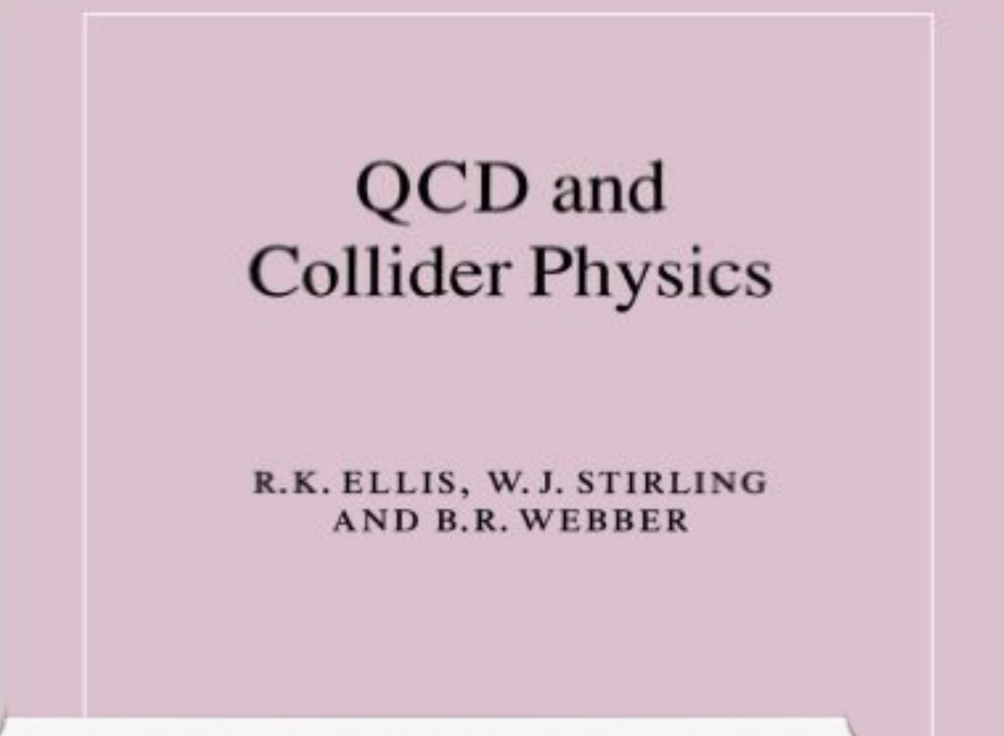
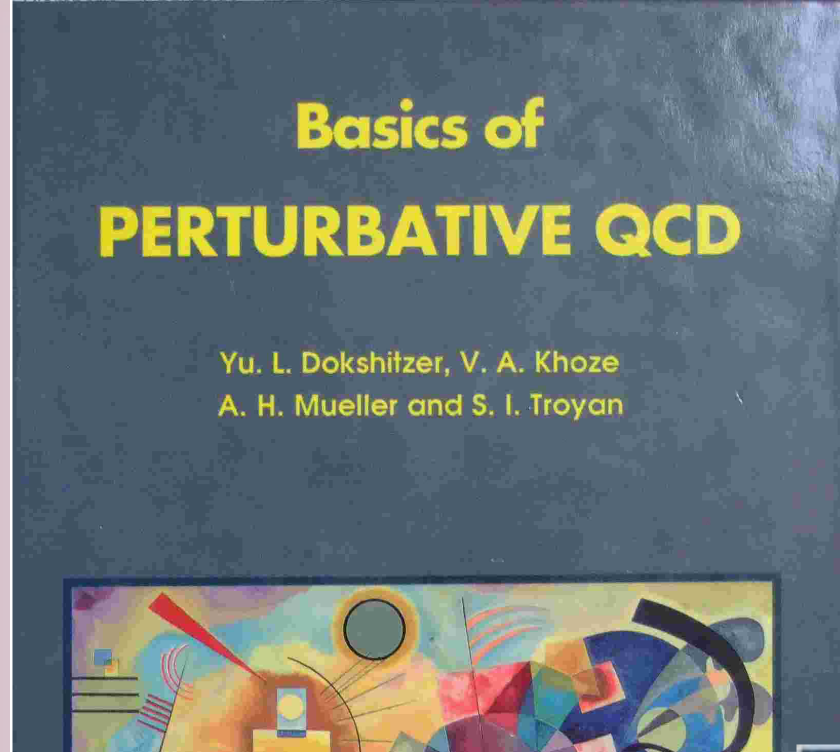
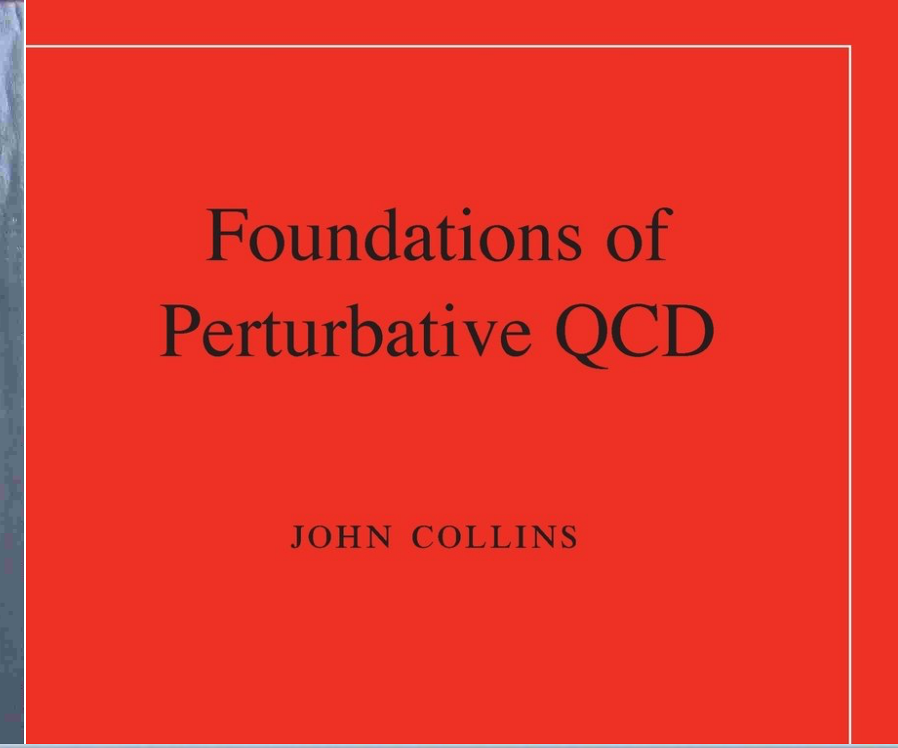
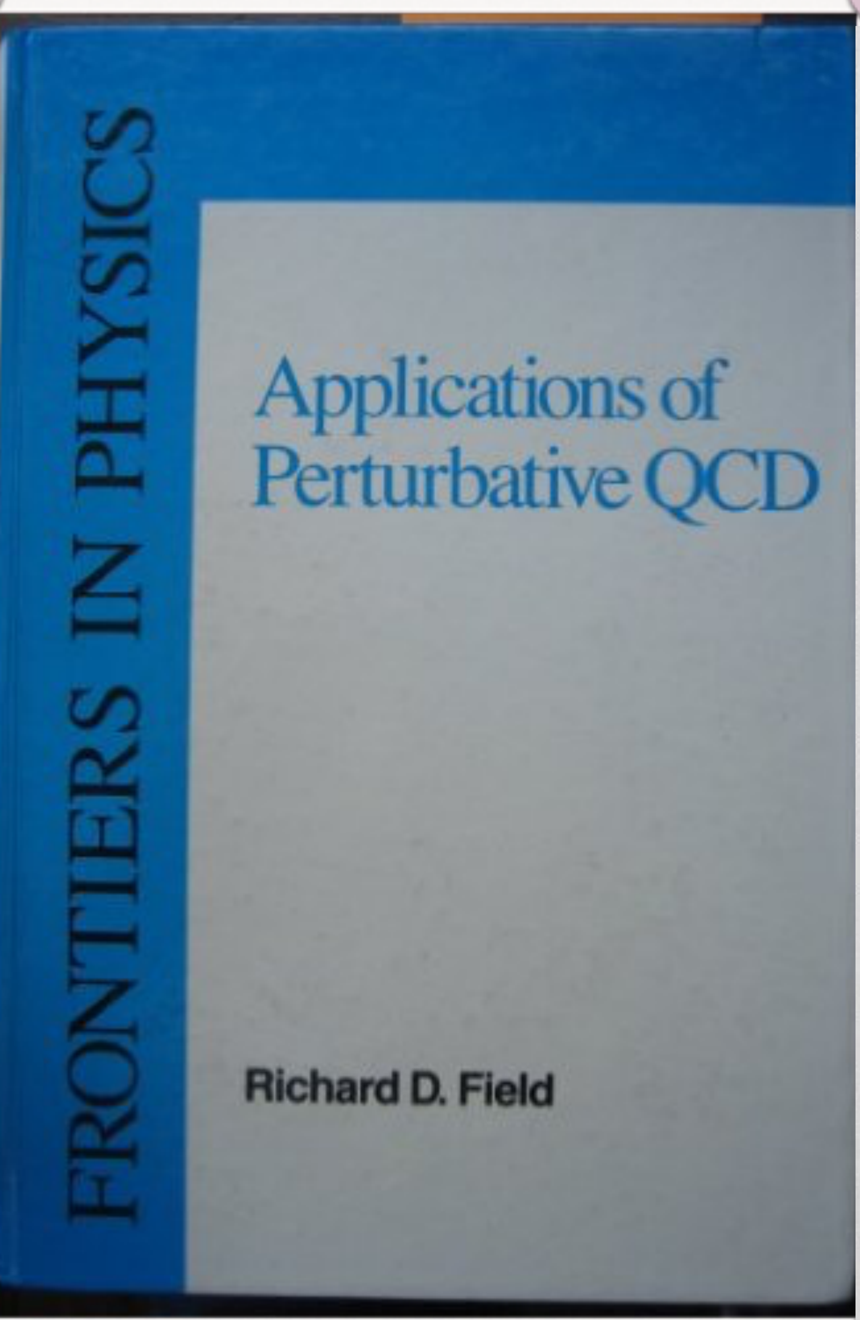
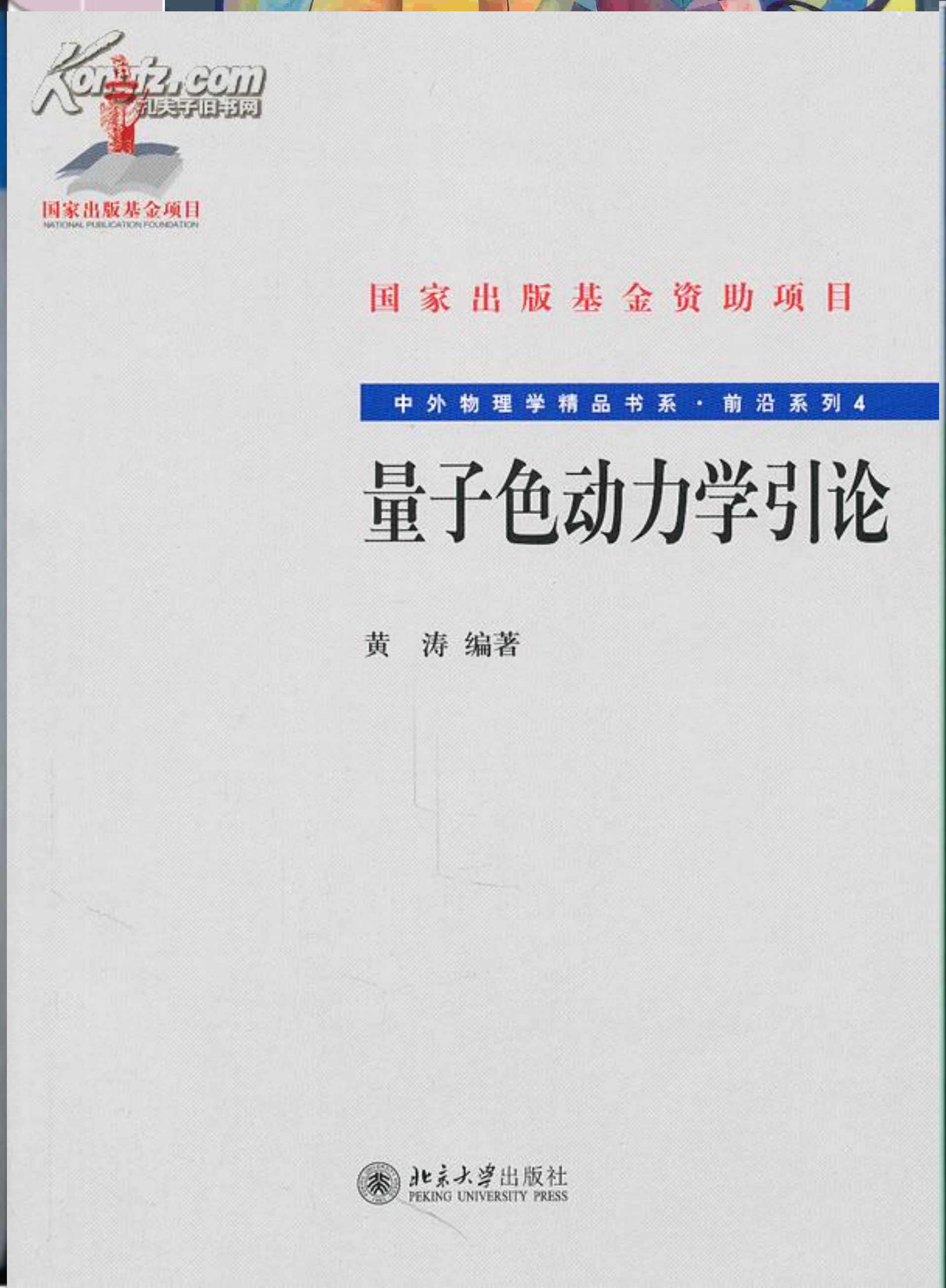
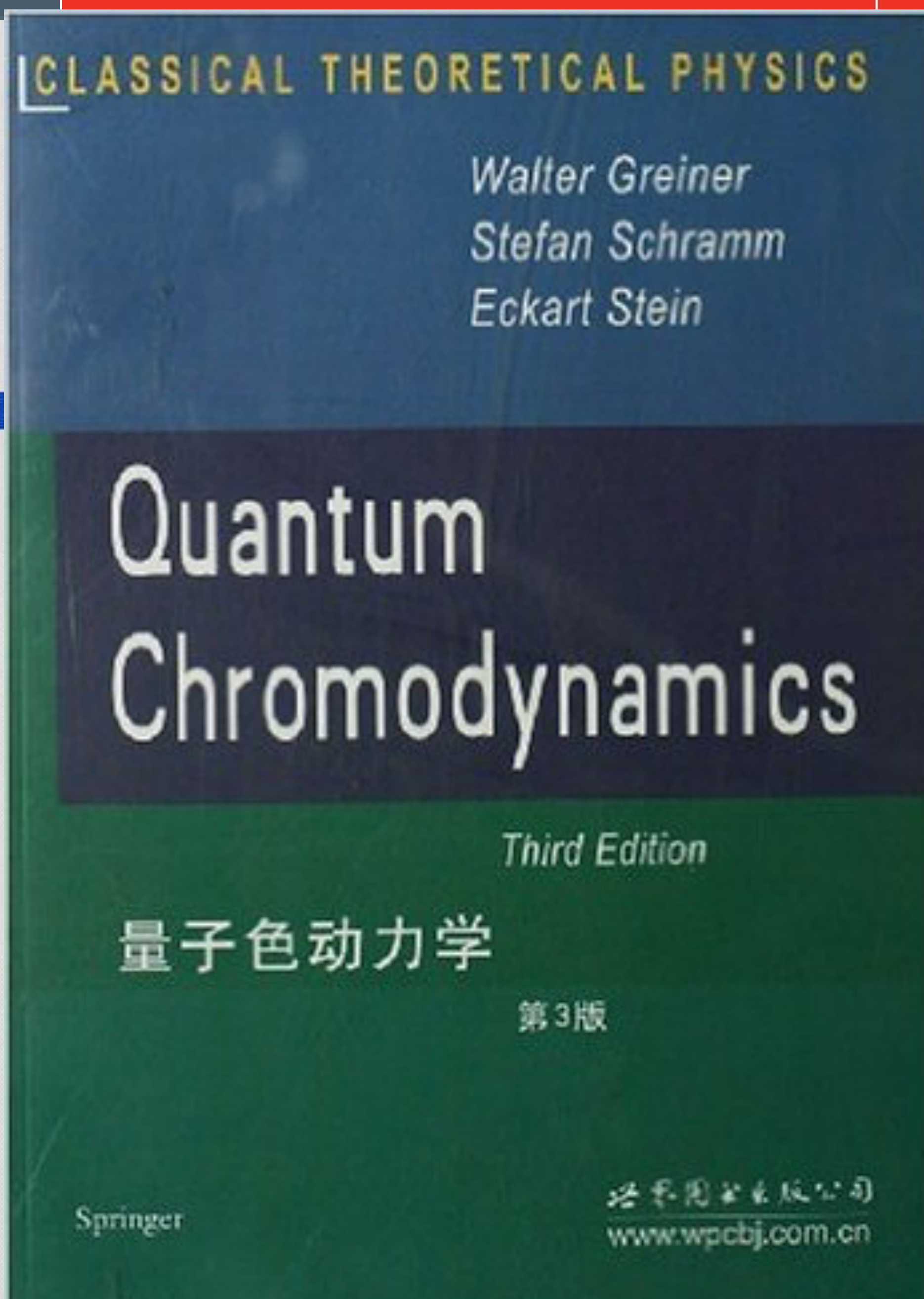

