Naturalness in the LHC Era

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Contents

1 Introduction **2**
1.1 "Just-so story" vs. hierarchy problem 4

2 The Hierarchy Problem **6**
2.1 Two hierarchy problems 10
2.2 Three reasons 11
2.2.1 Things we want to believe 11
2.2.2 Things we should believe 16
2.2.3 Things we must believe 17
2.3 Naturalness in Nature 19

3 Custodial Symmetries **21**
3.1 Supersymmetry 22
3.2 Global symmetry 27
3.3 Scale invariance? 33

1 Introduction

Lectures about naturalness of the weak scale. Incredibly important topic in the LHC era. A driver of the program for physics beyond the Standard Model. Count ATLAS, CMS searches in SUSY and Exotica groups: 170 of 226 searches motivated by naturalness. Need to understand why this is so important and make sure program at LHC is complete.

Higgs boson was the theory problem of the last 40 years. Naturalness of the weak scale is the theory problem of our time.

The starting point for all phenomenology, the Standard Model consists of an $SU(3) \times SU(2) \times U(1)$ gauge theory with the following matter content:

Table 1: Standard Model field content (×3 generations); conventions are those for LH Weyl fermions. Here the bars are just part of the field name, not conjugation

Field	$SU(3)$	$SU(2)$	$U(1)$
$Q = \begin{pmatrix} u \\ d \end{pmatrix}$	$\square$	$\square$	$\frac{1}{6}$
$\bar{u}$	$\overline{\square}$	-	$-\frac{2}{3}$
$\bar{d}$	$\overline{\square}$	-	$\frac{1}{3}$
$L = \begin{pmatrix} \nu \\ e \end{pmatrix}$	-	$\square$	$-\frac{1}{2}$
$\bar{e}$	-	-	1
$\bar{\nu}$?	-	-	0
H	-	$\square$	$-\frac{1}{2}$

Notably, the interactions of the Standard Model are the most general marginal and relevant operators allowed by gauge invariance and Lorentz invariance. They are encoded in gauge and matter kinetic terms, as well as in Yukawa couplings and a potential for the Higgs:

$$\begin{aligned}
\mathcal{L}_{SM} &= \mathcal{L}_{kin} + \mathcal{L}_{yuk} + \mathcal{L}_{higgs} && (1)\\
\mathcal{L}_{kin} &= -\frac{1}{4}(G^a_{\mu\nu})^2 - \frac{1}{4}(W^a_{\mu\nu})^2 - \frac{1}{4}B^2_{\mu\nu} + (D_\mu H)^\dagger (D_\mu H) + i\psi^\dagger_I \bar{\sigma}^\mu (D_\mu \psi)_I && (2)\\
\mathcal{L}_{yuk} &= -\mathbf{y}^e(\epsilon^{ij} H_i L_j)\bar{e} - \mathbf{y}^d(\epsilon^{ij} H_i Q_{j\alpha})\bar{d}^\alpha - \mathbf{y}^u(H^{\dagger i} Q_{i\alpha})\bar{u}^\alpha + \text{h.c.} && (3)\\
\mathcal{L}_{higgs} &= m^2 H^\dagger H - \lambda(H^\dagger H)^2 && (4)
\end{aligned}$$

The yukawas are 3×3 matrices. The sign of m^2 leads to spontaneous symmetry breaking.

Remarkably, if the Standard Model is the complete theory of the universe, we have now measured all of its parameters (the quartic coupling is inferred from knowledge of the Higgs vev).

The fact that all marginal and relevant couplings allowed by the symmetries of the theory are present is a marvelous example of Gell-Mann's *totalitarian principle*, namely

Everything not forbidden is compulsory.

The fact that the totalitarian principle seems to be born out so completely in the context of the Standard Model will be important!

Can't write down a gauge-invariant mass term for any SM fermion respecting gauge & Lorentz symmetries. Fermion masses can only arise after electroweak symmetry is broken.

The Higgs, however, enjoys no such symmetry. The mass term

$$m^2 H^\dagger H$$

is a complete invariant under any gauge or global symmetry. It is also the only marginal operator in the Standard Model.

Observationally, the mass parameter is $m^2 \sim (89\,\text{GeV})^2$. Simply an experimental fact in the Standard Model. But a curious scale. The Standard Model is itself not a complete description of nature. We know there are also gravitational interactions, which we can express at low energies in terms of a quantum field theory with Lagrangian density of the form

$$\mathcal{L}_{EH} \sim M_P^2 \sqrt{-\det(g)} \mathrm{tr}\left[g^{\mu\nu} \partial_\mu \partial_\nu \exp\left(\frac{h_{\alpha\beta}}{M_P} \right) \right] \tag{5}$$

where $g_{\mu\nu} = \eta_{\mu\nu} + \frac{1}{M_P} h_{\mu\nu}$ where $\eta_{\mu\nu}$ is the Minkowski metric and $h_{\mu\nu}$ is a tensor field. Intrinsically non-renormalizable, with scale $M_P \sim 10^{19}$ GeV. Field theory contains an infinite number of irrelevant operators equally important at the scale M_P. We do not know what the complete theory at M_P is.

1.1 "Just-so story" vs. hierarchy problem

Worrisome; two dimensionful scales that differ by sixteen orders of magnitude. However, we already see a hierarchy of nearly six orders of magnitude in the known fermion masses (between the electron at 0.5 MeV and the top quark at 173 GeV), or something in the ballpark of twelve orders of magnitude between neutrino masses and the top quark.

However, there is a further complication at the quantum level. In the case of fermion masses the hierarchy is merely a just-so story. Consider a Dirac fermion Ψ. A mass for this fermion consists of a term of the form

$$m\bar{\Psi}\Psi \tag{6}$$

where $\bar{\Psi} = \Psi^\dagger \gamma_0$. This mass term is invariant under global abelian rotations of the form $\Psi \to e^{i\alpha}\Psi$. However, in the limit $m \to 0$, there is an additional symmetry, namely $\Psi \to e^{i\beta\gamma_5}\Psi$. Quantum corrections respect the symmetries of the quantum action, so when $m = 0$ implies that quantum corrections will not generate a mass term. Moreover, when the chiral symmetry is broken by $m \neq 0$, quantum corrections will be proportional to the symmetry-breaking term. Corrections to fermion masses are proportional to fermion masses.

