A resolution of the puzzle of low V_{us} values from inclusive flavor-breaking sum rule analyses of hadronic τ decay data*

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Abstract: Continuum and lattice methods are used to investigate systematic issues in the sum rule determination of V_{us} using inclusive hadronic τ decay data. Results for V_{us} employing assumptions for $D>4$ OPE contributions used in previous conventional implementations of this approach are shown to display unphysical dependence on the sum rule weight, w, and choice of upper limit, s_0, of the relevant experimental spectral integrals. Continuum and lattice results suggest a new implementation of the sum rule approach with not just $|V_{us}|$, but also $D>4$ effective condensates, fit to data. Lattice results are also shown to provide a quantitative assessment of truncation uncertainties for the slowly converging $D=2$ OPE series. The new sum rule implementation yields $|V_{us}|$ results free of unphysical s_0- and w-dependences and ~ 0.0020 higher than that obtained using the conventional implementation. With preliminary new experimental results for the $K\pi$ branching fraction, the resulting $|V_{us}|$ is in excellent agreement with that based on $K_{\ell 3}$, and compatible within errors with expectations from three-family unitarity.

Key words: V_{us}, lattice, sum rules

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1 Introduction

The conventional τ decay determination of $|V_{us}|$ employs finite-energy sum rules (FESRs) and flavor-breaking (FB) combinations of inclusive hadronic τ decay data [1]. With $\Pi^{(J)}_{V/A;ij}(s)$ the $J=0,1$ components of flavor $ij=ud,us$, vector (V) or axial vector (A) current 2-point functions, $\rho^{(J)}_{V/A;ij}(s)$ the corresponding spectral functions, and $\Delta\Pi_\tau \equiv \left[\Pi^{(0+1)}_{V+A;ud} - \Pi^{(0+1)}_{V+A;us}\right]$, the FESR relation

$$\int_0^{s_0} w(s)\Delta\rho_\tau(s)\,ds = -\frac{1}{2\pi i}\oint_{|s|=s_0} w(s)\Delta\Pi_\tau(s)\,ds\ , \quad (1)$$

is valid for any s_0 and any analytic $w(s)$. The spectral function of $\Delta\Pi_\tau$, $\Delta\rho_\tau$, is experimentally accessible in terms of the differential distribution, $dR_{V/A;ij}/ds$, of $R_{V/A;ij} \equiv \Gamma[\tau^- \to \nu_\tau\,\mathrm{hadrons}_{V/A;ij}(\gamma)]/\Gamma[\tau^- \to \nu_\tau e^-\bar{\nu}_e(\gamma)]$. Explicitly [2]

$$\frac{dR_{V/A;ij}}{ds} = c^{EW}_\tau |V_{ij}|^2 \left[w_\tau(s)\rho^{(0+1)}_{V/A;ij}(y_\tau) - w_L(y_\tau)\rho^{(0)}_{V/A;ij}(s)\right] \quad (2)$$

with $y_\tau = s/m_\tau^2$, $w_\tau(y) = (1-y)^2(1+2y)$, $w_L(y) = 2y(1-y)^2$, c^{EW}_τ a known constant, and V_{ij} the flavor ij CKM matrix element. The RHS of Eq. (1) is treated using the OPE.

One uses the $J=0+1$ FESR Eq. (1), rather than the analogue involving the spectral function combination in Eq. (2), because of the very bad behavior of the integrated $J=0$, $D=2$ OPE series [3]. $\rho^{(0+1)}_{V/A;ud,us}(s)$ is obtained after subtracting phenomenologically determined $J=0$ contributions from $dR_{V/A;ud;us}/ds$. This subtraction is dominated by the accurately known, non-chirally-suppressed π and K pole terms. Continuum $\rho^{(0)}_{V/A;ud}$ contributions are $\propto (m_d \mp m_u)^2$ and numerically negligible. Small, but not totally negligible, $(m_s \mp m_u)^2$-suppressed continuum $\rho^{(0)}_{V/A;us}$ contributions are determined using highly constrained dispersive and sum rule methods [4, 5]. With $|V_{ud}|$ fixed [6], $\Delta\rho_\tau(s)$ is determined by experimental data and $|V_{us}|$. $|V_{us}|$ is then obtained using the OPE on the RHS and data on the LHS of Eq. (1).