QCD and
Collider Physics
R.K. ELLIS, W.J. STIRLING
AND B.R. WEBBER


Basics of
PERTURBATIVE QCD
Yu. L. Dokshitzer, V. A. Khoze
A. H. Mueller and S. I. Troyan


Foundations of
Perturbative QCD
JOHN COLLINS


FRONTIERS IN PHYSICS
Applications of
Perturbative QCD
Richard D. Field


国家出版基金项目
国家出版基金资助项目
中外物理学精品书系 · 前沿系列 4
量子色动力学引论
黄 涛 编著
北京大学出版社
PEKING UNIVERSITY PRESS


CLASSICAL THEORETICAL PHYSICS
Walter Greiner
Stefan Schramm
Eckart Stein
Quantum
Chromodynamics
Third Edition
量子色动力学
第3版
Springer
www.wpcbj.com.cn

A NLO Calculation of pQCD

Qing-Hong Cao
Peking University

What I learned from my thesis advisor:

Stop arguing!
Just do it!
And you will understand after.

A NLO Calculation of pQCD:
Total Cross Section of $P\overline{P} \to W^+ + X$

C.–P. Yuan

Michigan State University

CCAST Summer School, June 2004

A good reference:
“**Handbook of pQCD**”, quite old but free

http://www.phy.pku.edu.cn/~qhcao/Class/Yuan/index_2014.html

Prof. C.-P. Yuan's Lecture on 'Perturbative QCD'

(May 31 - July 4, 2014, Peking University)

美国密西根州立大学的袁简鹏教授访问北大理论物理所，讲授《微扰QCD理论和标准模型重整化》的系列讲座，侧重于因子化定理和微扰QCD理论应用等基础知识。主要内容包括：

1）以Drell-Yan过程为例讲解次领头阶QCD辐射修正的详细计算，

2）以DIS过程为例讲解部分子分布函数，

3）Eikonal近似和跑动耦合常数。

整个课程中袁简鹏老师会在课前将详细大纲讲义发给同学。2014年7月8日-17日即将在北大举办第21届CTEQ暑期学校。袁老师这一系列讲座将帮助参加暑期学校的同学预习QCD相关的理论内容。袁老师办公室在物理学院南楼431，有问题同学可以在午后到袁老师办公室详谈。

上课地点：物理学院南楼408室

5月31日（周六，上午9-12）：第1课 量子色动力学的拉格朗日量：01，02

6月01日（周日，上午9-12）：第2课 部分子模型、因子化定理和次领头阶的微扰QCD计算（综述）

6月07日（周六，上午9-12）：第3课 次领头阶的微扰QCD计算：W玻色子产生过程的虚修正 01，02，03，04

6月08日（周日，上午9-12）：第4课 部分子分布函数简介：01

6月09日-6月22日：北京大学考试周

6月29日（周六，上午9-12）：第5课 次领头阶的微扰QCD计算：W玻色子产生过程的实修正

6月30日（周一，上午9-12）：第6课 部分子分布函数进阶和DGLAP部分子衍化 01，02

7月02日（周三，上午9-12）：第7课 Eikonal近似 01，02，03

7月04日（周五，上午9-12）：第8课 Running Coupling、小测验题目

Outline

1. Parton Model

$\Rightarrow$ Born Cross Section

2. Factorization Theorem

$\Rightarrow$ How to organize a NLO calculation of pQCD

3. Feynman rules and Feynman diagrams

$\Rightarrow$ "Cut diagram" notation

4. Immediate Problems (Singularities)

$\Rightarrow$ Dimensional Regularization

5. Virtual Corrections

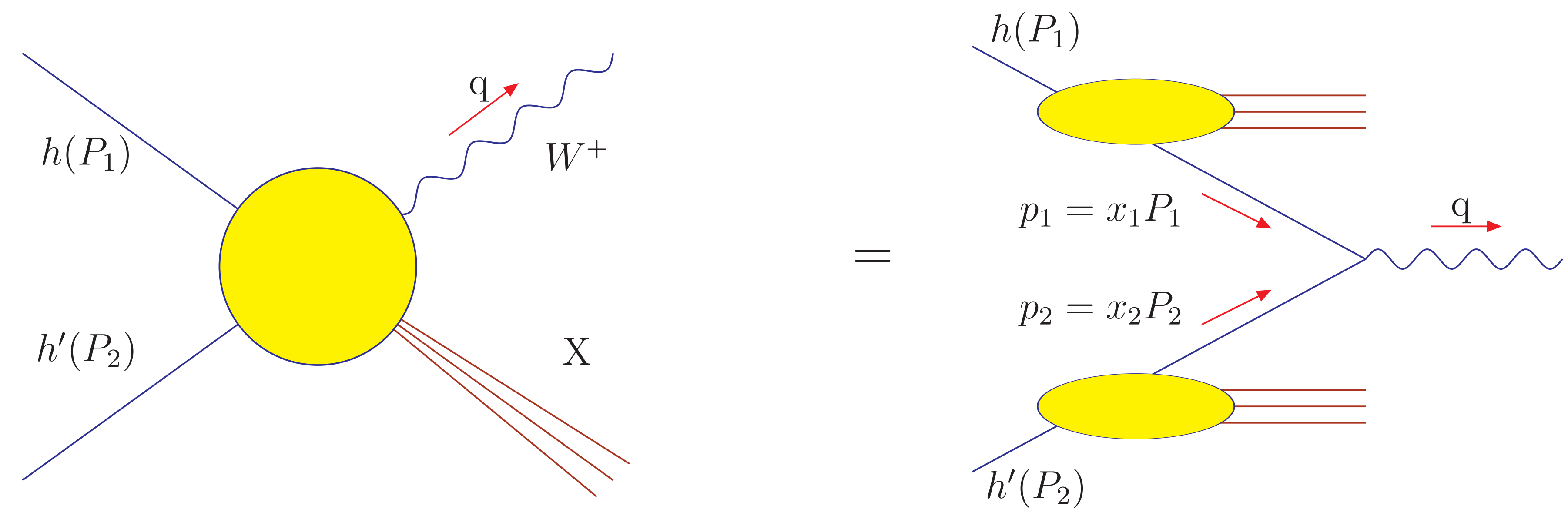
6. Real Emission Contribuation

7. Perturbative Parton Distribution Functions

8. Summary of NLO [$O(\alpha_s)$] Corrections

W-Boson Production at Hadron Colliders

$\sigma_{hh' \to W^+ X} =$



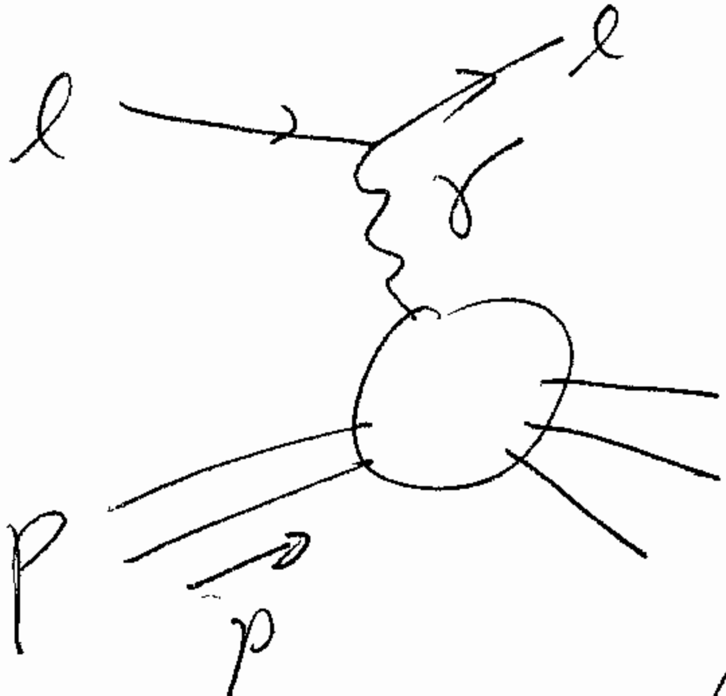
$$\sigma_{hh' \to W^+ X} = \sum_{f, f' = q, \bar{q}} \int_0^1 dx_1 dx_2 \left\{ \phi_{f/h}(x_1)\, \hat{\sigma}_{f f'} \phi_{\bar{f}'/h'}(x_2) + (x_1 \leftrightarrow x_2) \right\}$$

Partonic "Born"
Cross Section of $f \overline{f}' \to W^+$

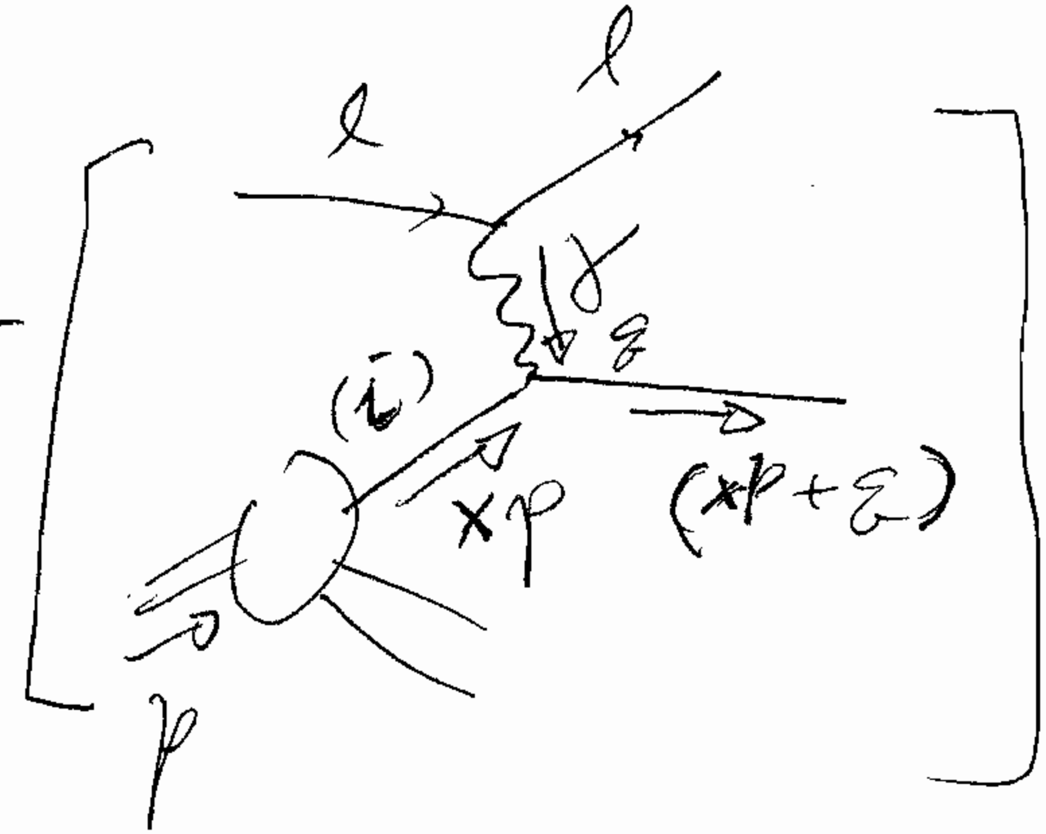
The probablility of finding a "parton" f with fraction x_1 of the hadron h momentum