Thus a large hierarchy between fermion masses is a curiosity, but not a deeply troubling one. If the fundamental theory of the universe generates fermions with very different masses, quantum corrections need not disturb the hierarchy.

The just-so story: why the mass term takes the specific value in the fundamental theory. But there is no preferred value.

This is a beautiful property not only of spin-1/2 particles, but of spin-1 particles as well. In the case of vector bosons, in the limit where the mass term

$$m^2 A_\mu A^\mu$$

goes to zero, there is an enhanced symmetry – gauge invariance under $A_\mu \to A_\mu + \partial_\mu \alpha$. This likewise guarantees that radiative corrections to gauge boson masses are proportional to the mass itself.

We call these *custodial symmetries*.

The same does not hold for scalar mass terms. As we noted earlier, the mass term

$$m^2 H^\dagger H$$

is a complete invariant under any gauge or global symmetry. In particular, no symmetry is enhanced when the mass is zero. Thus we are without any argument to justify the stability of the Higgs mass parameter against radiative corrections.

You usually see this phrased as something terrible like "there is a hierarchy problem because the Higgs mass is quadratically divergent in the Standard Model, and the contribution at one loop up to a cutoff Λ is

$$\delta m_H^2 = \frac{\Lambda^2}{16\pi^2}\left(-6y_t^2 + \frac{9}{4}g^2 + \frac{3}{4}g'^2 + 6\lambda\right)$$

and the cutoff runs all the way up to the Planck scale." Here the loops are loops of top quarks, electroweak gauge bosons, and the Higgs itself.

This poses a *Naturalness Problem* because the UV contributions to the Higgs mass greatly exceed its observed value; there must be large cancellations between contributions, which are non-generic.

But you should be very suspicious of this argument. The RHS depends on a hard cutoff, but we should be able to compute physical quantities in any renormalization scheme. If we repeated this calculation in $\overline{MS}$, all of the divergent $1/\epsilon$ pieces would be soaked up into the counterterm. We would be left only with finite terms like

$$\delta m_H^2(\overline{MS}) = -\frac{6}{16\pi^2} m_t^2 \log(\mu/m_t)$$

which does not look frightening at all; this is a correction that is numerically of order tens of GeV.

There is also nothing wrong with running over large energy hierarchies in the Standard Model. You can compute the beta function for the Higgs mass in the Standard Model, and at one loop you find in $\overline{MS}$

$$\beta_{m_H^2} = \frac{m_H^2}{16\pi^2}\left[6y_t^2 - \frac{9}{4}g^2 - \frac{3}{4}g'^2 - 6\lambda\right]$$

Again, there is nothing here to suggest a hierarchy problem. The Higgs mass in the Standard Model runs proportional to itself, and given a renormalized mass at one scale μ_1, we can find the renormalized mass at another scale μ_2 with corrections that are merely of the form

$$\delta m_H^2 \sim \frac{6y_t^2}{16\pi^2} m_H^2 \log(\mu_1/\mu_2)$$

These corrections are parametrically small with respect to m_H^2, so there is no problem apparent in the Standard Model.

So where is the hierarchy problem? Our goal in these lectures is to answer this question and explore its implications.

2 The Hierarchy Problem

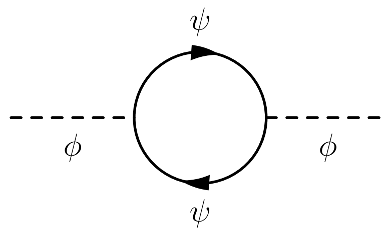
We encounter infinities all the time in quantum field theory all the time. Accommodate them by adopting a suitable form of *regularization*, and we liberate physical quantities from dependence on the regulator by adopting a *renormalization scheme*.

Various ways of accommodating these infinities. Simplest is to add counterterms to the Lagrangian which are formally infinite and contribute to physical processes. The form of the counterterms is fixed by a renormalization condition which defines couplings in the renormalized theory in terms of an observable. Then use QFT to relate observables at different energies and make predictions.

Concretely, consider as a toy model a scalar coupled to a Dirac fermion,

$$\mathcal{L} = -\frac{1}{2}\phi(\Box + m^2)\phi + \lambda\phi\bar{\psi}\psi + \bar{\psi}(i\not{\partial} - M)\psi \qquad (7)$$

We can go ahead and compute radiative corrections to the scalar mass in perturbation theory from diagrams of the form



which gives a correction to the scalar self-energy of the form

$$i\Sigma_2(p^2) = -4\lambda \int \frac{d^4k}{(2\pi)^4} \int_0^1 dx \left[\frac{1}{k^2 - \Delta} + \frac{2\Delta}{[k^2-\Delta]^2} \right] \tag{8}$$

where $\Delta = M^2 - p^2 x(1-x)$. Now first we notice that this integral is divergent. We tame the divergence by picking a regulator. For example, we could pick a hard cutoff Λ on the Euclidean momentum. Then we would find

$$\Sigma_2(p^2) = \frac{3\lambda^2}{4\pi^2} \int_0^1 dx \left([M^2 - p^2x(1-x)] \log \left(\frac{M^2 - p^2x(1-x)}{\Lambda^2} \right) + \Lambda^2 \right) + \text{finite} \tag{9}$$

where the finite terms are functions of M and p. But this is just a parameterization of the divergence. We could pick another regularization scheme, such as dimensional regularization. In the hard cutoff we deformed the theory in the UV to make integrals finite, whereas in dim reg we deform the theory by non-integer dimension (e.g. $d = 4 - \varepsilon$) to tame divergences.

In dim reg we instead obtain something that looks like

$$\Sigma_2(p^2) = \frac{3\lambda^2}{4\pi^2} \left(-\frac{2M^2}{\varepsilon} + \frac{p^2}{3\varepsilon} + \int_0^1 dx \left([M^2 - p^2x(1-x)] \log \left(\frac{M^2 - p^2x(1-x)}{4\pi\mu^2 e^{-\Gamma_E}} \right) \right) \right) \tag{10}$$

Gone is explicit dependence on the cutoff, but the divergences are present in $1/\varepsilon$ poles.