From the distribution $dR^{(0+1)}_{V+A;ud,us}/ds$, obtained after subtracting $J=0$ contributions, re-weighted $J=0+1$ versions, $R^w_{V+A;ij}(s_0) \equiv \int_0^{s_0} ds\,\frac{w(s)}{w_\tau(s)}\,\frac{dR^{(0+1)}_{V+A;ij}(s)}{ds}$, of $R_{V+A;ud,us}$, may be constructed for any w and $s_0 \le m_\tau^2$. With $\delta R^{w,OPE}_{V+A}(s_0)$ the OPE representation of

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$\delta R^w_{V+A}(s_0) \equiv \frac{R^w_{V+A;ud}(s_0)}{|V_{ud}|^2} - \frac{R^w_{V+A;us}(s_0)}{|V_{us}|^2}$, one then has

$$|V_{us}| = \sqrt{R^w_{V+A;us}(s_0) / \left[\frac{R^w_{V+A;ud}(s_0)}{|V_{ud}|^2} - \delta R^{w,OPE}_{V+A}(s_0)\right]} \; . \tag{3}$$

The resulting $|V_{us}|$ should be independent of s_0 and the choice of weight, w, provided all experimental data, and any assumptions employed in evaluating $\delta R^{w,OPE}_{V+A}(s_0)$, are reliable. Since integrated $D = 2k+2$ OPE contributions scale as $1/s_0^k$, problems with assumptions about higher D non-perturbative contributions, e.g., will produce an unphysical s_0-dependence in $|V_{us}|$.

The conventional implementation of Eq. (3) [1] employs $w = w_\tau$ and $s_0 = m_\tau^2$. With this choice, the spectral integrals $R^{w_\tau}_{V+A;ud,us}(m_\tau^2)$ are determinable from inclusive non-strange and strange hadronic τ branching fractions, but assumptions about higher dimension $D = 6, 8$ OPE contributions, in priniciple present for a degree 3 weight like w_τ, are unavoidable. Using a single w and single s_0 precludes subjecting these assumptions to w- and s_0-independence tests. It is a long-standing puzzle that this implementation produces inclusive τ $|V_{us}|$ determinations $> 3\sigma$ below 3-family-unitarity expectations (the most recent version, $|V_{us}| = 0.2176(21)$ [7], e.g., lies 3.6σ below the current unitarity expectation, $|V_{us}| = 0.2258(9)$ [6]). Tests of the conventional implementation, however, show sizeable s_0- and w-dependence [8] (see also, e.g., the left panel, and solid lines in the right panel, of Fig. 1), indicating the existence of systematic problems in the conventional implementation. The dashed lines in the right panel show the results of the alternate implementation discussed below.