Parton Model

1. What's parton?



$$= \sum_i \int dx \, e_i^2 \left[\; \right]$$

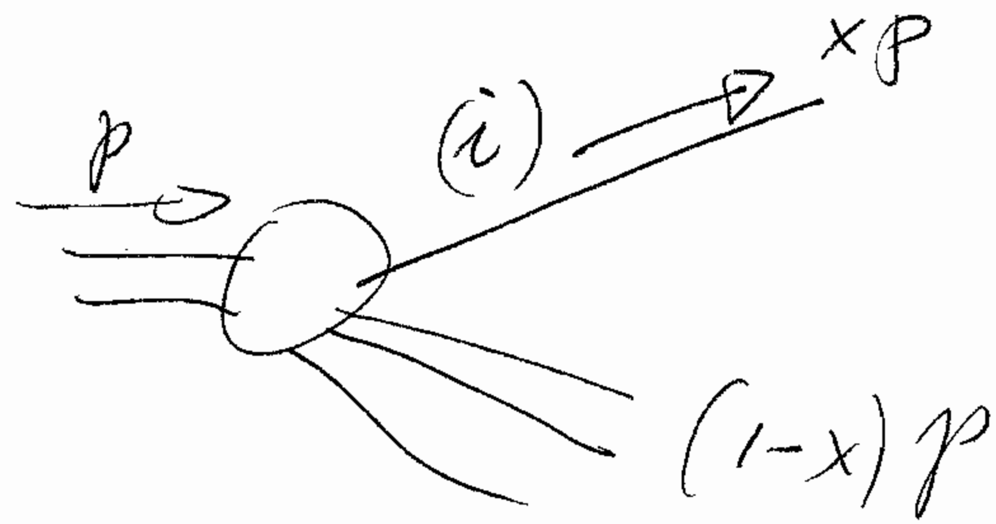


$i = u, d, s, c, b, g$
$\bar{u}, \bar{d}, \bar{s}, \bar{c}, \bar{b}$

x: fraction of the proton momentum (p) carried by the parton (i)

2) Parton distribution function (PDF)

$f_i(x) \equiv$



↳ The probability for the struck parton (i) to carry a fraction x of the proton momentum.

$$\sum_i \int dx \cdot \left[x f_i(x) \right] = 1 \qquad \text{(Total Momentum of proton.)}$$

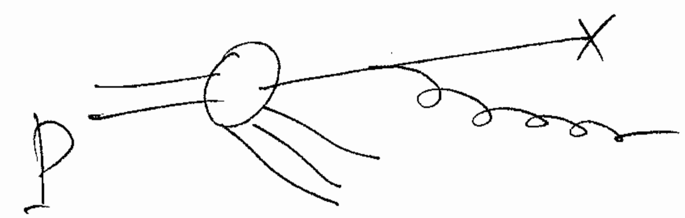
↳ $i = u, \bar{u}, d, \bar{d}, s, \bar{s}, b, \bar{b}, g$

3) In parton model,

$$F_2(x) = \sum_i (e_i)^2 \, x \cdot f_i(x)$$ (1969)

which is independent of Q ⇒ Bjorken scaling
($x = \frac{1}{\omega}$)

<u>Note</u>: In QCD parton model, the violation of Bjorken scaling is logarithmic, and is a signature of gluon emission.

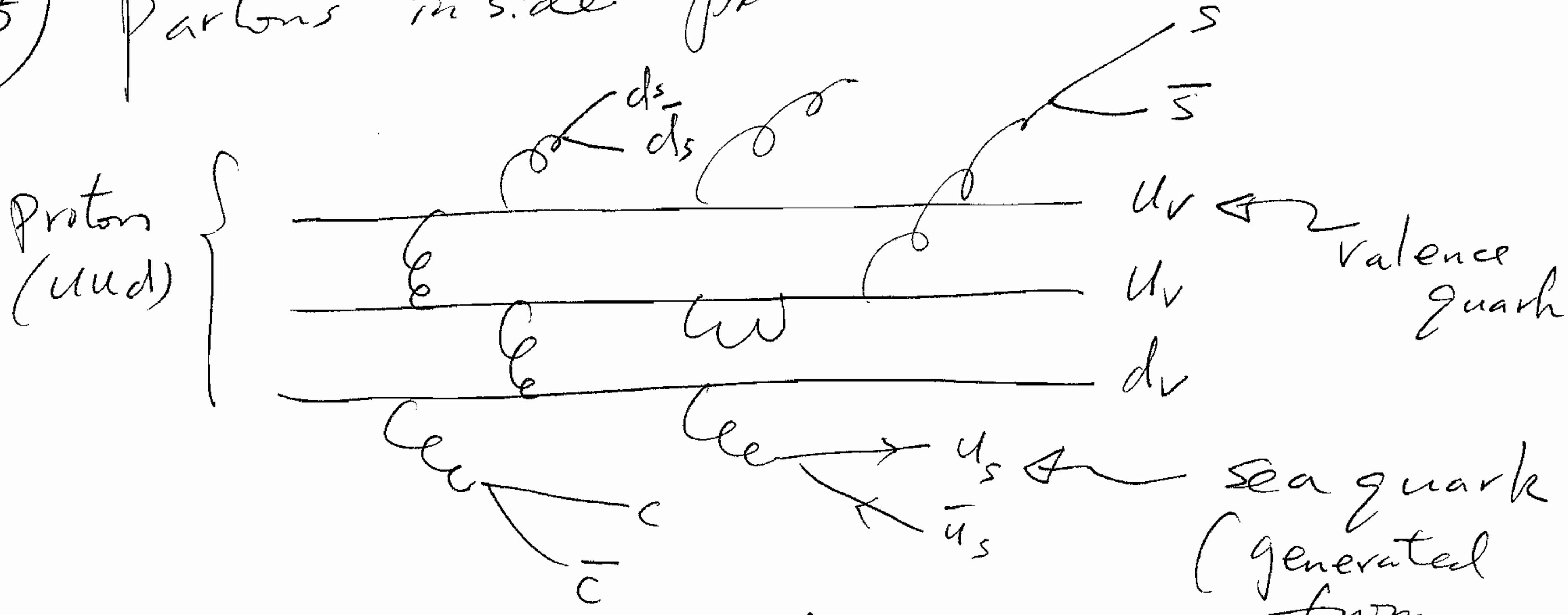


4) $$F_1(x) = \frac{1}{2x} F_2(x)$$ (Callan-Gross relation)

proves quark is spin-$\frac{1}{2}$ parton.

[For a spin-0 parton, $F_1(x) = 0$.]

5) Partons inside proton.



$$u(x) = u_v(x) + u_s(x)$$
$$d(x) = d_v(x)\, d_s(x)$$

$$\left(\begin{array}{l} \bar{u}_s = u_s \\ \bar{d}_s = d_s \end{array} \right)$$

$s, \bar{s}, c, \bar{c}, b, \bar{b}$ are "assumed" to be all coming from gluon splitting (i.e. sea quarks).

$$\int_0^1 dx \left[u(x) - \bar{u}(x) \right] = \int_0^1 dx \left[u_v(x) \right] = 2$$
$$\int dx \left[d(x) - \bar{d}(x) \right] = \int_0^1 dx \left[d_v(x) \right] = 1$$

P=(uud)

6) Where are the gluons?

For $Q \gtrsim 1.5$ GeV, Charm quark is a "heavy" quark.

Hence, the total momentum of proton gives

$$P = \int_0^1 dx \cdot (xp)\left[u + \bar{u} + d + \bar{d} + s + \bar{s} + g\right]$$

$$\Rightarrow \quad \int_0^1 dx \cdot x \cdot \left(u + \bar{u} + d + \bar{d} + s + \bar{s}\right) = 1 - \varepsilon_g,$$

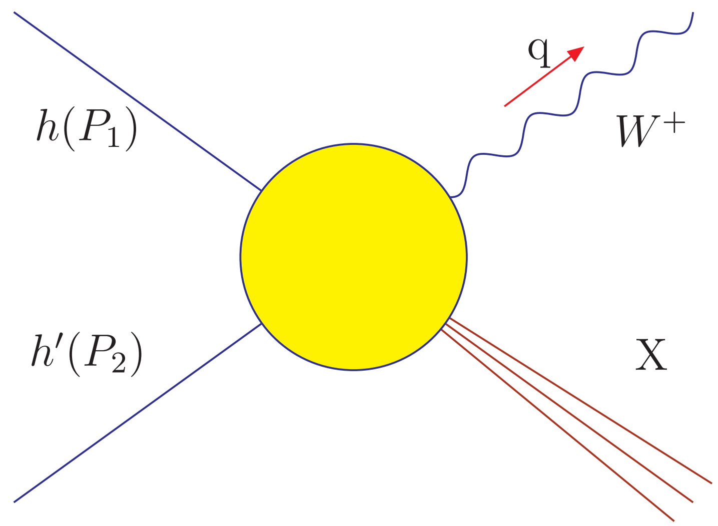
$$\varepsilon_g \equiv \frac{P_g}{P} \equiv \frac{\text{total amount of momentum carried by gluon}}{\text{the momentum of proton}}$$

Experimental data tells us $\varepsilon_g \simeq 0.46$

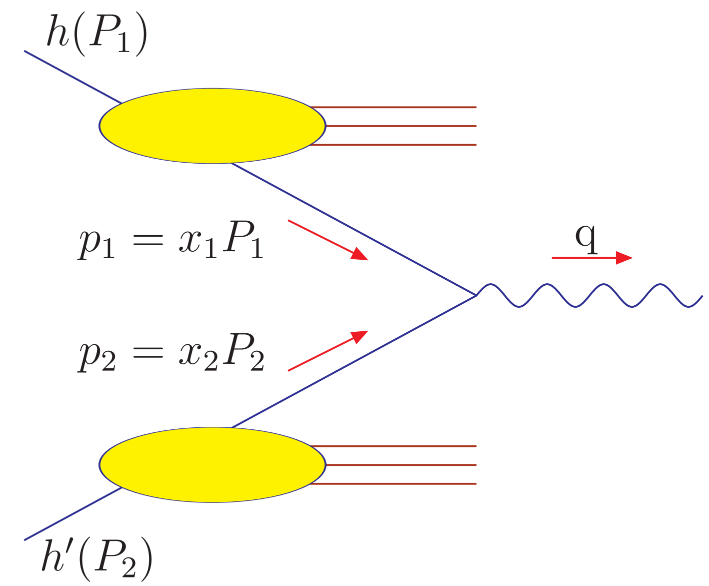
Hence, gluon carries about 50% of the proton momentum.

Naive Parton Model

$$\sigma_{hh'\to W^+X} =$$



$$=$$



$$\sigma_{hh'\to W^+X} = \sum_{f,f'=q,\bar{q}} \int_0^1 dx_1 dx_2 \left\{ \phi_{f/h}(x_1)\, \hat{\sigma}_{ff'} \phi_{\bar{f}'/h'}(x_2) + (x_1 \leftrightarrow x_2) \right\}$$

Partonic "Born"
Cross Section of $f\overline{f}' \to W^+$

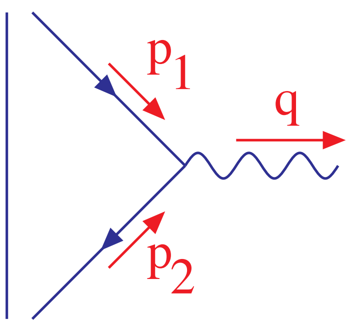
The probablility of finding a "parton" f with fraction x_1 of the hadron h momentum

Born Cross Section

$$\hat{\sigma}_{q\bar{q}'} = \frac{1}{2\hat{s}} \int \frac{d^3q}{(2\pi)^3\, 2q_0} (2\pi)^4\, \delta^4\,(p_1 + p_2 - q) \cdot \overline{|\mathcal{M}|^2}$$

where

$$\overline{|\mathcal{M}|^2} = \underbrace{\left(\frac{1}{3} \cdot \frac{1}{3}\right)\left(\frac{1}{2} \cdot \frac{1}{2}\right)}_{\text{average color and spin}} \sum_{\substack{spin \\ color}} \left| \quad \right|^2$$