In both cases, we next obtain physical predictions by introducing counterterms in the form of field strength and mass renormalization for the scalar. That is to say, we define the renormalized theory by the addition of counterterms,

$$\mathcal{L} = -\frac{1}{2}\phi\Box\phi - \frac{1}{2}\delta_\phi\phi\Box\phi + m_R^2\phi^2 + (\delta_\phi + \delta_m)m_R^2\phi^2 + \dots \tag{11}$$

which introduce new Feynman rules,

$$i(p^2\delta_\phi - (\delta_m + \delta_\phi)m_R^2) \tag{12}$$

that we also need to include in computing the self-energy. Now we fix the form of the counterterms by some renormalization scheme. A convenient one is an on-shell scheme, where we have measured the pole mass of the scalar and we set the pole of the propagator at the renormalized mass with residue 1. The resummed scalar propagator takes the form

$$iG(p^2) = \frac{i}{p^2 - m^2 + \Sigma(p^2)} \tag{13}$$

with $\Sigma(p^2) = \Sigma_2(p^2) + p^2\delta_\phi - (\delta_m + \delta_\phi)m_R^2$. The on-shell conditions are $\Sigma(m_P^2) = \Sigma'(m_P^2) = 0$ at $m_R = m_P$. Then

$$\delta_m = \frac{1}{m_P^2}\Sigma_2(m_P^2) \qquad \delta_\phi = -\left.\frac{d\Sigma_2(p^2)}{dp^2}\right|_{p^2=m_P^2} \tag{14}$$

We then use this to work out the counterterms, which will differ for the two different regulators but in both cases will cancel all of the divergences, whether a cutoff Λ or a $1/\varepsilon$ pole. The result is, for example, a perfectly finite self-energy.

Of course, this uses a measured pole mass to soak up divergences, so perhaps you are concerned that we are sweeping something under the rug. However, we could pick a different scheme. A good choice is dimensional regularization with modified minimal subtraction. Can talk about an $\overline{MS}$ mass which is a Lagrangian parameter, and study how loops relate this to a pole mass.

The difference between the pole mass and the $\overline{MS}$ mass is finite – the scheme still subtracts out divergences – but nonzero, and keeps track of the sorts of radiative corrections we just computed.

The pole and MS masses differ by an amount

$$m_P^2 - m_{\overline{MS}}^2(\mu) = \frac{\lambda^2}{4\pi^2}M^2 + \dots \tag{15}$$

This is perfectly finite, but notably it's proportional to the mass of the fermion. The analogue of this in the SM is the correction to the Higgs mass proportional to the top yukawa.

Already, though, something important is clear. Consider scalar loop corrections to fermions in our Yukawa toy model:

$$i\Sigma_2(\not{p}) = \lambda^2 \int \frac{d^4k}{(2\pi)^4} \frac{\not{p} + \not{k} + M}{[(p-k)^2 - m^2][k^2 - M^2]} \tag{16}$$

$$= \lambda^2 \int dx \int \frac{d^4\ell}{(2\pi)^4} \frac{(1+x)\not{p} + M}{[\ell^2 - \Delta]^2} \tag{17}$$

where $\Delta = xm^2 + (1-x)M^2 - x(1-x)p^2$. If we evaluate this integral with dimensional regularization, we obtain

$$i\Sigma_2(\not{p}) \propto \frac{i\lambda^2}{16\pi^2}\frac{M}{\varepsilon} + \dots \tag{18}$$

Now the counterterm that absorbs this divergence is proportional to M, the physical source of chiral symmetry breaking. Alternately, we could do this integral with a hard cutoff, in which case we obtain terms of the form

$$i\Sigma_2(\not{p}) \propto \frac{i\lambda^2}{16\pi^2}\frac{\Lambda^2}{M} + \dots \tag{19}$$

Now we can absorb this divergence with our renormalization procedure, via a ounterterm that is not strictly proportional to the physical source of chiral symmetry breaking. Made the mistake of choosing a regulator that violated a symmetry of the theory. Can still make sensible predictions, but only at the price of adding counterterms that compensate for our mistake. In general, much easier and wiser to simply choose a regulator that preserves the symmetries of the quantum action in the first place.

In this case, can also compute difference between pole mass and $\overline{MS}$ mass like we did for the scalar. Difference here is

$$M_P - M_{\overline{MS}} = \frac{\lambda^2}{16\pi^2}M + \dots \tag{20}$$

i.e., proportional to fermion mass, not scalar mass.

At this point, you should be very skeptical about the hierarchy problem. If we are only allowed to speak about observables, and we construct our perturbation theory in such a way as to absorb all divergences, then it is not at all clear that a hierarchy problem exists.

2.1 Two hierarchy problems

But can see two senses in which there is a problem.

1. Physical states correct Higgs mass proportional to their own mass, e.g. $\delta m^2 \propto M^2$. Independent of regularization and renormalization scheme. Threshold corrections. Long way from weak scale to Planck scale, lots of reasons to expect new physics. In fact, *forced upon you.* SM is an incomplete theory, must have more physics.

2. Higgs mass is incalculable in SM. Just a parameter. Can't predict it, only measure it and relate it. Expect instead in fundamental theory it is finite, calculable. What then?

Two problems often related but different in nature. First problem is clear. We saw that particles with mass M give $\delta m^2 \propto M^2$, independent of how we regularize theory. Given Clear that we can compute in lots of different ways if we know the specific physics.

Second problem also pretty clear. We can extend the SM to include symmetries that make the Higgs mass calculable in terms of fundmental parameters. Requires new particles, and will have $\delta m^2 \propto M^2$. Implies hierarchy problem in any calculable framework.

Original line of thinking (Wilson): if theory is finite, cutoff Λ has physical meaning (i.e., lattice spacing). Then no need for counterterms to avoid infinities; all cutoff-dependent terms are physical. Then it is physically meaningful to compute

$$\delta m_H^2 = \frac{\Lambda^2}{16\pi^2}\left(-6y_t^2 + \frac{9}{4}g^2 + \frac{3}{4}g'^2 + 6\lambda\right)$$

Don't absorb this into counterterms; it's a physical correction from high-momentum modes probing the physical cutoff.

But if you know the detailed physics, can do even better; compute in dim reg or in mass-dependent scheme; new states at the cutoff contribute in the same way.