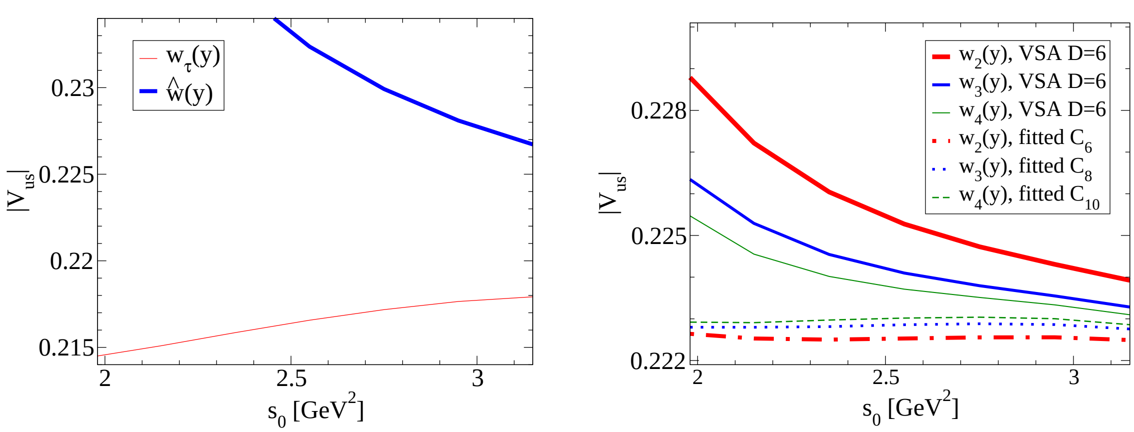

Fig. 1. Left panel: $|V_{us}|$ from the w_τ and $\hat{w}$ FESRs with standard [1] OPE treatment (including CIPT for the $D = 2$ series). Right panel: Comparison of conventional implementation results with those obtained using central fitted $C_{6,8,10}$ values and the FOPT $D = 2$ prescription favored by lattice results, for the weights $w_{2,3,4}$ defined in the text.

Two obvious theoretical systematic issues exist which might account for the observed w- and s_0-instabilities. The first concerns the treatment of $D = 6$, 8 OPE contributions. Both the conventional implementation and generalized versions just mentioned [8], estimate $D = 6$ contributions using the vacuum saturation approximation (VSA) and neglect $D = 8$ contributions. The VSA $D = 6$ estimate is very small due to significant cancellations, both in the individual ud and us V+A sums and in the subsequent FB difference of these sums. With sizeable channel-dependent VSA breaking observed in the flavor ud V and A channels [9], such strong cancellations make the VSA estimate potentially quite unreliable. The second possibility concerns the slow convergence of the $D = 2$ OPE series for $\Delta\Pi_\tau$. With $\bar{a} = \alpha_s(Q^2)/\pi$, and $m_s(Q^2)$, $\alpha_s(Q^2)$ the running strange quark mass and coupling in the $\overline{MS}$ scheme, one has, to four loops [10] (neglecting $O(m^2_{u,d}/m^2_s)$ corrections)

$$[\Delta\Pi_\tau(Q^2)]^{OPE}_{D=2} = \frac{3}{2\pi^2}\frac{m_s(Q^2)}{Q^2}\left[1 + \frac{7}{3}\bar{a} + 19.93\bar{a}^2 + 208.75\bar{a}^3 + \cdots\right] \; . \tag{4}$$

Since $\bar{a}(m_\tau^2) \simeq 0.1$, convergence at the spacelike point on $|s| = s_0$ is marginal at best, raising questions concerning the choice of truncation order and truncation error estimates for the corresponding integrated series. The $D = 2$ convergence/truncation issue is also evident in the significant difference (increasing from ~ 0.0010 to ~ 0.0020 between 3- and 5-loop truncation order) in $|V_{us}|$ results obtained using alternate (fixed-order (FOPT) and contour-improved (CIPT)) prescriptions (prescriptions differing only by terms beyond the common truncation order) for the truncated integrated $D=2$ series [8].

In what follows, we first investigate the treatment of the $D = 2$ OPE series using lattice data for $\Delta\Pi_\tau$, then test the $D = 6, 8$ assumptions of the conventional implementation by comparing FESR results for a judiciously chosen pair of weights, $w_\tau(y)$ and $\hat{w}(y) = (1-y)^3$, $y = s/s_0$. Results obtained employing an alternate implementation of the FB FESR approach suggested by these investigations are then presented.

2 Lattice and continuum investigations of the OPE representation of $\Delta\Pi_\tau$

Data for $\Delta\Pi_\tau(Q^2)$ can be generated over a wide range of Euclidean Q^2 using the lattice. The (tight) cylinder cut which must be applied to avoid lattice artifacts at higher Q^2 has been determined, for the ensemble employed here, in a recent analysis aimed at using lattice current-current two-point function data to determine α_s [11]. We first consider data at high enough Q^2 that $[\Delta\Pi_\tau]_{OPE}$ will be safely dominated by the leading $D = 2$ and 4 contributions. The latter are determined by light and strange quark masses and condensates and hence known. We use FLAG results for physical quark masses [12] and GMOR for the light condensate. $\langle\bar{s}s\rangle$ then follows from $\langle\bar{s}s\rangle/\langle\bar{u}u\rangle$ (the HPQCD physical-m_q version of this ratio [13] is easily translated to the m_q of the ensemble employed using NLO ChPT [14]). We then consider various combinations of truncation order and log-resummation schemes for the $D=2$ OPE series, investigating whether a choice exists which produces a good match between the resulting $D = 2+4$ OPE sum and the lattice data in the high-Q^2 region.