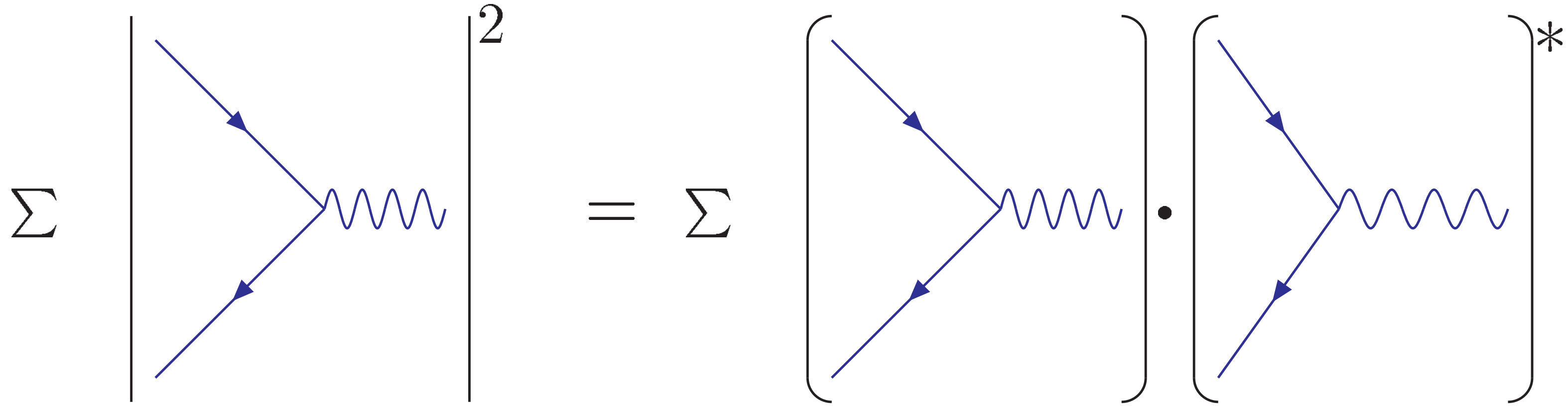


average color and spin

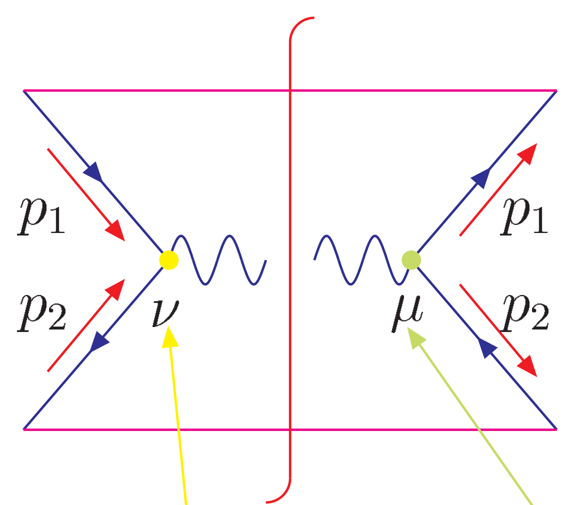
$$\left[\text{ Or, } \; -i\mathcal{M} = \bar{v}\,(p_2)\,\frac{ig_w}{\sqrt{2}}\gamma_\mu \frac{1}{2}\,(1-\gamma_5)\,u\,(p_1) \right]$$

Born Cross Section

"**cut-diagram**" notation



$=$



$\frac{ig_w}{\sqrt{2}}\gamma_\nu P_L$ $\quad$ $-\frac{ig_w}{\sqrt{2}}\gamma_\mu P_L$ $\quad$ $P_L \equiv \frac{1}{2}(1-\gamma_5)$

$$= \left(\frac{g_W}{\sqrt{2}}\right)^2 Tr\left[\not{p}_1\gamma_\mu P_L \; \not{p}_2\gamma_\nu P_L\right] \cdot \left(-g^{\mu\nu} + \frac{q^\mu q^\nu}{M^2}\right) \cdot Tr\, I_{3\times 3}$$

Doesn't contribute for $m_q = 0$, due to Ward identity

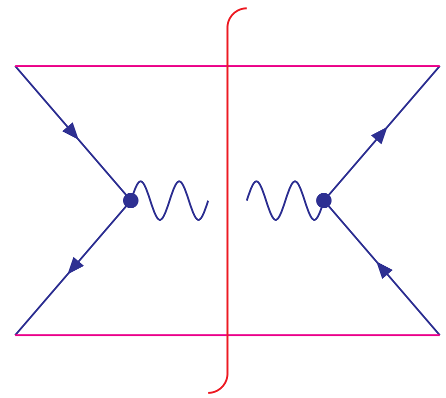
Color

Matrix element square

$$Tr\left[\not{p}_1 \gamma_\mu P_L \not{p}_2 \gamma^\mu P_L\right](-1)$$
$$= Tr\left[\not{p}_1 \gamma_\mu \not{p}_2 \gamma^\mu P_L\right](-1)$$
$$= (-2)\, Tr\left[\not{p}_1 \not{p}_2 P_L\right](-1)$$
$$= (-2) \cdot \frac{1}{2} \cdot 4\,(p_1 \cdot p_2)\,(-1)$$
$$= +2\hat{s}$$

$$P_L P_L = P_L = \frac{1}{2}(1-\gamma_5) \qquad Tr\left[I_{3\times 3}\right] = 3$$
$$\gamma_\mu \not{p}_2 \gamma^\mu = -2 \not{p}_2$$
$$Tr\left(\not{p}_1 \not{p}_2\right) = 4\,(p_1 \cdot p_2)$$
$$Tr\left(\not{p}_1 \not{p}_2 \gamma_5\right) = 0$$

$\left(\hat{s} \equiv (p_1+p_2)^2 = q^2 \text{ and } p_1^2 = p_2^2 = 0\right)$



$$= \left(\frac{g_w}{\sqrt{2}}\right)^2 \cdot (+2\hat{s})\,(3) = 3\, g_w^2\, \hat{s}$$

Phase space integration

$$\int \frac{d^3q}{2q_0}\delta^4(p_1+p_2-q) = \int d^4q\,\delta^4(p_1+p_2-q)\,\delta^+\left(q^2-M^2\right) = \delta\left(q^2-M^2\right)$$

Hard process cross section

$$\hat{\sigma}_{q\bar{q}'} = \frac{1}{2\hat{s}}(2\pi)\cdot\delta\left(\hat{s}-M^2\right)\cdot\left(\frac{1}{3}\right)\left(\frac{1}{2}\cdot\frac{1}{2}\right)\cdot g_w^2\hat{s}$$
$$= \frac{\pi}{12} g_w^2 \delta\left(\hat{s}-M^2\right)$$
$$= \frac{\pi}{12\hat{s}} g_w^2 \delta\left(1-\hat{\tau}\right)$$

$\hat{\tau} = M^2/\hat{s}$, $\hat{s} = x_1 x_2 S$ for
$S = (P_1+P_2)^2$ and $P_1^2 = P_2^2 = 0$

Factorization Theorem

$$\sigma_{hh'} = \sum_{i,j} \int_0^1 dx_1 dx_2 \, \phi_{i/h}\left(x, Q^2\right) H_{ij}\left(\frac{Q^2}{x_1 x_2 S}\right) \phi_{j/h'}\left(x_2, Q^2\right)$$

Nonperturbative, but universal, hence, measurable

IRS, Calculable in pQCD

Procedure:

(1) Compute σ_{kl} in pQCD with k, l partons (not h, h' hadron)

$$\sigma_{kl} = \sum_{i,j} \int_0^1 dx_1 dx_2 \, \phi_{i/k}\left(x_1, Q^2\right) H_{ij}\left(\frac{Q^2}{x_1 x_2 S}\right) \phi_{j/l}\left(x_2, Q^2\right)$$

(2) Compute $\phi_{i/k}, \phi_{j/l}$ in pQCD

(3) Extract H_{ij} in pQCD

H_{ij} IRS $\Rightarrow$ H_{ij} indepent of k, l

$\Rightarrow$ same H_{ij} with $(k \to h, l \to h')$

(4) Use H_{ij} in the above equation with $\phi_{i/h}, \phi_{j/h'}$

Extracting H_{ij} in pQCD

$$\sigma_{kl} = \sum_{i,j} \int_0^1 dx_1 dx_2\, \phi_{i/k}\left(x_1, Q^2\right) H_{ij}\left(\frac{Q^2}{x_1 x_2 S}\right) \phi_{j/l}\left(x_2, Q^2\right)$$

- Expansions in α_s:

$$\sigma_{kl} = \sum_{n=0}^{\infty} \left(\frac{\alpha_s}{\pi}\right)^n \sigma_{kl}^{(n)} \qquad \phi_{i/k}(x) = \delta_{ik}\delta(1-x) + \sum_{n=1}^{\infty} \left(\frac{\alpha_s}{\pi}\right)^n \phi_{i/k}^{(n)}$$

$$H_{ij} = \sum_{n=0}^{\infty} \left(\frac{\alpha_s}{\pi}\right)^n H_{ij}^{(n)}$$

$\uparrow$ $\phi_{i/k}^{(0)}$ ($\alpha_s = 0 \Rightarrow$ Parton k " stays itself ")

$$H_{ij}^{(0)} = \sigma_{ij}^{(0)} = \text{"Born"}$$

suppress "^" from now on

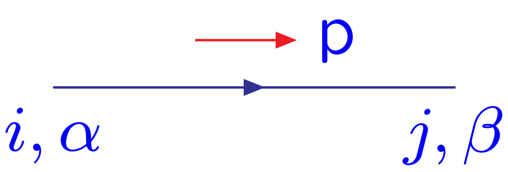
$$H_{ij}^{(1)} = \sigma_{ij}^{(1)} - \left[\, \sigma_{il}^{(0)} \phi_{l/j}^{(1)} + \phi_{k/i}^{(1)} \sigma_{kj}^{(0)} \,\right]$$

Computed from Feynman diagrams (process dependent)

Computed from the definition of perturbative parton distribution function (process independent, scheme dependent)

Feynman Rules

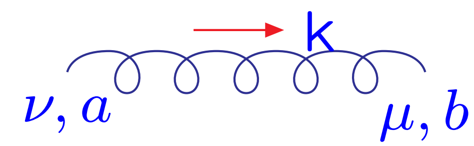
Quark propagator



$$\frac{i(\not p + m)_{\beta\alpha}}{p^2 - m^2 + i\epsilon}\delta_{ij}$$

(i,j=1,2,3)

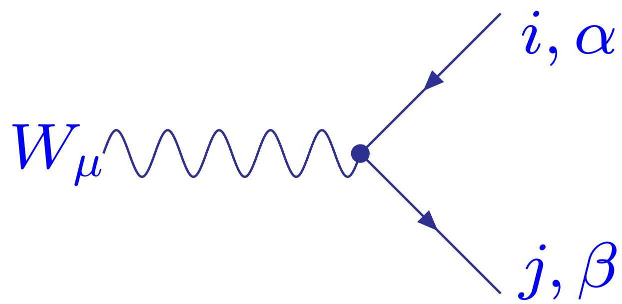
Gluon propagator



$$\frac{i(-g_{\mu\nu})}{k^2 + i\epsilon}\delta_{ab}$$

(a,b=1,2...,8)

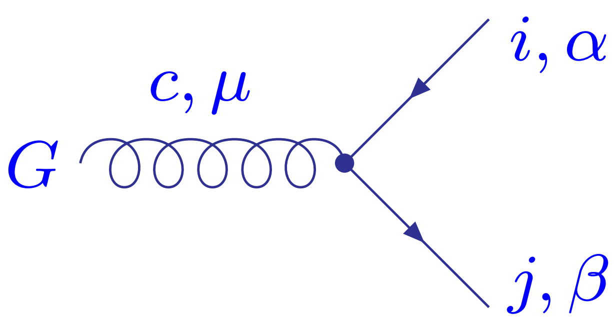
Quark-W vertex



$$i\frac{g_W}{\sqrt{2}}(\gamma_\mu)_{\beta\alpha}\frac{(1-\gamma_5)}{2}\delta_{ij}$$

$g_w = \frac{e}{\sin\theta_w}$, weak coupling

Quark-Gluon vertex

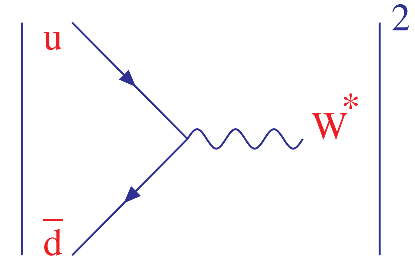


$$-ig\,(t_c)_{ji}\,(\gamma_\mu)_{\beta\alpha}$$

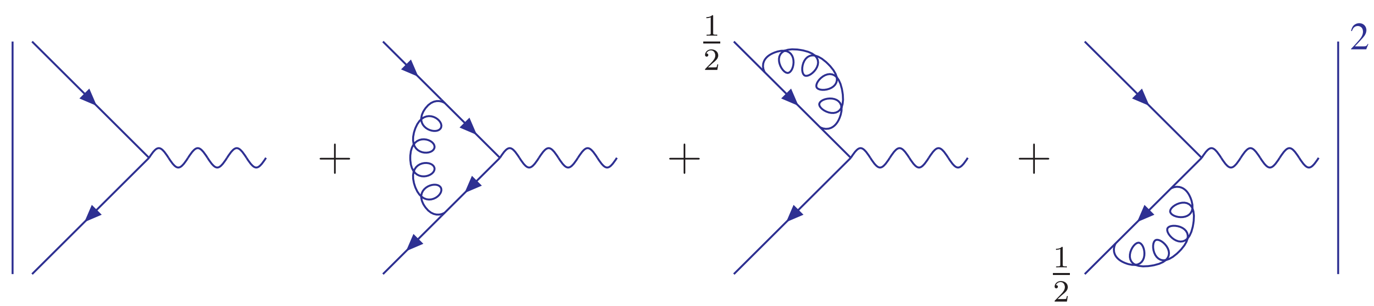
t_c is the $SU(N)_{N\times N}$ generator

Feynman Diagrams

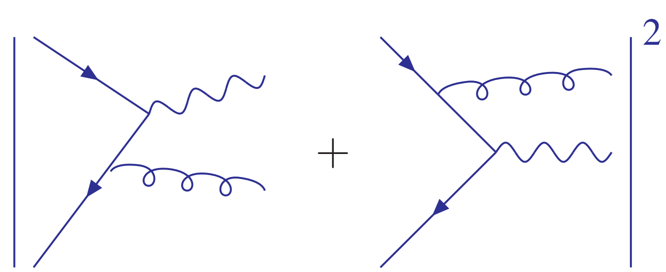
Born level $\alpha_s^{(0)}$ $(q\overline{q'})_{Born}$



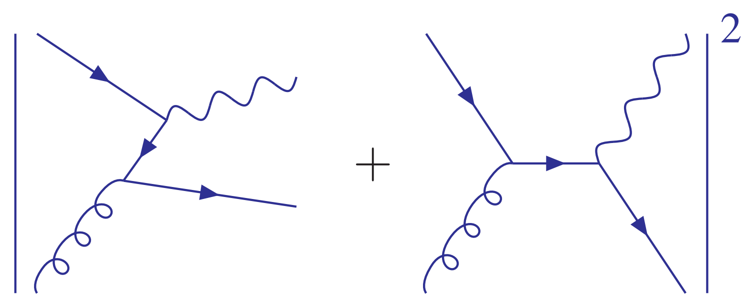
NLO: $(\alpha_s^{(1)})$ virtual corrections $(q\overline{q'})_{virt}$