So cutoff picture is a good way to parameterize our ignorance. Gives the same answer as a physical calculation in other regularization schemes. Only subtlety is that in general the cutoff Λ varies from loop to loop.

Makes clear that the *Naturalness Problem* is robust: new physics gives contributions to the Higgs mass, typically from different sources, and these must cancel to fine precision to give observed Higgs mass.

2.2 Three reasons

Two senses of hierarchy problem. Will address the latter sense (calculability) when we get to custodial symmetries. But in the meantime, first we can explore how much of a challenge is posed by physical contributions from different possible states beyond the Standard Model.

Three categories:

- things we want to believe (hints of BSM physics);
- things we should believe (very probable consequences of physics at Planck scale);
- things we must believe (problem of SM even if gravity does not provide cutoff).

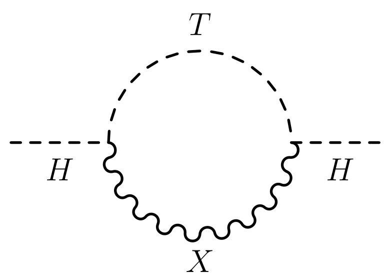
2.2.1 Things we want to believe

We have lots of reasons to expect field-theoretic scales above the weak scale. When do these introduce hierarchy problems? Focus on four things: unification, neutrino masses, flavor, and dark matter.

Unification One of the first concrete settings in which the hierarchy problem became apparent was that of grand unification. In grand unified theories there are heavy gauge bosons associated with the scale of unification that interact with the

Higgs boson.

Details depend on the precise model of unification, and the representation into which the Higgs is embedded. For example, in $SU(5)$ unification the SM gauge bosons are embedded into the 24 of $SU(5)$, which decomposes into the SM gauge bosons plus X gauge bosons transforming in the $(3,2)_{-5/6}$ + conjugate representation. Moreover, the Higgs is embedded in a $\bar{5}$ of $SU(5)$. In this case there are loops involving a triplet scalar Higgs and X boson of the form



In general, these loops of heavy bosons give corrections of order

$$\delta m_H^2 \sim \frac{\alpha_{GUT}}{4\pi} M_{GUT}^2 \tag{21}$$

The original apparent scale of unification in nonsupersymmetric theories was $\mathcal{O}(10^{15})$ GeV, while bounds on proton decay now imply $M_{GUT} \gtrsim 10^{16}$ GeV. So grand unification implies a huge hierarchy problem.

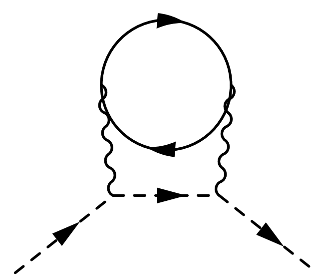
Hierarchy problems can be even worse than the one we see in quantum effects; it can be classical. For example, in SUSY GUT models there are Higgs multiplets in the 5 and $\bar{5}$ of $SU(5)$, and the triplet states must be heavy (on the order of the GUT scale) to avoid dimension-5 proton decay. Problem is unified mass term $\bar{5}^\dagger \bar{5}$.

Moreover, now the symmetries of the theory admit couplings to the heavy scalar Φ that breaks the $SU(5)$ unified symmetry, i.e. $\bar{5}\Phi 5$, and Φ acquires a GUT-scale vev to break the unified symmetry. This generically implies the masses of doublet and triplet Higgs bosons are on the order of the GUT scale from tree-level effects![1]

[1] This can be ameliorated in more complicated GUT models such as $SO(10)$ via the Dimopoulos-Wilczek mechanism, or in orbifold GUTs.

So *classical* naturalness problems are often even more of a threat than quantum ones. More generally, this implies that any scalars acquiring large vevs must have tremendously small couplings to the Higgs in order to avoid introducing new fine-tuning problems.

Let's now turn to the effects of new fermions. Very generally, consider adding a new fermion Ψ to the Standard Model, charged under $SU(2)_L \times U(1)_Y$. Even before trying to include Yukawa couplings to the Higgs, it gives corrections to the Higgs mass at two loops via diagrams of the form



which corrects the Higgs mass by an amount

$$\delta m_H^2 \sim \left(\frac{\alpha}{4\pi}\right)^2 \times g\left(\frac{m_W^2}{m_\Psi^2}\right) \times m_\Psi^2 \tag{22}$$

where g is a $\mathcal{O}(1)$ dimensionless function. Such states, if they exist, should be lighter than about 10 TeV in order to avoid introducing a fine-tuning problem.

We could further imagine that this new fermion couples to the Higgs directly with a Yukawa interaction. For example, there could be fermions Ψ, ψ such that the coupling

$$y\psi H\Psi \tag{23}$$

is allowed, or Ψ could be a new fermion with electroweak quantum numbers and ψ could be an existing SM chiral fermion. In this case there is a one-loop diagram feeding into the Higgs mass, with

$$\delta m_H^2 \sim C\frac{y^2}{16\pi^2}m_\psi^2 \tag{24}$$

Avoiding fine-tuning from this requires $ym_\psi \lesssim$ TeV.

Neutrino masses Now we can have a perfectly consistent universe without new electroweak fermions, but there are scenarios that favor the existence of new fermions. For example, the generation of neutrino masses may strictly be due to a dimension-five operator,

$$\mathcal{L} \supset \frac{(L^i H)(L^j H))}{M} + \text{h.c.} \tag{25}$$

without further ado. However, we expect that if the theory is genuinely renormalizable, this interaction arose from integrating out heavier states with mass $\sim M$. In particular, the Type-I seesaw entails right-handed neutrinos N with couplings

$$\mathcal{L} \supset -\frac{M_R^{ij}}{2} N_i N_j - y_{ij} L^i N^j H + \text{h.c.} \tag{26}$$

This provides a very concrete example of new fermions coupling to the Higgs. The leading one-loop correction to the Higgs mass is

$$\delta m_H^2 = -\frac{1}{4\pi^2} \sum_{ij} |y_{ij}|^2 M_j^2 \tag{27}$$

If all the RH neutrinos have a common mass M, the bound will be dominated by the combination of yukawas giving the heaviest SM neutrinos. In this case the naturalness bound is $M \lesssim 10^4$ TeV. This has amusing implications because thermal leptogenesis requires much higher values of M, on the order of $M \gtrsim 10^6$ TeV. So in this case naturalness would rule out thermal leptogenesis in a Type 1 see-saw.