For this high-Q^2 study, we employ the RBC/UKQCD $n_f = 2+1$, $32^3 \times 64$, $1/a = 2.38$ GeV, $m_\pi \sim 300$ MeV domain wall fermion ensemble [15]. We find that 3-loop $D = 2$ truncation with fixed-scale choice (the analogue of the FOPT FESR prescription) provides an excellent OPE-lattice match over a wide range of Q^2, extending from near ~ 10 GeV^2 down to just above ~ 4 GeV^2. The comparison is shown, for fixed scale choice $\mu^2 = 4\ GeV^2$, in the left panel of Fig. 2, where, for ease of display, we plot results for the product $Q^2\,\Delta\Pi_\tau(Q^2)$ (this removes a factor of $1/Q^2$ present in the $D = 2$ OPE contributions). The right panel of Fig. 2 shows the analogous comparison for the alternate local-scale ($\mu^2 = Q^2$) choice (analogous to the CIPT FESR prescription). It is clear that the Q^2 dependence of the lattice data prefers the fixed-scale treatment of the $D=2$ series.

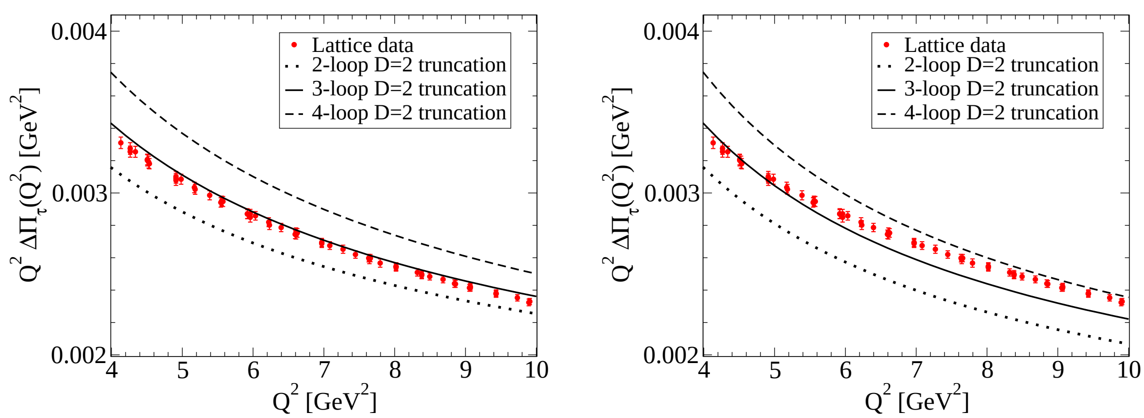

Fig. 2. Comparison of lattice data and OPE $D=2+4$ expectations for $Q^2\,\Delta\Pi_\tau(Q^2)$, for various truncation orders and either the fixed-scale treatment (left panel) or local-scale treatment (right panel) of the $D=2$ series.

The lattice data also provides us with the possibility of investigating the reliability of conventional methods for estimating the theoretical error to be associated with the truncated OPE. This is of particular relevance given the very slow convergence of the $D = 2$ series, which might raise doubts about the suitability of such conventional estimates in the case of the $D = 2$ truncation uncertainty. Fig. 3 shows the $D = 2+4$ OPE error band obtained using the 3-loop-truncated, fixed-scale $D=2$ OPE treatment and such conventional OPE error estimates, taking into account uncertainties in the input OPE parameters and using the magnitude of the last term kept to estimate the $D = 2$ series truncation uncertainty. One sees that, despite the very slow convergence of the $D=2$ series, the resulting conventionally determined error turns out to be extremely conservative.

