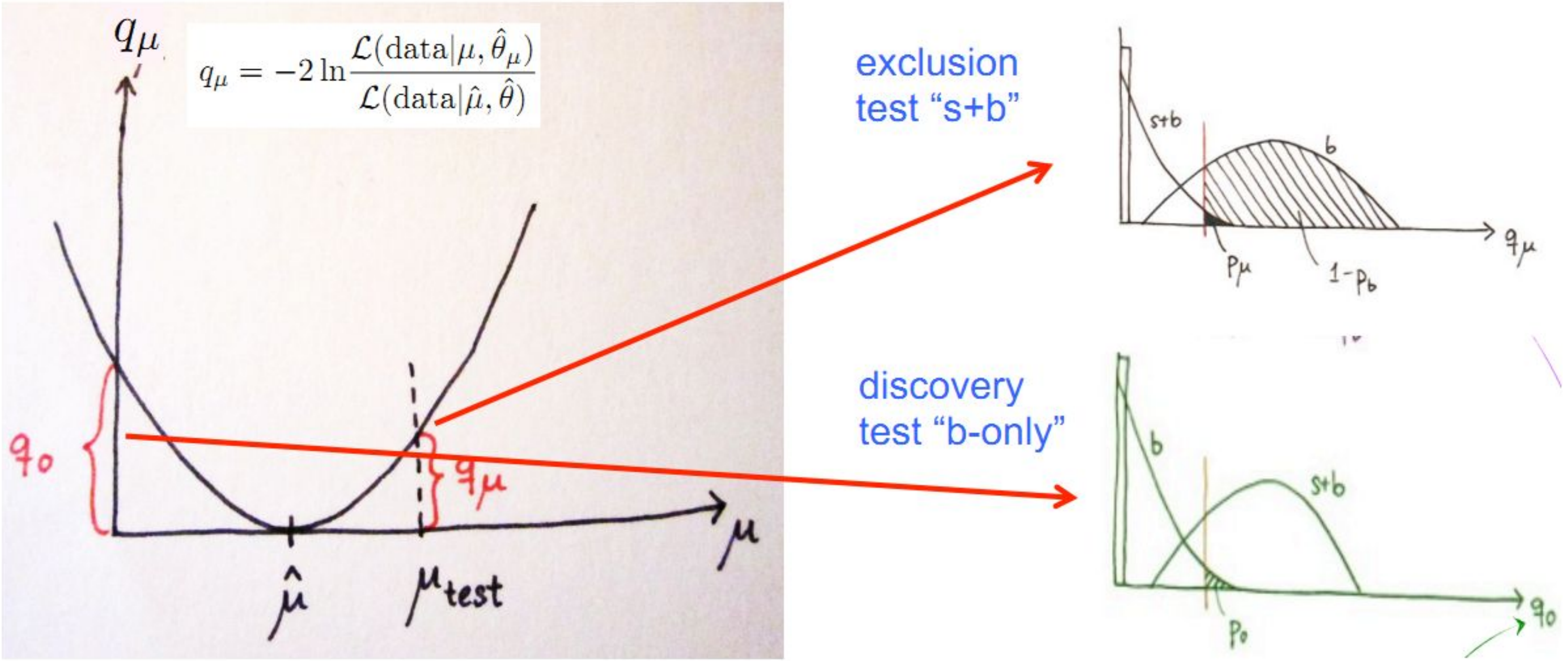
discovery

- if significance is 5 sigma or p-value <2.9e-7,a new particle is discovered---it is to say we have probability of 1-p to reject null hypo.

- test statistic: $q_0 = -2Ln\frac{L(0,\hat{\hat{\theta}})}{L(\hat{\mu},\hat{\theta})}$ if mu^hat>0

- also parameters with one hat are the best fit and paramters with two hat is the best fit with mu=0
- so q_0 is only decided by dataset x and $f(x|\mu,\theta)$

if I am right...



- where is the line s+b in lower right plot from?