Of course, people are infinitely clever, and there are other models for neutrino masses. Possible to generate Type II or Type III masses. All in tension with thermal leptogenesis.

Flavor New physics is also implied by flavor. A canonical example is the Froggatt-Nielsen mechanism which employs charges under a global $U(1)$ to generate viable flavor textures. The $U(1)$ is spontaneously broken by a scalar vev which, after integrating out some heavy fermions, leads to effective Yukawa interactions.

Concretely, the top and bottom masses can be generated by

$$\mathcal{L} \supset i\bar{D} \not{\partial} D - M_D \bar{D} D + y_1 \bar{Q}_3 D H + y_2 \bar{D} b_R \phi + \text{h.c.} \tag{28}$$

where ϕ is the scalar breaking the $U(1)$, and $D, \bar{D}$ is a vector-like set of quarks. The charges $[Q_3] = +1, [t_R] = +1, [D] = +1, [b_R] = 0, [H] = 0, [\phi] = +1$ allow us to write down the top yukawa as a marginal interaction, while the bottom yukawa is only generated once we integrate out $D, \bar{D}$, giving a bottom yukawa of order

$$y_b = y_1 y_2 \frac{\langle \phi \rangle}{M_D} \tag{29}$$

The same procedure can be repeated to generate yukawa couplings for lighter generations. The y_1 yukawa means we have new contributions to the Higgs mass at one loop, giving a correction

$$\delta m_H^2 = -\frac{6}{8\pi^2} |y_1|^2 M_D^2 \tag{30}$$

which requires $y_1 M_D \lesssim$ TeV. Can do it, but suggests new physics associated with flavor enters at a low scale.

Dark matter Know there is dark matter! Lots of possibilities. Minimal option is for new electroweak multiplet, gives conventional WIMP abundance. Now contributes to Higgs mass at least via two loops.

Quantum numbers			DM could	DM mass	$m_{\mathrm{DM}^\pm} - m_{\mathrm{DM}}$	Finite naturalness	σ_{SI} in
$\mathrm{SU}(2)_L$	$\mathrm{U}(1)_Y$	Spin	decay into	in TeV	in MeV	bound in TeV	$10^{-46}\,\mathrm{cm}^2$
2	1/2	0	EL	0.54	350	$0.4 \times \sqrt{\Delta}$	$(0.4 \pm 0.6)\,10^{-3}$
2	1/2	1/2	EH	1.1	341	$1.9 \times \sqrt{\Delta}$	$(0.3 \pm 0.6)\,10^{-3}$
3	0	0	HH^*	$2.0 \to 2.5$	166	$0.22 \times \sqrt{\Delta}$	0.12 ± 0.03
3	0	1/2	LH	$2.4 \to 2.7$	166	$1.0 \times \sqrt{\Delta}$	0.12 ± 0.03
3	1	0	HH, LL	$1.6 \to ?$	540	$0.22 \times \sqrt{\Delta}$	0.001 ± 0.001
3	1	1/2	LH	$1.9 \to ?$	526	$1.0 \times \sqrt{\Delta}$	0.001 ± 0.001
4	1/2	0	HHH^*	$2.4 \to ?$	353	$0.14 \times \sqrt{\Delta}$	0.27 ± 0.08
4	1/2	1/2	(LHH^*)	$2.4 \to ?$	347	$0.6 \times \sqrt{\Delta}$	0.27 ± 0.08
4	3/2	0	HHH	$2.9 \to ?$	729	$0.14 \times \sqrt{\Delta}$	0.15 ± 0.07
4	3/2	1/2	(LHH)	$2.6 \to ?$	712	$0.6 \times \sqrt{\Delta}$	0.15 ± 0.07
5	0	0	(HHH^*H^*)	$5.0 \to 9.4$	166	$0.10 \times \sqrt{\Delta}$	1.0 ± 0.2
5	0	1/2	stable	$4.4 \to 10$	166	$0.4 \times \sqrt{\Delta}$	1.0 ± 0.2
7	0	0	stable	$8 \to 25$	166	$0.06 \times \sqrt{\Delta}$	4 ± 1

In some cases, agreement between thermal abundance and naturalness bounds, though very constraining for larger $SU(2)_L$ multiplets.

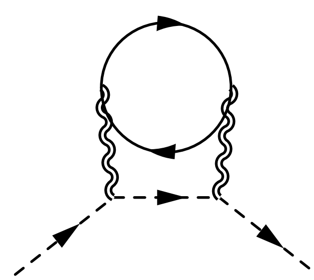
Already we see that vanilla field theory candidates for new physics are strongly constrained by naturalness arguments.

2.2.2 Things we should believe

But maybe we are willing to give up on all of these things. Neutrino masses could be dim-5 operator, dark matter a total SM singlet, unification an illusion, flavor a fact about matrices.

But some UV completion is forced upon us. We have already encountered the physics of quantum gravity at a scale $M_P \sim 10^{19}$ GeV. Do not have a complete theory of quantum gravity, although it is likely that the answer lies in string theory. Not yet able to compute the mass of the Higgs in a complete string theory, the expectation is that string theory contains heavy states whose masses are close to the Planck scale that would give corrections to the Higgs mass.

Clear that this is a problem, but make it even more apparent. Even new states coupling to the Higgs through loops of perturbative gravitons give a large threshold correction. For example, imagine there is some massive Dirac fermion Ψ with mass m_Ψ and it coupled to the Standard Model only gravitationally. Then as long as we are at energies $E \ll M_{Pl}$ we can compute loop diagrams including gravitons. The correction to the Higgs mass in this case arises at two loops,

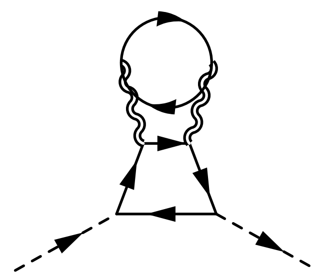


and gives a correction parametrically of order

$$\delta m_H^2 \sim \frac{m_H^2}{(16\pi^2)^2} \frac{m_\Psi^4}{M_{Pl}^4}$$

This correction is small because the graviton coupling to a massless, on-shell particle at zero momentum vanishes, and so the result is proportional to m_H.