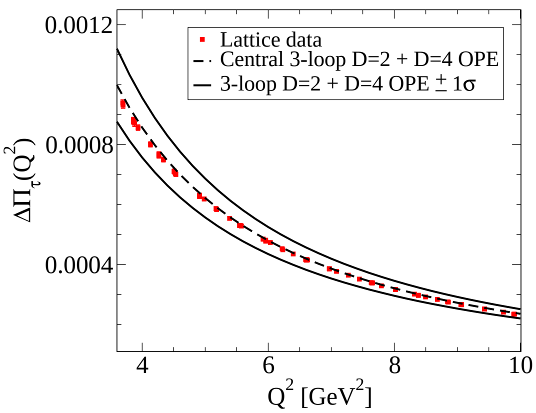

Fig. 3. Lattice data and the $D=2+4$ OPE sum, with conventional OPE error estimates, for the 3-loop-truncated, fixed-scale $D=2$ treatment

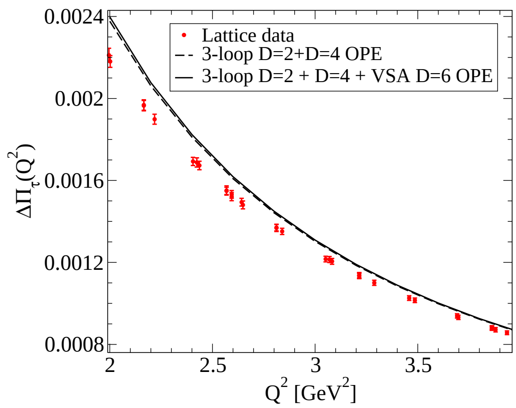

Fig. 4. Comparison of lower-Q^2 lattice data with fixed-scale $D=2$ OPE based expectations, one employing just the sum of $D=2$ and 4 contributions, the other supplementing this with the estimated $D=6$ contribution obtained using the VSA.

We now switch our attention to lower Q^2, the goal being to use lattice data to test the assumptions about $D>4$ contributions underlying the standard implementation, namely that $D>6$ contributions are safely negligible and $D=6$ contributions can be reasonably well approximated using the VSA. Fig. 4 shows the comparison of lattice data for $\Delta\Pi_\tau(Q^2)$ in the region below $\sim 4\,GeV^2$ with two versions of the truncated OPE. The dashed line represents the 3-loop-truncated, fixed-scale $D=2+4$ OPE sum discussed above, which provides an excellent match to the lattice data at higher scales, while the solid line shows the result of supplementing the $D=2+4$ sum with the VSA estimate for the $D=6$ contribution. The results show clear evidence for the onset of $D>4$ contributions below $Q^2\sim 4\ GeV^2$ significantly larger than those obtained using the VSA estimate for $D=6$ and neglecting $D>6$ contributions. The VSA $D=6$ estimate is not only far too small in magnitude to bring the low-Q^2 truncated OPE into agreement with the lattice data but, in fact, moves the truncated OPE sum slightly in the wrong direction. Unfortunately, the Euclidean lattice data provides no means of selectively isolating contributions of different $D>4$ in this lower Q^2 region. Further investigation of the higher D question thus requires continuum FESR methods.

Our continuum FESR studies employ the $D=2,4$ OPE treatment favored by lattice data, detailed above. Spectral integral input is as follows: $\pi_{\mu 2}$, $K_{\mu 2}$ and Standard Model expectations for the π and K pole contributions, recent ALEPH data for the continuum ud V+A distribution [16], BaBar [17] and Belle [18] results for the $K^-\pi^0$ and $\bar{K}^0\pi^-$ distributions, BaBar results [19] for the $K^-\pi^+\pi^-$ distribution, Belle results [20] for the $\bar{K}^0\pi^-\pi^0$ distribution and 1999 ALEPH results [21] for the combined distribution of those strange modes not remeasured by the B-factory experiments.

The BaBar and Belle exclusive mode distributions are unit-normalized and must have their overall scales fixed using experimental branching fractions. In the results quoted below, HFAG strange exclusive mode branching fractions have been used, with the exception of the $K^-\pi^0$ mode, for which the updated version from the recent BaBar Adametz thesis [22] has been employed. A corresponding (very) small rescaling is applied to the continuum ud V+A distribution to restore unitarity.