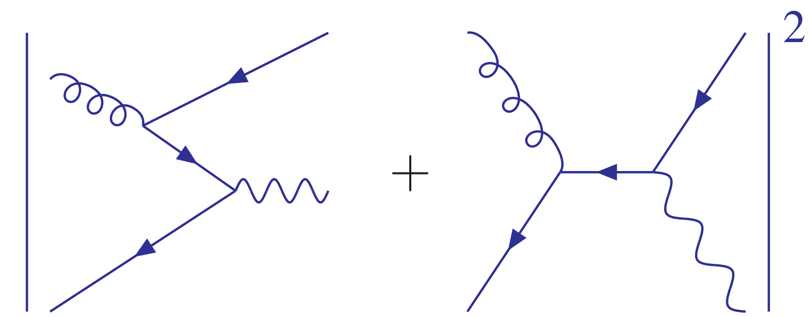
$(\alpha_s^{(1)})$ real emission diagrams $(q\overline{q'})_{real}$



$(\alpha_s^{(1)})$ real emission diagrams $(qG)_{real}$

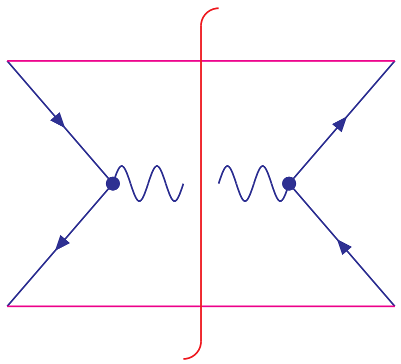


$(\alpha_s^{(1)})$ real emission diagrams $(G\overline{q'})_{real}$

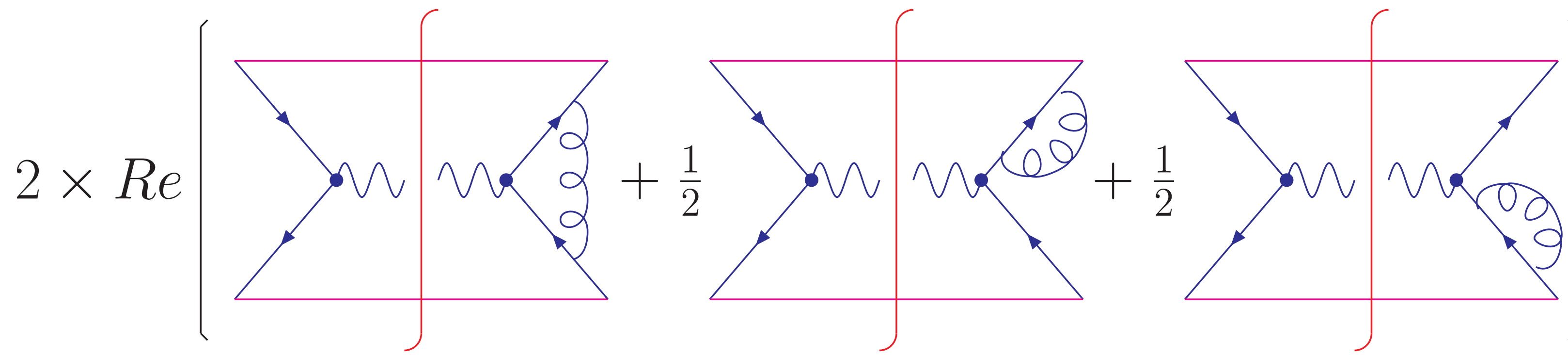


In “cut-diagram” Notation

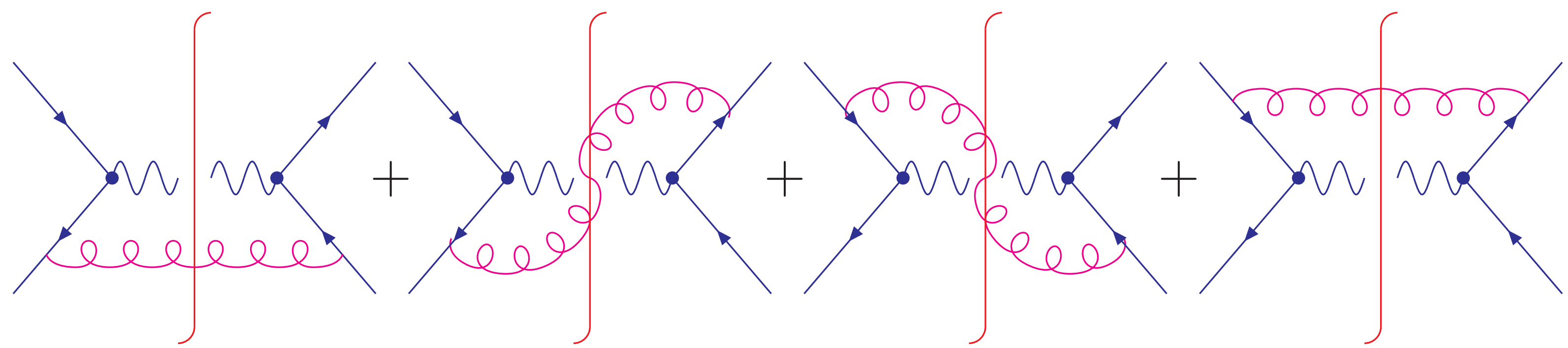
- $(q\overline{q'})_{Born}$



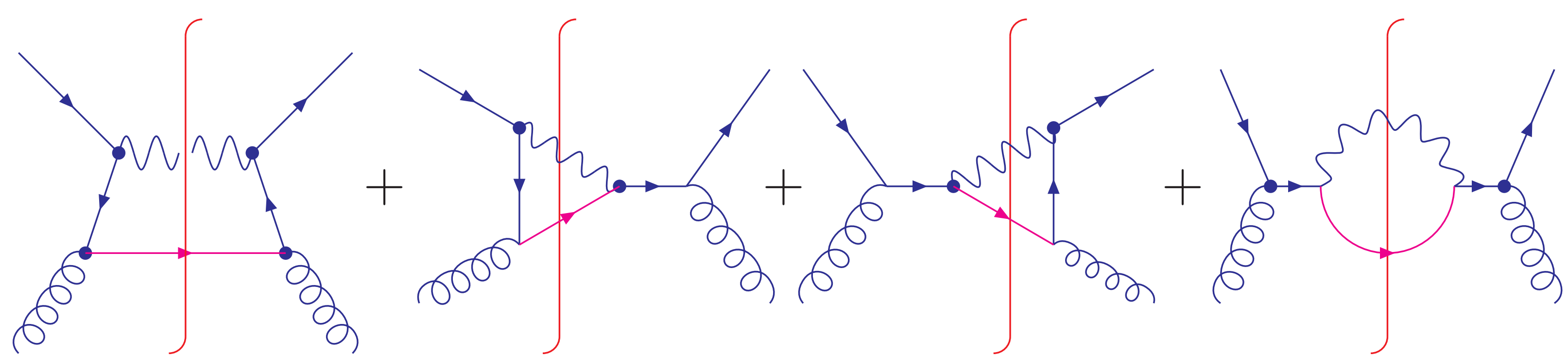
- $(q\overline{q'})_{virt}$



- $(q\overline{q'})_{real}$



- $(qG)_{real}$

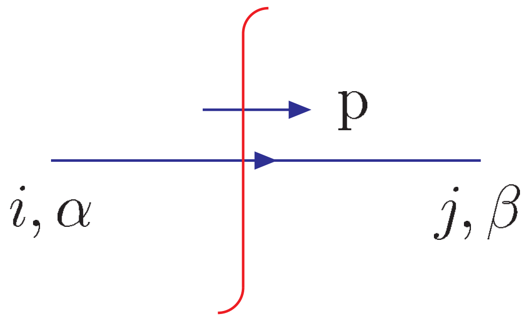


- $(G\overline{q'})_{real}$

Same as $(qG)_{real}$ after replacing q by $\overline{q'}$.

Feynman Rules for Cut-diagrams

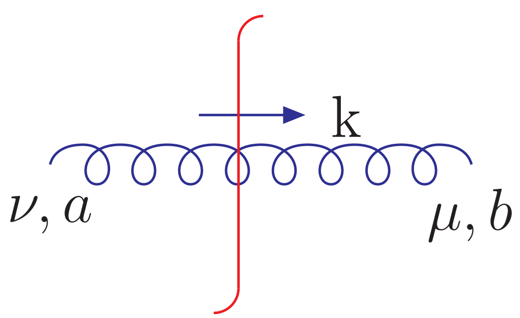
Quark line



$$(2\pi)\delta^+(p^2-m^2)(\not p + m)_{\beta\alpha}\delta_{ij}$$

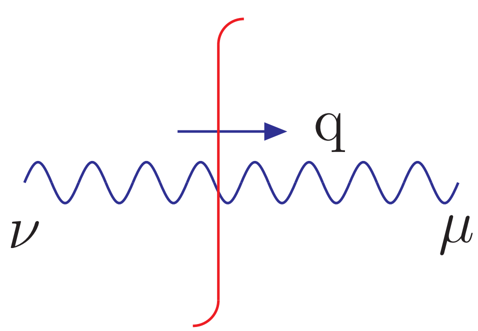
$$\delta(p^2-m^2)\theta(p_0)$$

Gluon line



$$(2\pi)\delta^+(k^2)(-g_{\mu\nu})\delta_{ab}$$

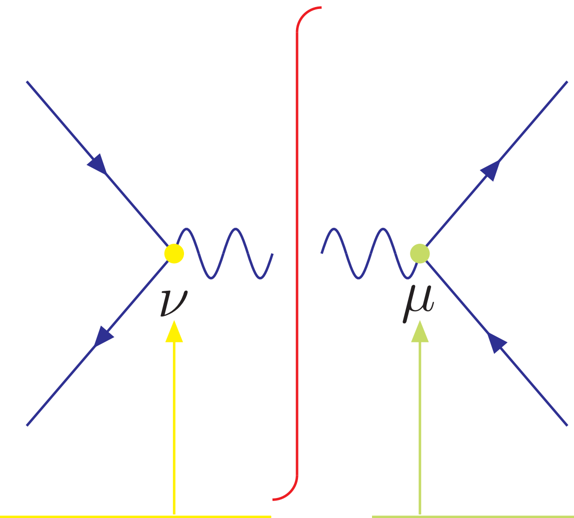
W-boson line



$$(2\pi)\delta^+(q^2-M^2)\left(-g_{\mu\nu}+\frac{q_\mu q_\nu}{M^2}\right)$$

Doesn't contribute for $m_f = 0$ because of Ward identity

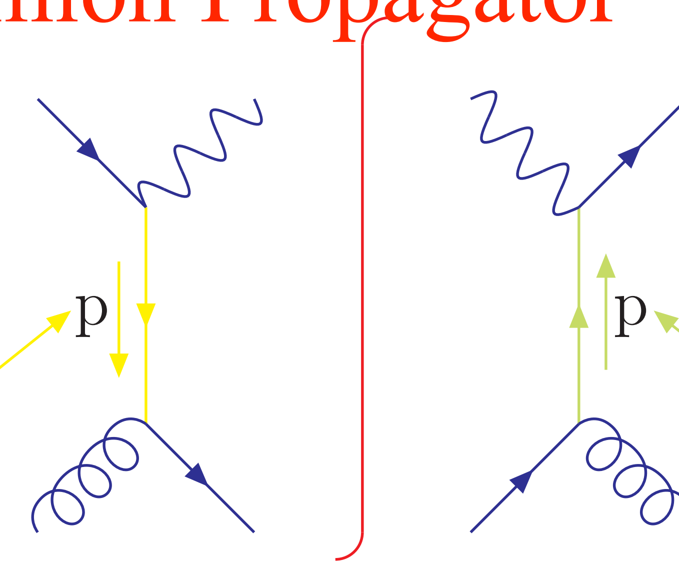
Vertex



$$i\frac{g_W}{\sqrt{2}}\gamma_\nu\frac{1}{2}(1-\gamma_5)$$

$$-i\frac{g_W}{\sqrt{2}}\gamma_\mu\frac{1}{2}(1-\gamma_5)$$

Fermion Propagator

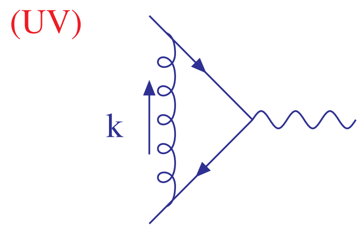


$$\frac{i(\not p+m)}{p^2-m^2+i\epsilon}$$

$$\frac{-i(\not p+m)}{p^2-m^2-i\epsilon}$$

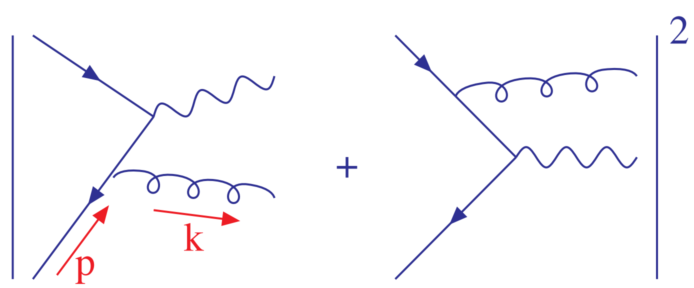
Intermediate Problems (Singularities)

- Ultraviolet singularity



$$\sim \int^{\infty} d^4k \frac{\not{k}\ \not{k}}{(k^2)\,(k^2)\,(k^2)} \rightarrow \infty$$

- Infrared singularities



$\rightarrow \infty$

(Similar singularities also exist in virtual diagrams.)

as $k^\mu \rightarrow 0$ (soft divergence)

or $k^\mu \parallel p^\mu$ (collinear divergence)

$$\frac{1}{(p-k)^2 - m^2} = \frac{1}{-2p \cdot k} \quad (\text{for } m = 0 \text{ or } m \neq 0)$$

$p \cdot k \rightarrow 0$ as

$k \rightarrow 0$ or $k^\mu \parallel p^\mu$ (for $m = 0$)

$k \rightarrow 0$ (for $m \neq 0$)

- Solutions

Compute H_{ij} in pQCD in $n = 4 - 2\varepsilon$ dimensions
(dimensional regularization)

(1) $n \neq 4 \Rightarrow$ UV & IR divergences appear as $\frac{1}{\varepsilon}$ poles
in $\sigma_{ij}^{(1)}$ (Feynman diagram calculation)

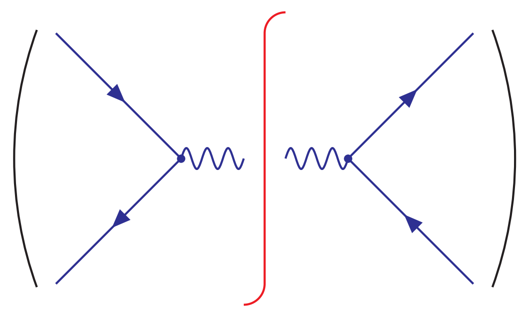
(2) H_{ij} is IR safe $\Rightarrow$ no $\frac{1}{\varepsilon}$ in H_{ij}
(H_{ij} is UV safe after "renormalization".)