However, we could also have a three-loop diagram where the graviton couples to a loop of top quarks,



The correction from this diagram is parametrically of the form

$$\delta m_H^2 \sim \frac{6y_t^2}{(16\pi^2)^3}\frac{m_\Psi^6}{M_{Pl}^4}$$

and is much larger because now the gravitons are coupling to off-shell states.

If $m_\Psi \sim M_{Pl}$, correction is $\sim \frac{6y_t^2}{16\pi^2}\frac{M_{Pl}^2}{(16\pi^2)^2}$. Of course at this point we doubt the validity of our gravity EFT, but this parametrically validates our naive expectation from the cutoff argument, now with $\Lambda \sim M_{Pl}/16\pi^2$. So even gravitational physics is sufficient to feed through threshold corrections to the Higgs mass.

The conclusion is that if there are *any* other states out there, even ones that only couple to the Higgs gravitationally, they give a threshold correction to the Higgs mass that is proportional to the mass scale of the new states. We can see these corrections in $\overline{MS}$ or any other scheme; they are physical threshold corrections and have unambiguous value. The result using a hard cutoff was merely a placeholder for threshold corrections, which we could only see in $\overline{MS}$ if we had actual physical states in the theory.

2.2.3 Things we must believe

Finally, one might hope that the theory of quantum gravity somehow decouples in such a way as to avoid inducing new scales for the Standard Model. If we are so fortunate as to imagine that gravity does not introduce a physical cutoff at the Planck scale, then we are faced with another problem.

Now there is nothing to cut off the running of Standard Model couplings as we go to higher and higher energies. This is problematic because the Standard Model contains an abelian gauge factor, hypercharge, whose beta function coefficient can only be positive. In the SM it is

$$16\pi^2\beta_{g_Y} = \frac{41}{10}g_Y^3 + \dots$$

Any additional physics simply makes it more positive. Running it up to the UV, the coupling grows in the ultraviolet and eventually hits a Landau pole around 10^{41} GeV. The Standard Model is truly an effective theory, and we expect that this Landau pole is UV completed into some new physics around 10^{41} GeV. Physical thresholds at this scale reintroduce a hierarchy problem.

One might hope that the situation is not so dire. We do not know for certain what a Landau pole implies, so perhaps something innocuous happens here as well. However, we can study strong coupling in an abelian gauge group on the lattice.

The evidence **hep-th/9712244** suggests that strong coupling of hypercharge leads to confinement and chiral symmetry breaking in the ultraviolet before the nominal Landau pole is reached. This generates both a nonzero fermion mass $m_f \sim \Lambda$ and a nonzero chiral condensate $\langle\psi\bar{\psi}\rangle \sim \Lambda$. This would break electroweak symmetry and give a huge contribution to the Higgs mass. So, indeed, the Landau pole is a problem, and new physics must enter to avoid it.

How to avoid if Planck scale does not provide a cutoff? Embed hypercharge into an asymptotically free non-abelian symmetry. Has to happen at low scale since we know

$$\delta m^2 \propto \frac{\alpha_{GUT}}{4\pi}M_{GUT}^2$$

Obvious candidates ($SU(5), SO(10)$, etc.) are in bad trouble as we have seen, since bounds on proton decay force scale to 10^{16} GeV.

Other options: to avoid proton decay,

- Pati-Salam, $SU(2)_L \times SU(2)_R \times SU(4)_{PS}$
- Trinification, $SU(3)_L \times SU(3)_R \times SU(3)_C$

Need at low scale to avoid fine-tuning. Many pheno consequences. New Higgs bosons, heavy vector bosons. Flavor constraints force $M_H \gtrsim 3$ TeV. Precision electroweak forces $M_{Z'} \gtrsim 4, 2$ TeV for Pati-Salam and trinification. Already bumping up on limits, naturalness bounds.

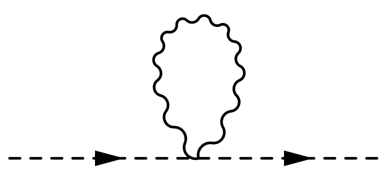
If this solves the problem, implies accessible physics near the weak scale. Seem unable to get around the hierarchy problem giving signs of new physics.

2.3 Naturalness in Nature

At this point it's reasonable to wonder if this is just wild theoretical prejudice or a sensible proposition. There are, in fact, many examples in nature where the expectation of naturalness has turned out to be sharp.

Favorite is the mass splitting between the charged and neutral pions, which differ by about 5 MeV. These states are all goldstones of the spontaneously broken chiral symmetries of QCD, and these symmetry arguments lead one to expect the pions to be nearly degenerate. The answer is that we have radiative corrections from the explicit breaking of chiral symmetries by QED. Charged pions and kaons can get a mass contribution from electromagnetic loops.

If we compute the photon loop that would give a mass correction, using a hard cutoff to estimate the threshold correction we get



$$\sim \frac{3e^2}{16\pi^2}\Lambda^2 = \delta m^2 \tag{31}$$

Given the size of the charged-neutral meson splittings, $m_{\pi^\pm}^2 - m_{\pi_0}^2 \sim (35.5\,\text{MeV})^2$, we expect the loop should be cut off around 850 MeV if electromagnetic loops explain the mass difference.

In fact, the ρ meson enters at 770 MeV, which provides a cutoff for the effective theory. Thus there is perfect agreement between the size of the mass correction based on cutoff-based arguments and the scale at which new physics enters.

Another beautiful example is the mass difference between the K_L^0 and K_S^0 states.

Computed in the effective theory at the scale of the kaons, the splitting is

$$\frac{M_{K_L^0} - M_{K_S^0}}{M_{K_L^0}} = \frac{G_F^2 f_K^2}{6\pi^2} \sin^2\theta_c \Lambda^2 \tag{32}$$

where $f_K = 114$ MeV is the kaon decay constant and $\sin\theta_C = 0.22$ is the Cabibbo angle. Requiring this correction to be smaller than the measured value $(M_{K_L^0}?M_{K_S^0})/M_{K_L^0} = 7\times10^{-15}$ gives $\Lambda < 2$ GeV. And lo, the charm quark enters with mass $m_c \sim 1.2GeV$ to modify the short-distance behavior of the theory by implementing the GIM mechanism. Moreover, this is not merely rationalization; this was the actual argument used by Gaillard and Lee to compute the mass of the charm quark before its discovery.