Neglecting α_s-suppressed logarithmic corrections, $D>4$ OPE contributions to $\Delta\Pi_\tau(Q^2)$ can be written $\sum_{D>4} C_D/Q^D$ with C_D an effective dimension D condensate. The degree 3 weights $w_\tau(y)=1-3y^2+2y^3$ and $\hat{w}(y)=1-3y+3y^2-y^3$ generate integrated OPE contributions up to $D=8$ only. The integrated $D=6,8$ results,

$$-\frac{3C_6}{s_0^2}-\frac{2C_8}{s_0^3}\ \text{ for } w_\tau \qquad \text{and} \qquad \frac{3C_6}{s_0^2}+\frac{C_8}{s_0^3}\ \text{ for } \hat{w}, \quad (5)$$

have $D=6$ contributions identical in magnitude but opposite in sign, and a $\hat{w}$ $D=8$ contribution similarly opposite in sign but half in magnitude that of w_τ. Were the assumptions of the conventional implementation to be correct, with $D=6,8$ contributions numerically negligible in the w_τ FESR, this will necessarily also be the case for the $\hat{w}$ FESR. The two FESRs should then produce results for $|V_{us}|$ which not only agree, but are both s_0-independent. In contrast, if the $D=6$ and/or 8 contributions to the w_τ FESR are not, in fact, negligible, the results for $|V_{us}|$ from the two FESRs, obtained assuming they are, should show s_0-instabilities of opposite sign, decreasing in magnitude with increasing s_0 for both. The left panel of Fig. 1 shows the latter scenario to be the one actually realized. The sizeable s_0- and weight-choice dependences demonstrate unambiguously the breakdown of the assumptions underlying the conventional implementation. The 3σ low $|V_{us}|$ results obtained employing them are thus afflicted with significant previously unquantified systematic uncertainties.

3 An alternate implementation of the FB FESR approach

With previously employed approaches to estimating $D>4$ effective OPE condensates shown to be untenable, our only option is to fit these condensates to data. This can be done only by exploiting the fact that integrated OPE contributions of different D scale differently with s_0, and hence requires working with FESRs involving a range of values of the variable s_0. This precludes determining the required spectral integrals solely in terms of hadronic τ decay branching fractions.

Considering FESRs based on different weights provides further tests of possible theoretical systematics. To suppress duality violating contributions, we restrict our attention to weights having at least a double zero at $s=s_0$. The weights $w_N(y)=1-\frac{N}{N-1}y+\frac{1}{N-1}y^N$, $N\geq 2$ [23] are particularly convenient in this regard since the corresponding integrated OPE involves a single $D>4$ contribution (with $D=2N+2$). With $D=2+4$ OPE contributions under control (as discussed above) this leaves $|V_{us}|$ and the effective condensate C_{2N+2} as the only parameters to be determined. These are obtained through a fit to the set of w_N-weighted spectral integrals in the chosen s_0 fit window. Further tests of the analysis are provided by verifying (i) that the $|V_{us}|$ obtained from the different w_N FESRs are in good agreement and (ii) that the fitted C_D are physically plausible, in the sense of showing FB cancellation relative to the results of Ref. [9] for the corresponding effective flavor ud channel condensates. We have analyzed the w_N FESRs for $N=2,3,4$ and verified that the results pass these self-consistency tests.

In the right panel of Fig. 1 we display, as dashed/dotted lines, the results which follow from taking as input the central value for the condensate, C_{2N+2}, obtained from the w_N FESR analysis, and solving Eq. (3) for $|V_{us}|$ as a function of s_0. The results are displayed for each of the w_2, w_3 and w_4 FESR cases. The results make clear (i) the underlying excellent match between the fitted OPE and spectral integral sets, (ii) the excellent agreement between results for $|V_{us}|$ obtained from the different w_N FESR analyses, and (iii) the dramatic decrease in the s_0- and weight-dependence of the results for $|V_{us}|$ produced by using $D>4$ OPE effective condensates fit to data in place of those based on the assumptions of the conventional implementation. One also sees that, as expected, the fitted $|V_{us}|$ lie between the s_0-unstable results produced by the conventional implementation of the w_τ and $\hat{w}$ FESRs, and are ~ 0.0020 higher than the results of the conventional w_τ implementation.