- anyway，we can calculate p-value as usual

$$p_{obs} = \int_{q_{obs}}^{\infty} f(q_0 \mid 0) dq_0$$

$$p_{\exp ect} = \int_{q_{\exp ect}}^{\infty} f(q_0 \mid 0) dq_0$$

significance

$$Z = \Phi^{-1}(1-p), \Phi(x) = \int_{-\infty}^{x} Gaussian(y,0,1)dy$$

In large sample limit, q_0~"half chi-2 distribution"(from Cowan's lecture)

$$f(q_0|0) = \frac{1}{2}\delta(q_0) + \frac{1}{2}\frac{1}{\sqrt{2\pi}}\frac{1}{\sqrt{q_0}}e^{-q_0/2}$$

CDF of q_0 $F(q_0|0) = \Phi\left(\sqrt{q_0}\right)$ p-value $p_0 = 1 - F(q_0|0)$

So Z can be calculate $Z = \Phi^{-1}(1 - p_0) = \sqrt{q_0}$

the following three pages are copied from Cowan's lecture.

$s/\sqrt{b}$ for expected discovery significance

For large $s + b$, $n \rightarrow x \sim$ Gaussian(μ,σ) , $\mu = s + b$, $\sigma = \sqrt{(s + b)}$.

For observed value x_{obs}, p-value of $s = 0$ is Prob($x > x_{obs} \mid s = 0$),:

$$p_0 = 1 - \Phi\left(\frac{x_{\rm obs} - b}{\sqrt{b}}\right)$$

Significance for rejecting $s = 0$ is therefore

$$Z_0 = \Phi^{-1}(1 - p_0) = \frac{x_{\rm obs} - b}{\sqrt{b}}$$

Expected (median) significance assuming signal rate s is

$$\mathrm{median}[Z_0|s + b] = \frac{s}{\sqrt{b}}$$

Better approximation for significance

Poisson likelihood for parameter *s* is

$$L(s) = \frac{(s+b)^n}{n!} e^{-(s+b)}$$

For now no nuisance params.

To test for discovery use profile likelihood ratio:

$$q_0 = \begin{cases} -2\ln\lambda(0) & \hat{s} \geq 0\,, \\ 0 & \hat{s} < 0\,. \end{cases} \qquad \lambda(s) = \frac{L(s, \hat{\hat{\boldsymbol{\theta}}}(s))}{L(\hat{s}, \hat{\boldsymbol{\theta}})}$$

So the likelihood ratio statistic for testing $s = 0$ is

$$q_0 = -2\ln\frac{L(0)}{L(\hat{s})} = 2\left(n\ln\frac{n}{b} + b - n\right) \quad \text{for } n > b, \; 0 \text{ otherwise}$$

Approximate Poisson significance (continued)

For sufficiently large $s + b$, (use Wilks' theorem),

$$Z = \sqrt{2\left(n \ln \frac{n}{b} + b - n\right)} \quad \text{for } n > b \text{ and } Z = 0 \text{ otherwise.}$$

To find median[$Z|s$], let $n \rightarrow s + b$ (i.e., the Asimov data set):

$$Z_{\rm A} = \sqrt{2\left((s+b)\ln\left(1+\frac{s}{b}\right) - s\right)}$$

This reduces to $s/\sqrt{b}$ for s << b.

$$Z_A = \sqrt{2\left((s+b)\ln\left(1+\frac{s}{b}\right) - s\right)}$$

- if $s << b, \ln(1+\frac{s}{b}) \sim \frac{s}{b} - \frac{s^2}{2b^2}$

$$(s+b)(\frac{s}{b} - \frac{s^2}{2b^2}) - s = \frac{s^2}{b} - \frac{s^3}{2b^2} + s - \frac{s^2}{2b} - s$$

$$= \frac{s^2}{2b} - \frac{s^3}{2b^2} \sim \frac{s^2}{2b}$$

- Z~ $\frac{s}{\sqrt{b}}$

back up

Test statistic for discovery

Try to reject background-only ($\mu = 0$) hypothesis using

$$q_0 = \begin{cases} -2\ln\lambda(0) & \hat{\mu} \geq 0 \\ 0 & \hat{\mu} < 0 \end{cases}$$

Here data in critical region (high q_0) only when estimated signal strength $\hat{\mu}$ is positive.