Dimensional Regularization

(Revisit the Born Cross Section in n dimension)

$$\hat{\sigma}^{(0)}_{q\bar{q}'} = \frac{1}{2\hat{s}} \int \frac{d^{n-1}q}{(2\pi)^{n-1}\, 2q_0} (2\pi)^n \cdot \delta^n \left(p_1 + p_2 - q\right) \cdot \overline{|m|^2}$$

$$\overline{|m|^2} = \left(\frac{1}{3} \cdot \frac{1}{3}\right) \left(\frac{1}{2} \cdot \frac{1}{2}\right) \cdot$$



In n-dim, the polarization degree of freedom is (2) for a quark, and (n-2) for a gluon.

Using the Naive-γ^5 prescription

$$\begin{aligned}
& Tr\left[\not{p}_1 \gamma_\mu P_L \ \not{p}_2 \gamma^\mu P_L\right](-1) \\
&= Tr\left[\not{p}_1 \gamma_\mu \ \not{p}_2 \gamma^\mu P_L\right](-1) \\
&= (-2)\left(1-\varepsilon\right) Tr\left[\not{p}_1 \ \not{p}_2 P_L\right](-1) \\
&= (-2)\left(1-\varepsilon\right) \cdot \frac{1}{2} \cdot 4\left(p_1 \cdot p_2\right)(-1) \\
&= 2\left(1-\varepsilon\right)\hat{s}
\end{aligned}$$

$$\gamma_\mu \ \not{p}_2 \gamma^\mu = -2\left(1-\varepsilon\right) \ \not{p}_2$$

In n dimensions

$$\hat{\sigma}^{(0)}_{q\bar{q}'} = \frac{\pi}{12\hat{s}} g_w^2 \cdot \left(1-\varepsilon\right) \cdot \delta\left(1-\hat{\tau}\right) \equiv \sigma^{(0)} \cdot \delta(1-\hat{\tau})$$

Strong Coupling *g* in *n* Dimensions

In *n* dimensions

$$\int d^n x \mathcal{L} \longrightarrow \int d^n x \left\{ \bar{\psi} i \not{\partial} \psi - \frac{1}{4} G_{\mu\nu} G^{\mu\nu} + g t^a \bar{\psi} \gamma^\mu G_\mu \psi + \cdots \right\}$$

The dimension in mass unit (μ)

$$[\psi] \sim \mu^{\frac{n-1}{2}} \qquad [G] \sim \mu^{\frac{n-2}{2}}$$

$$[\bar{\psi} G \psi] \sim \mu^{\frac{n-1}{2} \times 2 + \frac{n-2}{2}} = \mu^{\frac{3n}{2} - 2}$$

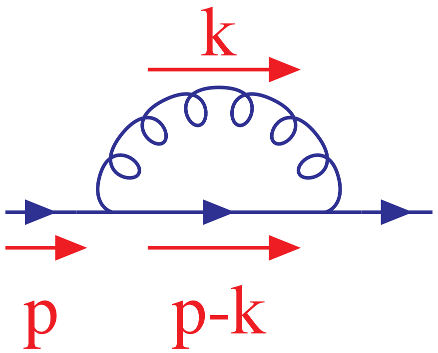
Since $[g\bar{\psi} G \psi] \sim \mu^n$, so $[g] \sim \mu^{\frac{-n}{2} + 2} = \mu^\varepsilon$ $\qquad n = 4 - 2\varepsilon$

g has a dimension in mass when $\varepsilon \neq 0$

Feynman rules should reads $g \rightarrow g\mu^\varepsilon$

Details of Virtual and Real Quantum Corrections

Virtual Corrections (in Feynman Gauge)



$$= \quad 0$$

$\frac{1}{\varepsilon_{IR}}$ and $\frac{1}{\varepsilon_{UV}}$ poles cancel when $\varepsilon_{UV} = -\varepsilon_{IR} \equiv \varepsilon$

$$\int \frac{d^n k}{(2\pi)^n} \frac{\gamma_\mu \left(\not{p} - \not{k}\right) \gamma^\mu}{(k^2 + i\epsilon)\left((p-k)^2 + i\epsilon\right)} \rightarrow \int \frac{d^n k}{(2\pi)^n} \int_0^1 dx \frac{(2-n)\left(\not{p} - \not{k}\right)}{\left[k^2 - 2k \cdot xp\right]^2}$$

$(l \equiv k - xp)$

$$= \int \frac{d^n l}{(2\pi)^n} \int_0^1 dx \frac{(2-n)\left[(1-x)\ \not{p} - \not{l}\right]}{\left[l^2 + i\epsilon\right]^2}$$

$$= \left[\left(1 - \frac{n}{2}\right)\ \not{p}\right] \cdot \underbrace{\int \frac{d^n l}{(2\pi)^n} \frac{1}{\left[l^2 + i\epsilon\right]^2}}_{=0}$$

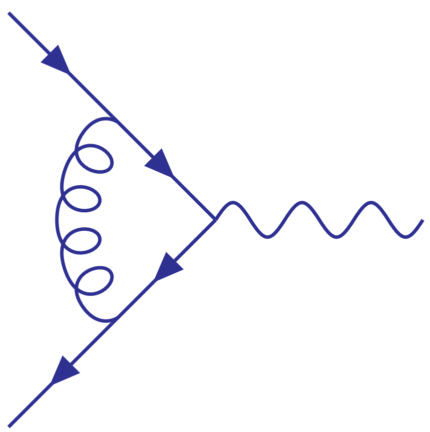
Trick: $A = A - B + B$

$$= \int \frac{d^n l}{(2\pi)^n} \left\{ \underbrace{\left[\frac{1}{(l^2)^2} - \frac{1}{(l^2 - \Lambda^2)^2}\right]}_{\textbf{IR div}} + \underbrace{\left[\frac{1}{(l^2 - \Lambda^2)^2}\right]}_{\textbf{UV div}} \right\}$$

$$n - 4 = 2\varepsilon_{IR}$$
$$4 - n = 2\varepsilon_{UV}$$

$$= \frac{i}{16\pi^2}\left(\frac{1}{\varepsilon_{IR}}\right) + \frac{i}{16\pi}\left(\frac{1}{\varepsilon_{UV}}\right)$$

Virtual Corrections (in Feynman Gauge)



$\frac{1}{\varepsilon_{UV}}$ cancel $\Rightarrow$ Electroweak coupling is not renormalized by QCD interactions at one-loop order (Ward identity, a renormalizable theory)

$\frac{1}{\varepsilon_{IR}}$ poles remain

$\sigma_{virt}^{(1)}$ is free of ultraviolent singularity.

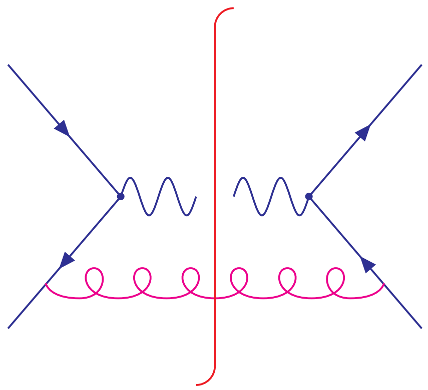
$$\sigma_{virt}^{(1)} = \sigma^{(0)} \frac{\alpha_s}{2\pi} \delta\left(1-\hat{\tau}\right) \left(\frac{4\pi\mu^2}{M^2}\right)^{\varepsilon} \frac{\Gamma\left(1-\varepsilon\right)}{\Gamma\left(1-2\varepsilon\right)} \cdot \left\{-\frac{2}{\varepsilon^2} - \frac{3}{\varepsilon} - 8 + \frac{2\pi^2}{3}\right\} \cdot \left(C_F\right)$$

$-\frac{2}{\varepsilon^2}$ soft and collinear singularities

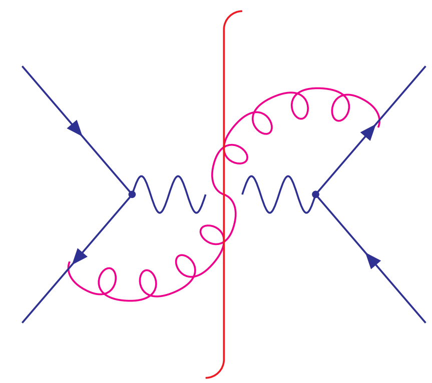
$-\frac{3}{\varepsilon}$ soft or collinear singularities

$$\sigma^{(0)} \equiv \frac{\pi}{12\hat{s}} g_w^2 \cdot (1-\varepsilon)$$

Real Emission Contribution $(q\bar{q}')_{real}$



$\sim \frac{1}{\varepsilon}$ Collinear



$\sim \frac{1}{\varepsilon^2}$ Soft and Collinear

$$\sigma^{(1)}_{\text{real}}\left(q\bar{q}'\right) = \sigma^{(0)}\frac{\alpha_s}{2\pi}\left(\frac{4\pi\mu^2}{M^2}\right)^{\varepsilon}\frac{\Gamma\left(1-\varepsilon\right)}{\Gamma\left(1-2\varepsilon\right)}\cdot C_F$$
$$\cdot\left\{\frac{2}{\varepsilon^2}\delta\left(1-\hat{\tau}\right)-\frac{2}{\varepsilon}\frac{1+\hat{\tau}^2}{\left(1-\hat{\tau}\right)_+}+4\left(1+\hat{\tau}^2\right)\left(\frac{\ln\left(1-\hat{\tau}\right)}{1-\hat{\tau}}\right)_+-2\frac{1+\hat{\tau}^2}{1-\hat{\tau}}\ln\hat{\tau}\right\}$$

Note: $[\cdots]_+$ is a distribution

$$\int_0^1 dz\, f\left(z\right)\left[\frac{1}{1-z}\right]_+ = \int_0^1 dz\frac{f\left(z\right)-f\left(1\right)}{1-z}$$ which is finite

In the soft limit, $\hat{\tau}\to 1 \quad \left(\hat{\tau}=\frac{M^2}{\hat{s}}\right)$,

$$\sigma^{(1)}_{\text{real}}\left(q\bar{q}'\right)\longrightarrow\sigma^{(0)}\frac{\alpha_s}{2\pi}\left(\frac{4\pi\mu^2}{M^2}\right)^{\varepsilon}\frac{\Gamma\left(1-\varepsilon\right)}{\Gamma\left(1-2\varepsilon\right)}\cdot C_F$$
$$\cdot\left\{\frac{2}{\varepsilon^2}\delta\left(1-\hat{\tau}\right)-\frac{4}{\varepsilon}\frac{1}{\left(1-\hat{\tau}\right)_+}+8\left(\frac{\ln\left(1-\hat{\tau}\right)}{1-\hat{\tau}}\right)_+\right\}$$

$\left(q\bar{q}'\right)_{virt} + \left(q\bar{q}'\right)_{\text{real}}$ at NLO

$$\sigma^{(1)}_{q\bar{q}'} = \sigma^{(1)}_{virt}\left(q\bar{q}'\right) + \sigma^{(1)}_{\text{real}}\left(q\bar{q}'\right)$$

$$= \sigma^{(0)}\frac{\alpha_s}{2\pi}\left(\frac{4\pi\mu^2}{M^2}\right)^{\varepsilon}\frac{\Gamma(1-\varepsilon)}{\Gamma(1-2\varepsilon)}\cdot C_F$$

$$\cdot\left\{\frac{-2}{\varepsilon}\left(\frac{1+\hat{\tau}^2}{1-\hat{\tau}}\right)_+ - 2\frac{1+\hat{\tau}^2}{1-\hat{\tau}}\ln\hat{\tau} + 4\left(1+\hat{\tau}^2\right)\left(\frac{\ln(1-\hat{\tau})}{1-\hat{\tau}}\right)_+\right.$$

$$\left. + \left(\frac{2\pi^2}{3} - 8\right)\delta(1-\hat{\tau})\right\}$$

Where we have used

$$\frac{-2}{\varepsilon}\left[\frac{1+\hat{\tau}^2}{(1-\hat{\tau})_+} + \frac{3}{2}\delta(1-\hat{\tau})\right] = \frac{-2}{\varepsilon}\left(\frac{1+\hat{\tau}^2}{1-\hat{\tau}}\right)_+$$

All the soft singularities $\left(\frac{1}{\varepsilon^2}, \frac{1}{\varepsilon}\right)$ cancel in $\sigma^{(1)}_{q\bar{q}'}$