Table 1. Error budgets for the w_2, w_3 and w_4 determinations of $|V_{us}|$, using the 3-loop-truncated, fixed-scale treatment of the $D=2$ OPE series

Error source	$\delta\|V_{us}\|$ (w_2 FESR)	$\delta\|V_{us}\|$ (w_3 FESR)	$\delta\|V_{us}\|$ (w_4 FESR)
$\delta\alpha_s$	0.00001	0.00004	0.00004
$\delta m_s(2\ GeV)$	0.00017	0.00019	0.00019
$\delta\langle m_s\bar{s}s\rangle$	0.00035	0.00035	0.00035
$\delta(long\ corr)$	0.00009	0.00009	0.00009
Experimental (ud)	0.00027	0.00028	0.00028
Experimental (us)	0.00226	0.00227	0.00227

In Table 1, we give the error budgets for the $|V_{us}|$ determinations based on the w_2, w_3 and w_4 FESRs, using the 3-loop, fixed-scale treatment of the $D=2$ OPE series favored by lattice data. The errors in the first half are those associated with input uncertainties on the theory side, with "long corr" labelling those associated with the sum rule/dispersive determinations of the small, doubly-chirally-suppressed continuum us V and A channel $J=0$ subtractions. Combining these errors in quadrature yields a total theory error of 0.0004 for each of the three cases. The errors listed in the second half of the table are those induced by the errors and covariances of the flavor ud and us V+A distributions. The experimental and total errors are both strongly dominated by the uncertainty on the us V+A spectral integrals.

The excellent s_0-stability and agreement between the results from the different w_N FESRs allows us to arrive at a final result for $|V_{us}|$ obtained by performing a combined fit to the w_2, w_3 and w_4 FESRs. We find

$$|V_{us}|=0.2228(23)_{exp}(5)_{th}\ . \qquad (6)$$

This is in excellent agreement with the results, $0.2235(4)_{exp}(9)_{th}$ and $0.2231(4)_{exp}(7)_{th}$, obtained using the 2014 FlaviaNet experimental $K_{\ell 3}$ update [25] and most recent $n_f = 2+1$ [26] and $n_f = 2+1+1$ [27] lattice results for $f_+(0)$. It is also compatible within errors with (i) the results, $0.2251(3)_{exp}(9)_{th}$ and $0.02250(3)_{exp}(7)_{th}$ obtained using the 2014 update for the experimental ratio $\Gamma[K_{\mu 2}]/\Gamma[\pi_{\mu 2}]$ [25] and the most recent $n_f = 2+1$ [28] and $n_f = 2+1+1$ [29] lattice determinations of f_K/f_π and (ii) the expectations of 3-family unitarity.*

It is worth noting that, among the methods mentioned above, the one having the smallest theoretical error is, in fact, the FB FESR determination. This error is, moreover, as we have seen, a very conservative one. At present the experimental error on the FB FESR determination (resulting almost entirely from uncertainties in the us exclusive mode distributions) is larger than those of the competing methods. We note, however, that the us spectral integral error is currently dominated by the uncertainty on the branching fraction normalizations for the exclusive strange modes, and hence systematically improvable through improvements in these branching fraction values in the near future. In the longer term, it will be important to complete the analysis of the exclusive mode us distributions not yet remeasured by the B-factory experiments and to finalize the analyses of the covariances of those unit normalized exclusive distributions which are not yet complete at present.

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*An analogous analysis with $K\pi$ normalization not employing the $B[K^-\pi^0\nu_\tau]$ update of Ref. [22] yields $|V_{us}| = 0.2200(23)_{exp}(5)_{th}$. Of the 0.0024 difference between this result and the conventional implementation result [7] noted above, 0.0005 results from the use of $K_{\mu 2}$ for the K pole contribution; the remainder is due to presence of the $D = 6, 8$ contributions not correctly accounted for by the assumptions of the conventional implementation. Note that the normalization of the two-mode $K\pi$ sum produced by the $B[K^-\pi^0\nu_\tau]$ update is in good agreement with the results of the dispersive study of $K\pi$ detailed in Ref. [24].