Could also want two-sided critical region, e.g., if presence of signal process could lead to suppression (and/or enhancement) in number of events.

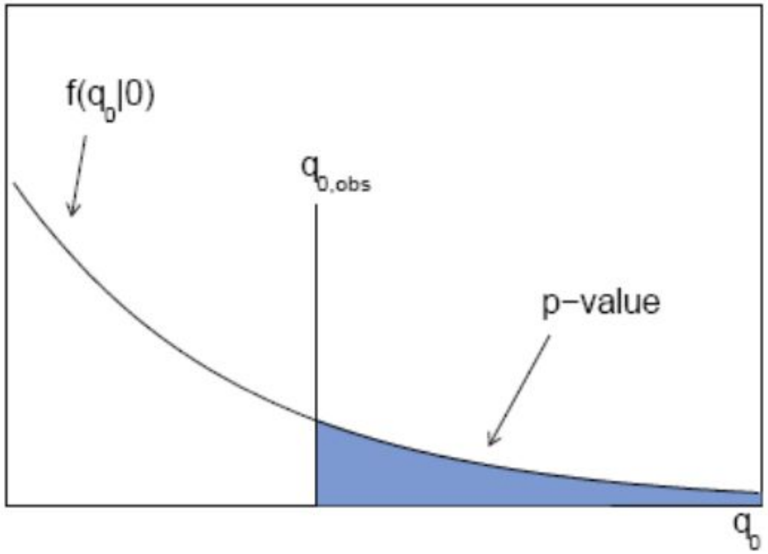
Note that even if physical models have $\mu \geq 0$, we allow $\hat{\mu}$ to be negative. In large sample limit its distribution becomes Gaussian, and this will allow us to write down simple expressions for distributions of our test statistics.

p-value for discovery

Large q_0 means increasing incompatibility between the data and hypothesis, therefore p-value for an observed $q_{0,\mathrm{obs}}$ is

$$p_0 = \int_{q_{0,\mathrm{obs}}}^{\infty} f(q_0|0)\, dq_0$$

will get formula for this later

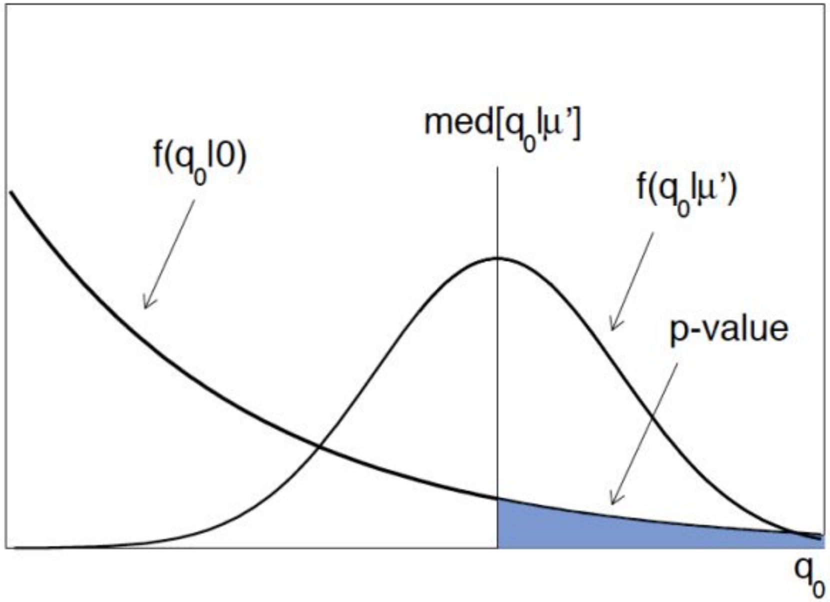


From p-value get equivalent significance,

$$Z = \Phi^{-1}(1 - p)$$

Expected (or median) significance / sensitivity

When planning the experiment, we want to quantify how sensitive we are to a potential discovery, e.g., by given median significance assuming some nonzero strength parameter μ'.



So for p-value, need $f(q_0|0)$, for sensitivity, will need $f(q_0|\mu')$,