On the other hand, the cosmological constant is a tremendous failure of naturalness. The observed value of the c.c. is on the order of $(2.4\times10^{-3}\text{ eV})^4$. The prediction obtained by simply computing vacuum loops up to a cutoff Λ is proportional to Λ^4 itself. There is no apparent new physics at the eV scale related to cutting off contributions to the vacuum, and even if these loops were cutoff not far above the Planck scale, there would be many orders of magnitude of unexplained hierarchy.

The successes and failures of naturalness in nature should make clear the fact that

Naturalness is a strategy, not a principle.

That is to say, it has often provided the correct guidance for new physics. However, it has also failed spectacularly in the case of the cosmological constant, and it's unclear precisely what lessons we should draw from the failure.

So now we come back to the Higgs mass and naturalness. Using a hard cutoff on Standard Model loops, we have again

$$\delta m_H^2 = \frac{\Lambda^2}{16\pi^2}\left(-6y_t^2 + \frac{9}{4}g^2 + \frac{3}{4}g'^2 + 6\lambda\right)$$

In order for this not to be much larger than the observed mass, we require $\Lambda \lesssim$ TeV.

3 Custodial Symmetries

Have now seen two senses of hierarchy problem:

1. Physical states correct Higgs mass proportional to their own mass, lots of room up to Planck scale and reasons for new physics.

2. Higgs mass is incalculable in SM, expect it is finite in fundamental theory.

Can address both these problems at same time with custodial symmetries.

So far, we have seen that the masses of scalars are special. They are not protected against radiative corrections by any symmetry, and so they are sensitive to all of the mass scales to which they are connected. This is in contrast to fermions and gauge bosons, which enjoy enhanced symmetry when mass is zero, enforcing $\delta m \propto m$.

Idea is to enlarge Standard Model so that Higgs also enjoys a custodial symmetry. Historically, not the only way to solve hierarchy problem. Could lower cutoff (large extra dimensions). Could break electroweak symmetry w/ strong condensate (technicolor). But now we have seen a light, apparently elementary Higgs scalar giving EWSB. Therefore our focus narrows to UV physics that solves hierarchy problem for a light, approximately elementary scalar.

What possible symmetries can we use? Coleman-Mandula theorem constrains options:

The Coleman-Mandula theorem (1967): *in a theory with non-trivial interactions (scattering) in more than 1+1 dimensions, the only possible conserved quantities that transform as tensors under the Lorentz group are the energy-momentum vector P_μ, the generators of Lorentz transformations $M_{\mu\nu}$, and possible scalar symmetry charges Z_i corresponding to internal symmetries, which commute with both P_μ and $M_{\mu\nu}$.* For theories with only massless particles, this can be extended to include generators of conformal transformations.

Generalizes to include spinorial symmetry charges: SUSY. Golfand and Likhtman, full set of possible generalizations were identified by Haag, Sohnius, and Lopuszanski.

So possible options seem to be: Spinorial internal symmetry (supersymmetry); scalar internal symmetry (global or gauge symmetry); and conformal symmetry.

3.1 Supersymmetry

Assume familiarity with SUSY. Idea is to extend Poincare algebra to include conserved supercharges $Q_\alpha, \tilde{Q}_{\dot{\alpha}}$. As a Weyl spinor, the transformation properties of Q_α with respect to the Poincare group are known, namely

$$[P_\mu, Q_\alpha] = [P_\mu, \tilde{Q}^{\dot{\alpha}}] = 0$$
$$[M^{\mu\nu}, Q_\alpha] = i(\sigma^{\mu\nu})_\alpha^\beta Q_\beta$$
$$[M^{\mu\nu}, Q^{\dot{\alpha}}] = i(\bar{\sigma}^{\mu\nu})^{\dot{\alpha}}_{\dot{\beta}} \tilde{Q}^{\dot{\beta}}$$

Also need anticommutators $\{Q, \tilde{Q}\}$ and $\{Q, Q\}$ to close the algebra. As we saw above, the only option is for $\{Q, \tilde{Q}\}$ to be proportional to $P_{\alpha\dot{\beta}}$, since this is the only conserved operator with the appropriate index structure. The choice of normalization gives us

$$\{Q_\alpha, \tilde{Q}_{\dot{\beta}}\} = 2P_\mu(\sigma^\mu)_{\alpha\dot{\beta}}$$

Finally, the simplest choice for $\{Q_\alpha, Q_\beta\}$ is

$$\{Q_\alpha, Q_\beta\} = \{\tilde{Q}_{\dot{\alpha}}, \tilde{Q}_{\dot{\beta}}\} = 0$$

Fields will be arranged into supermultiplets, transforming as irreducible representations of super-Poincare.

For example, chiral multiplet contains scalar and fermion related by infinitesimal SUSY rotation,

$$\phi \to \phi + \delta\phi \qquad \psi \to \psi + \delta\psi$$

where

$$\delta\phi = \epsilon^\alpha \psi_\alpha \tag{33}$$
$$\delta\psi_\alpha = -i(\sigma^\nu \epsilon^\dagger)_\alpha \partial_\nu \phi \tag{34}$$

where ϵ_α is a Grassmann variable that you can think of as an infinitestimal parameter multiplying a SUSY generator; it has mass dimension $[\epsilon] = -1/2$.

Relates a scalar to a fermion, and so relates a scalar mass to a fermion mass protected by chiral symmetry.

Note the Coleman-Mandula theorem told us that all global symmetries (and thus also all gauge symmetries obtained by gauging them) must commute with the generators of the Poincare group. However, it is not strictly necessary for them to commute with all the generators of the super-Poincare group.