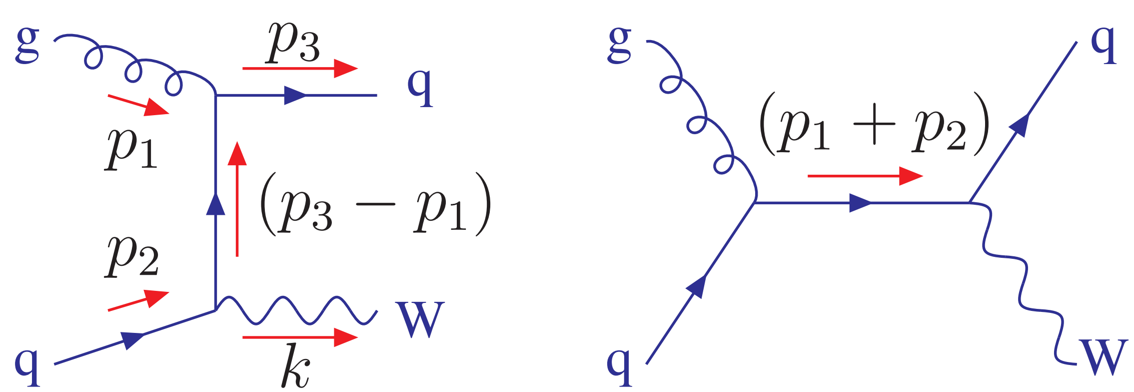
$\Rightarrow$ KLN theorem

(Kinoshita-Lee-Navenberg)

$$\Rightarrow \quad \sigma^{(1)}_{q\bar{q}'} \sim \frac{1}{\varepsilon}\,(\text{term}) + \text{finite}\;\;(\text{terms})$$

collinear singularity

Real Correction $qG \to q'W^+$



Define the Mandelstam variables

$$\hat{s} = (p_1 + p_2)^2 = 2p_1 \cdot p_2$$
$$\hat{t} = (p_1 - p_3)^2 = -2p_1 \cdot p_3$$
$$\hat{u} = (p_2 - p_3)^2 = -2p_2 \cdot p_3$$

After averaging over colors and spins

$$\overline{|\mathcal{M}|^2} = \underbrace{\left(\frac{1}{2(1-\varepsilon)}\frac{1}{2}\right)}_{\text{Spin}} \cdot \underbrace{\left(\frac{1}{3}\cdot\frac{1}{8}\right) \cdot Tr\left(t^a t^a\right)}_{\text{Color}} \cdot \left(g\mu^{\varepsilon}\right)^2 \cdot g_w^2 \cdot 2\left(1-\varepsilon\right)$$

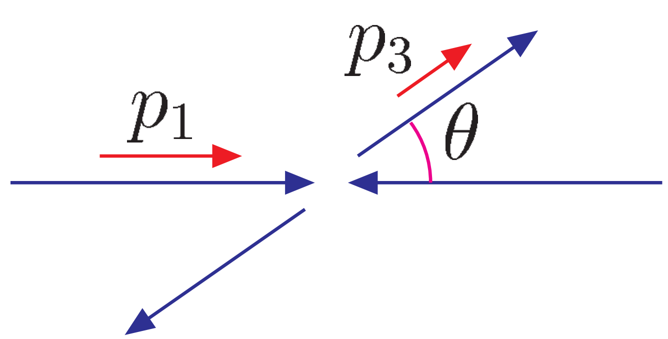
$$\times \left[(1-\varepsilon)\left(-\frac{\hat{s}}{\hat{t}} - \frac{\hat{t}}{\hat{s}}\right) - \frac{2\hat{u}M^2}{\hat{t}\hat{s}} + 2\varepsilon\right]$$

- In the parton c.m. frame, the constituent cross section

$$\hat{\sigma} = \frac{1}{2\hat{s}}\overline{|\mathcal{M}|^2}\cdot(PS_2) = \frac{1}{2\hat{s}}\cdot\left\{\frac{1}{4}\cdot\frac{1}{6}\cdot 2g_s^2\mu^{2\varepsilon}g_w^2\,(1-\varepsilon)\cdot\left[(1-\varepsilon)\left(-\frac{\hat{s}}{\hat{t}}-\frac{\hat{t}}{\hat{s}}\right)-\frac{2\hat{u}M^2}{\hat{t}\hat{s}}+2\varepsilon\right]\right\}$$

$$\times\left\{\frac{1}{8\pi}\left(\frac{4\pi}{M^2}\right)^{\varepsilon}\frac{1}{\Gamma(1-\varepsilon)}\hat{\tau}^{\varepsilon}\,(1-\hat{\tau})^{1-2\varepsilon}\int_0^1 dy\,[y\,(1-y)]^{-\varepsilon}\right\}$$

where $y \equiv \frac{1}{2}(1+\cos\theta)$



Using $\hat{t} = -\hat{s}\left(1-\frac{M^2}{\hat{s}}\right)(1-y)$ $\qquad \hat{u} = -\hat{s}\left(1-\frac{M^2}{\hat{s}}\right)y$

$$\int_0^1 dy\,y^{\alpha}\,(1-y)^{\beta} = \frac{\Gamma(1+\alpha)\,\Gamma(1+\beta)}{\Gamma(2+\alpha+\beta)}\ ,$$

we get

$$\hat{\sigma}_{qG} = \hat{\sigma}^{(0)}\frac{\alpha_s}{4\pi}\cdot\left\{2P^{(1)}_{q\leftarrow g}(\hat{\tau})\left[\frac{-1}{\varepsilon}\frac{\Gamma(1-\varepsilon)}{\Gamma(1-2\varepsilon)}+\ln\frac{M^2\,(1-\hat{\tau})^2}{4\pi\mu^2\hat{\tau}}\right]+\frac{3}{2}+\hat{\tau}-\frac{3}{2}\hat{\tau}^2\right\},$$

with $\quad P^{(1)}_{q\leftarrow g}(\hat{\tau}) = \frac{1}{2}\left[\hat{\tau}^2+(1-\hat{\tau})^2\right] \qquad \hat{\sigma}^{(0)} \equiv \frac{\pi}{12}g_w^2\frac{1}{\hat{s}}$

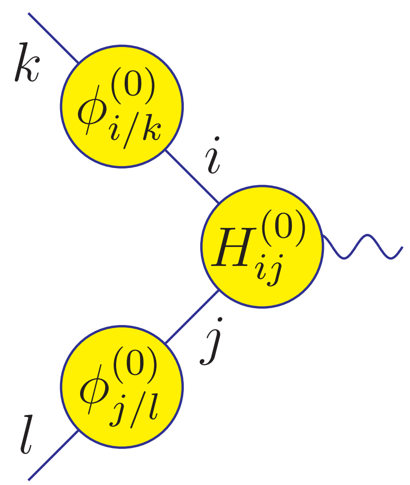
Factorization Theorem

- Perturbative PDF

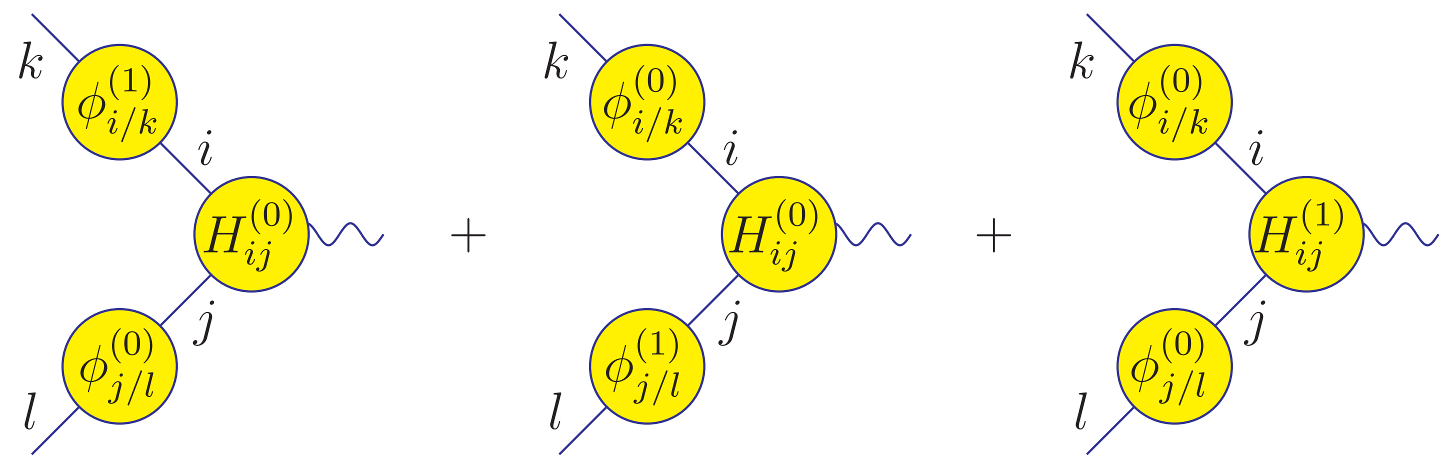
$$\phi_{i/k}^{(0)} = \delta_{ik}\delta(1-x) \qquad \frac{\alpha_s}{\pi}\phi_{i/k}^{(1)}$$

can be calculated from the definition of PDF (Process independent, but factorization scheme dependent)

$$\sigma_{kl}^{(0)} =$$



$$\sigma_{kl}^{(1)} =$$



$$\Rightarrow H_{kl}^{(0)} = \sigma_{kl}^{(0)}$$

Factorization scheme dependent

$$\Rightarrow H_{kl}^{(1)} = \sigma_{kl}^{(1)} - \left[\phi_{i/k}^{(1)} H_{il}^{(0)} + H_{kj}^{(0)} \phi_{j/l}^{(1)}\right]$$

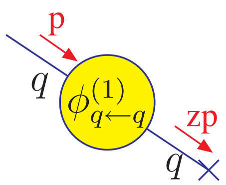
Finite

Divergent

Perturbative PDFs

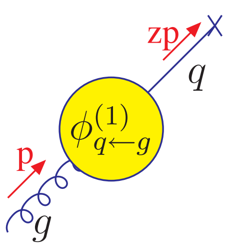
- In $\overline{MS}$-scheme (modified minimal subtraction)

$$\phi^{(1)}_{q/q}(z) = \phi^{(1)}_{\bar{q}/\bar{q}}(z) = \frac{-1}{\varepsilon}\frac{1}{2}\left(4\pi e^{-\gamma_E}\right)^{\varepsilon} P^{(1)}_{q\leftarrow q}(z)$$
$$\phi^{(1)}_{q/g}(z) = \phi^{(1)}_{\bar{q}/g}(z) = \frac{-1}{\varepsilon}\frac{1}{2}\left(4\pi e^{-\gamma_E}\right)^{\varepsilon} P^{(1)}_{q\leftarrow g}(z)$$

where the splitting kernel for 

is

$$P^{(1)}_{q\leftarrow q}(z) = C_F\left(\frac{1+z^2}{1-z}\right)_+ = C_F\left(\frac{1+z^2}{(1-z)_+} + \frac{3}{2}\delta(1-z)\right)$$

and for 

is $P^{(1)}_{q\leftarrow g}(z) = T_R\left(z^2 + (1-z)^2\right)$,

where $C_F = \frac{4}{3}$ and $T_R = \frac{1}{2}$.

Note: The Pole part in the $\overline{MS}$ scheme is

$$\frac{1}{\hat{\varepsilon}} = \frac{1}{\varepsilon}(4\pi e^{-\gamma_E})^{\varepsilon} = \frac{1}{\varepsilon} + \ln 4\pi - \gamma_E$$

Find $H^{(1)}_{q\bar{q}'}$ (in the $\overline{MS}$ scheme)

- Take off the factor $\left(\frac{\alpha_s}{\pi}\right)$

$$\sigma^{(1)}_{q\bar{q}'} = \sigma^{(0)} \left\{ P^{(1)}_{q\leftarrow q}(\hat{\tau}) \left[\ln\left(\frac{M^2}{\mu^2}\right) - \frac{1}{\varepsilon} + \gamma_E - \ln 4\pi \right] + C_F \left[-\frac{1+\hat{\tau}^2}{1-\hat{\tau}} \ln\hat{\tau} + 2\left(1+\tau^2\right)\left(\frac{\ln(1-\hat{\tau})}{1-\hat{\tau}}\right)_+ + \left(\frac{\pi^2}{3} - 4\right)\delta(1-\hat{\tau}) \right] \right\}$$

$$\begin{aligned} H^{(1)}_{q\bar{q}'}(\hat{\tau}) &= \sigma^{(1)}_{q\bar{q}'} - \left[2\phi^{(1)}_{q\leftarrow q}\sigma^{(0)}_{q\bar{q}'} \right] \\ &= \hat{\sigma}^{(0)} \cdot \left\{ P^{(1)}_{q\leftarrow q}(\hat{\tau}) \ln\left(\frac{M^2}{\mu^2}\right) \right. \\ &\left. + C_F \left[-\frac{1+\hat{\tau}^2}{1-\hat{\tau}} \ln\hat{\tau} + 2\left(1+\tau^2\right)\left(\frac{\ln(1-\hat{\tau})}{1-\hat{\tau}}\right)_+ + \left(\frac{\pi^2}{3} - 4\right)\delta(1-\hat{\tau}) \right] \right\} \end{aligned}$$

where $\hat{\tau} = \frac{M^2}{\hat{s}} = \frac{M^2}{x_1 x_2 S}$, $\quad \sigma^{(0)} = \hat{\sigma}^{(0)} \cdot (1-\varepsilon)\,, \quad \hat{\sigma}^{(0)} = \frac{\pi}{12\hat{s}} g_w^2 = \frac{\pi g_w^2}{12 S} \frac{1}{x_1 x_2}\,.$

Find $H_{qG}^{(1)}$ (in the $\overline{MS}$ scheme)

- Take off the factor $\left(\frac{\alpha_s}{\pi}\right)$

$$\sigma_{qG}^{(1)} = \sigma^{(0)} \cdot \frac{1}{4} \cdot \left\{ 2P_{q\leftarrow g}^{(1)}\left(\hat{\tau}\right)\left[\frac{-1}{\varepsilon}\frac{\Gamma\left(1-\varepsilon\right)}{\Gamma\left(1-2\varepsilon\right)} + \ln\frac{M^2\left(1-\hat{\tau}\right)^2}{4\pi\mu^2\hat{\tau}}\right] + \frac{1}{2} + 3\hat{\tau} - \frac{7}{2}\hat{\tau}^2 \right\}$$

$$\begin{aligned} H_{qG}^{(1)}\left(\hat{\tau}\right) &= \sigma_{qG}^{(1)} - \left[\sigma_{q\bar{q}'}^{(0)}\phi_{\bar{q}'\leftarrow G}^{(1)}\right] \\ &= \frac{\hat{\sigma}^{(0)}}{2} \cdot \left\{ P_{q\leftarrow g}^{(1)}\left(\hat{\tau}\right)\left[\ln\left(\frac{M^2}{\mu^2}\right) + \ln\left(\frac{\left(1-\hat{\tau}\right)^2}{\hat{\tau}}\right)\right] + \frac{1}{4} + \frac{3}{2}\hat{\tau} - \frac{7}{4}\hat{\tau}^2 \right\} \end{aligned}$$

Similarly,

$$H_{G\bar{q}'}^{(1)} = \sigma_{G\bar{q}'}^{(1)} - \left[\phi_{q\leftarrow G}^{(1)}\sigma_{q\bar{q}'}^{(0)}\right] = H_{qG}^{(1)}$$

Note: If we choose the renormalization scale $\mu^2 = M^2$,
then $\ln\left(\frac{M^2}{\mu^2}\right) = 0$

pQCD Prediction

$$\sigma_{hh'} = \left\{ \sum_{f=q,\bar{q}'} \int dx_1 dx_2 \phi_{f/h}\left(x_1,\mu^2\right) \left[\sigma^{(0)}\delta\left(1-\hat{\tau}\right)\right] \phi_{\bar{f}/h'}\left(x_2,\mu^2\right) \right.$$

$$+ \sum_{f=q,\bar{q}'} \int dx_1 dx_2 \phi_{f/h}\left(x_1,\mu^2\right) \left[\frac{\alpha_s\left(\mu^2\right)}{\pi} H^{(1)}_{f\bar{f}}\left(\hat{\tau}\right)\right] \phi_{\bar{f}/h'}\left(x_2,\mu^2\right)$$

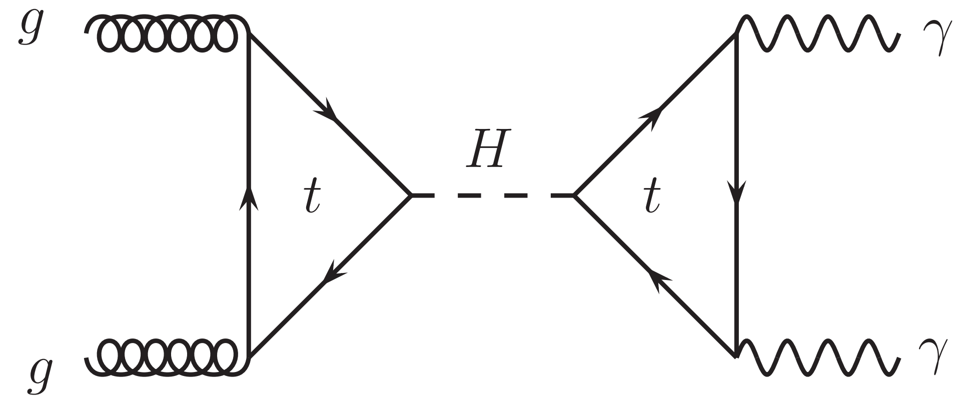
$$\left. + \sum_{f=q,\bar{q}'} \int dx_1 dx_2 \phi_{f/h}\left(x_1,\mu^2\right) \left[\frac{\alpha_s\left(\mu^2\right)}{\pi} H^{(1)}_{fG}\left(\hat{\tau}\right)\right] \phi_{G/h'}\left(x_2,\mu^2\right) + \left(x_1 \leftrightarrow x_2\right) \right\}$$

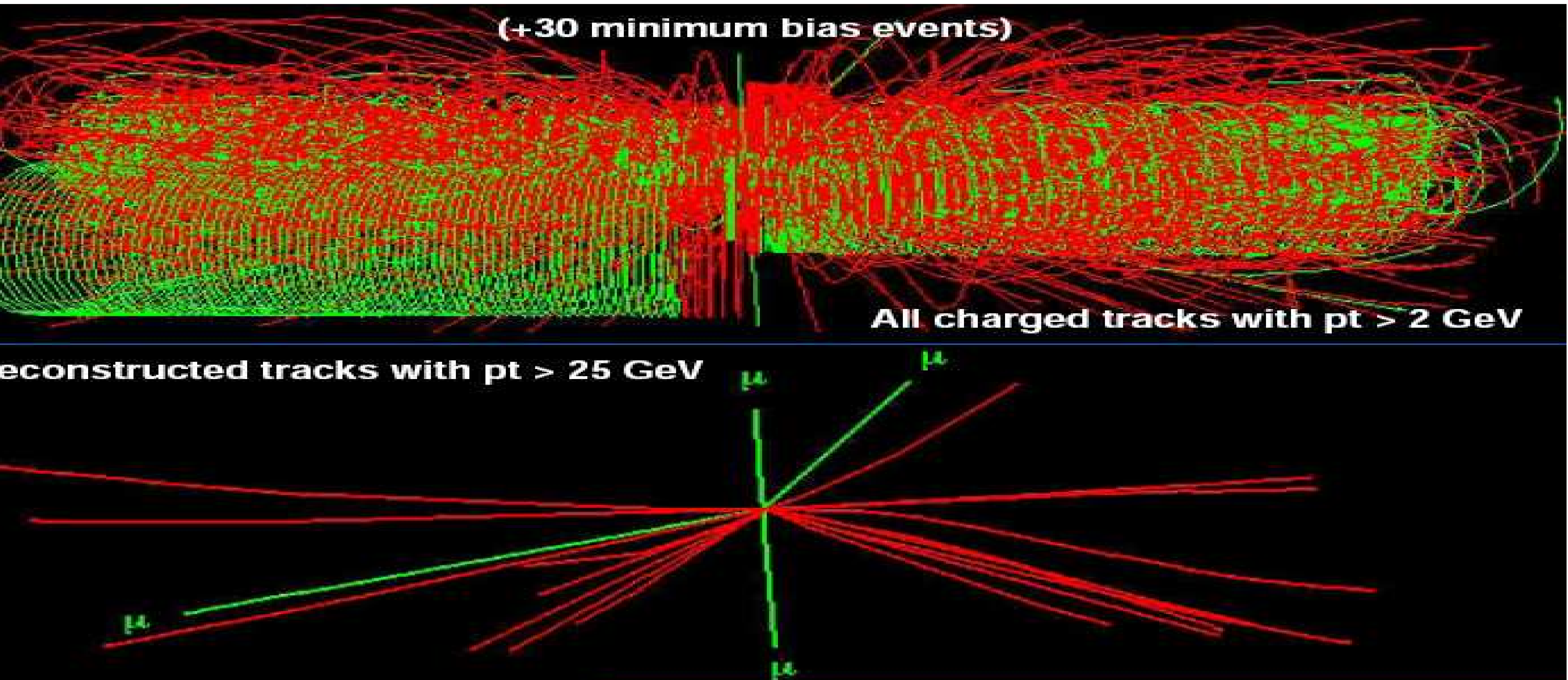
- $\phi_{f/h}(x,\mu^2)$ depends on scheme ($\overline{MS}$,DIS,...)
  $\Rightarrow H_{ij}$ **scheme dependent**

- Evolution equations allow us to perdict
  q^2-**dependent of** $\phi(x,q^2)$

- Essentially identical procedure for
  $hh' \rightarrow jets$, inclusive $Q\overline{Q}$,...
  But, when the Born level process involves strong interaction (eg. $q\overline{q} \rightarrow t\bar{t}$), it is also necessary to renormalize the strong coupling α_s, etc, to eliminate ultraviolate singularities

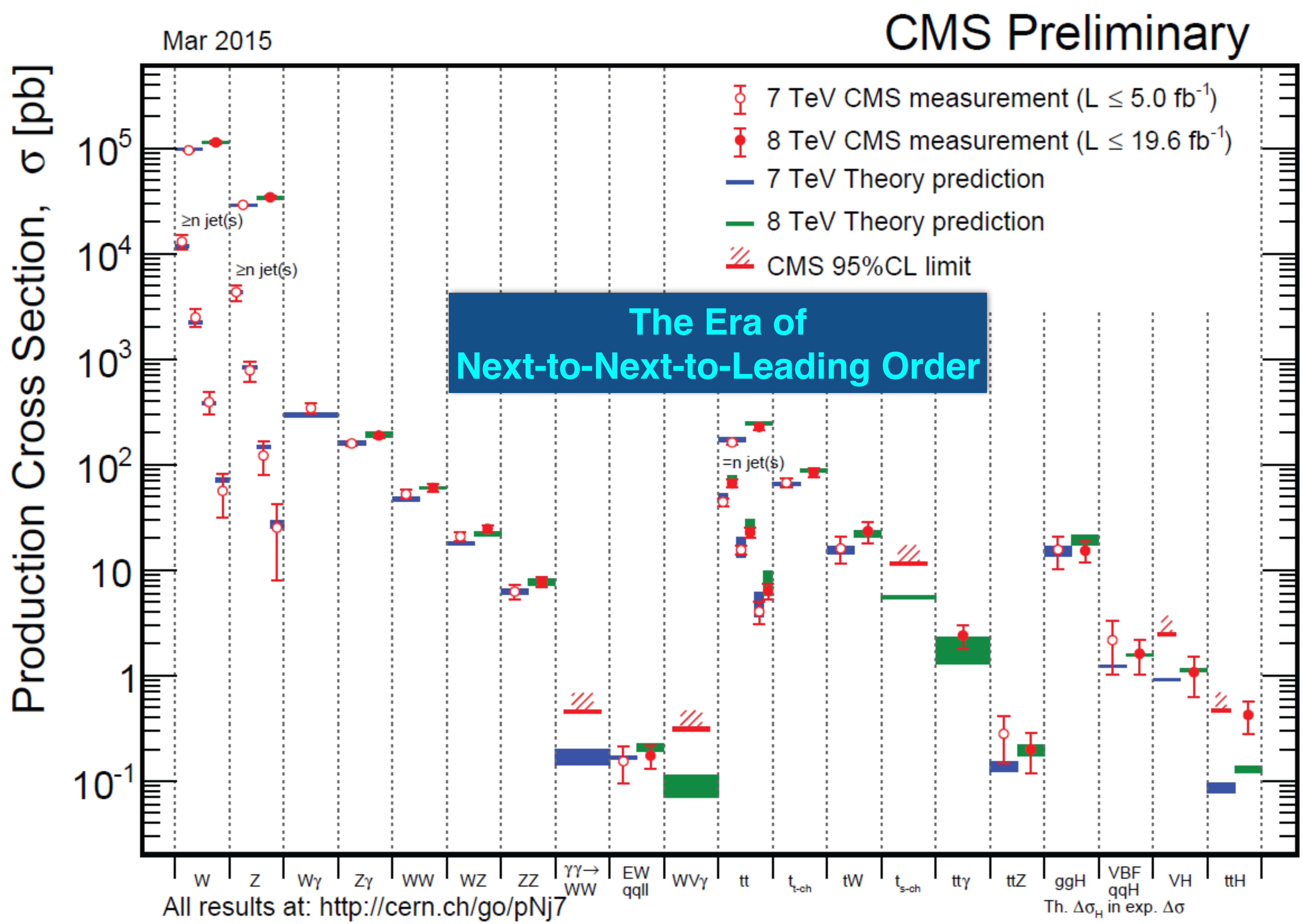
Glory of Perturbative QCD





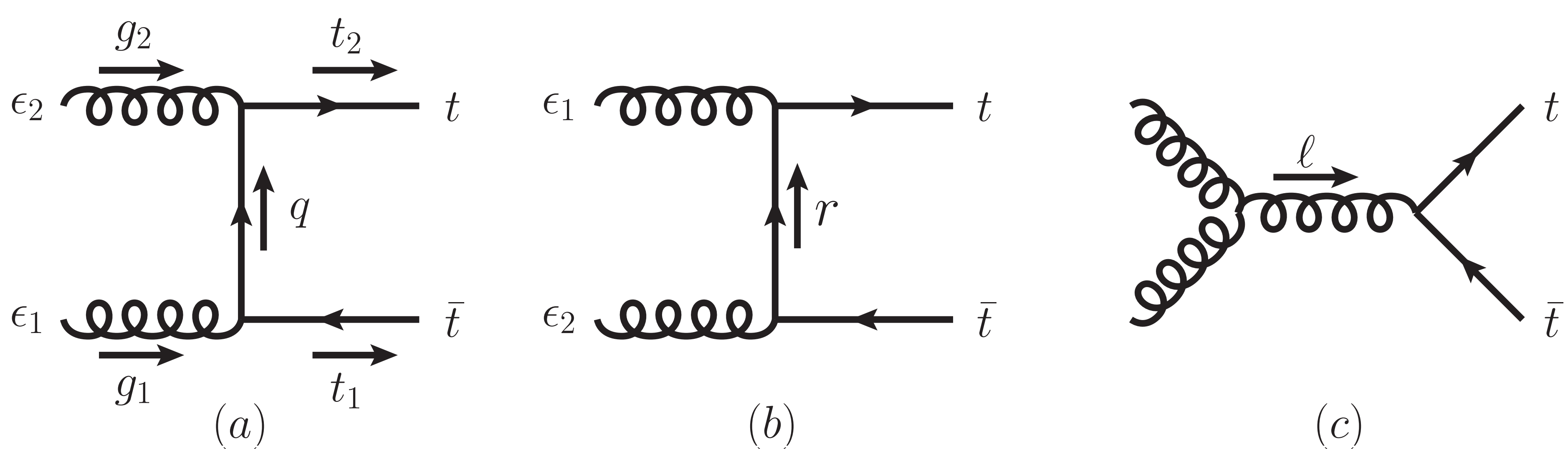
Higgs decay in 4 muons
1 in 10 ^{13} events

Glory of Perturbative QCD



Survey

Who has calculated top-quark pair production at the LHC?



If yes, how did you calculate it?

MadEvent? Pythia? Herwig or by hand?