Associativity of the super-Poincare algebra implies that in a $\mathcal{N}=1$ supersymmetric theory there can be at most one independent Hermitian $U(1)$ generator R that does not commute with the SUSY generators, with commutation relations

$$[R, Q_\alpha] = -Q_\alpha \qquad [R,Q^\dagger_{\dot\alpha}] = Q^\dagger_{\dot\alpha}$$

I won't prove this in total generality, but let's sketch how the argument works. Imagine there were a global symmetry algebra with hermitian generators T^a satisfying $[T^a, T^b] = if^{ab}_c T^c$ (where we are being very general, and not yet assuming anything about the Killing form) that didn't commute with supersymmetry,

$$[T^a, Q_\alpha] = h^a Q_\alpha$$

for some h^a. Now the Jacobi identity

$$[T^a, [T^b, Q]] + [T^b, [Q, T^a]] + [Q, [T^a, T^b]] = 0$$

implies $f^{ab}_c h^c = 0$. In general, any scalar symmetry algebra is a direct sum of a semi-simple algebra and an abelian algebra. For a semi-simple Lie algebra we can go to a basis where the Killing form is diagonal and f^{abc} is antisymmetric, so $0 = f_{abc}h^c = f^{bad} f_{abc} h^c \propto h^d$. Then only the components of h^a in the Abelian directions can be non-zero.

Of the abelian directions, we can form a single linear combination having a nonzero commutator with the Qs, as above. This argument extends to gauge symmetries, which we can think of as gauging global symmetries. Consistently gauging the R-symmetry takes us into supergravity.

For extended supersymmetries with more supercharges, we can have non-abelian R-symmetries; in 4D we can have an $SU(4)$ R-symmetry for $\mathcal{N}=4$ theories, and an $SU(2)\times U(1)$ R-symmetry for $\mathcal{N}=2$. But chiral states only possible for $\mathcal{N}=1$

Thus we learn that states related by SUSY carry the same gauge quantum numbers.

Names		spin 0	spin 1/2	$SU(3)_C,\ SU(2)_L,\ U(1)_Y$
squarks, quarks	Q	$(\widetilde{u}_L\ \ \widetilde{d}_L)$	$(u_L\ \ d_L)$	$(\,\mathbf{3},\ \mathbf{2}\,,\ \frac{1}{6})$
(×3 families)	$\overline{u}$	$\widetilde{u}^*_R$	$u^\dagger_R$	$(\,\overline{\mathbf{3}},\ \mathbf{1},\ -\frac{2}{3})$
	$\overline{d}$	$\widetilde{d}^*_R$	$d^\dagger_R$	$(\,\overline{\mathbf{3}},\ \mathbf{1},\ \frac{1}{3})$
sleptons, leptons	L	$(\widetilde{\nu}\ \widetilde{e}_L)$	$(\nu\ \ e_L)$	$(\,\mathbf{1},\ \mathbf{2}\,,\ -\frac{1}{2})$
(×3 families)	$\overline{e}$	$\widetilde{e}^*_R$	$e^\dagger_R$	$(\,\mathbf{1},\ \mathbf{1},\ 1)$
Higgs, higgsinos	H_u	$(H^+_u\ \ H^0_u)$	$(\widetilde{H}^+_u\ \ \widetilde{H}^0_u)$	$(\,\mathbf{1},\ \mathbf{2}\,,\ +\frac{1}{2})$
	H_d	$(H^0_d\ \ H^-_d)$	$(\widetilde{H}^0_d\ \ \widetilde{H}^-_d)$	$(\,\mathbf{1},\ \mathbf{2}\,,\ -\frac{1}{2})$

Table 2: Chiral supermultiplets in the Minimal Supersymmetric Standard Model. The spin-0 fields are complex scalars, and the spin-1/2 fields are left-handed two-component Weyl fermions.

Names	spin 1/2	spin 1	$SU(3)_C,\ SU(2)_L,\ U(1)_Y$
gluino, gluon	$\widetilde{g}$	g	$(\,\mathbf{8},\ \mathbf{1}\,,\ 0)$
winos, W bosons	$\widetilde{W}^\pm\ \ \widetilde{W}^0$	$W^\pm\ \ W^0$	$(\,\mathbf{1},\ \mathbf{3}\,,\ 0)$
bino, B boson	$\widetilde{B}^0$	B^0	$(\,\mathbf{1},\ \mathbf{1}\,,\ 0)$

Table 3: Gauge supermultiplets in the Minimal Supersymmetric Standard Model.

The supersymmetric extension of the Standard Model is fairly straightforward, entailing the incorporation of all Standard Model fields into corresponding supermultiplets, with the addition of a second Higgs multiplet. This is necessary on account of both anomalies and holomorphy.

You might be cheeky and try to put the Higgs scalar into a lepton chiral multiplet, so that the Higgs is effectively a sneutrino. This turns out to have various problems – lepton number non-conservation is one of them, as well as an unviably large prediction for (at least one) neutrino mass.

Of course, supersymmetry cannot be an exact symmetry of nature, otherwise we would have seen selectrons degenerate with electrons. So in general we must include soft terms, which can be worked out using the generalization of our spurion technique; in the case of the MSSM these take the form

$$\begin{aligned}\mathcal{L}_{\rm soft}^{\rm MSSM} &= -\frac{1}{2}\left(M_3\widetilde{g}\widetilde{g} + M_2\widetilde{W}\widetilde{W} + M_1\widetilde{B}\widetilde{B} + {\rm h.c.}\right) \\ &\quad -\left(\widetilde{\overline{u}}\,\mathbf{a_u}\,\widetilde{Q}H_u - \widetilde{\overline{d}}\,\mathbf{a_d}\,\widetilde{Q}H_d - \widetilde{\overline{e}}\,\mathbf{a_e}\,\widetilde{L}H_d + {\rm c.c.}\right) \\ &\quad -\widetilde{Q}^\dagger\,\mathbf{m_Q^2}\,\widetilde{Q} - \widetilde{L}^\dagger\,\mathbf{m_L^2}\,\widetilde{L} - \widetilde{\overline{u}}\,\mathbf{m_{\overline{u}}^2}\,\widetilde{\overline{u}}^\dagger - \widetilde{\overline{d}}\,\mathbf{m_{\overline{d}}^2}\,\widetilde{\overline{d}}^\dagger - \widetilde{\overline{e}}\,\mathbf{m_{\overline{e}}^2}\,\widetilde{\overline{e}}^\dagger \\ &\quad - m_{H_u}^2 H_u^* H_u - m_{H_d}^2 H_d^* H_d - (bH_uH_d + {\rm c.c.})\,. \end{aligned} \tag{35}$$

SUSY accomplishes everything we want from custodial symmetry. Places Higgs scalar in a supermultiplet with a Higgs fermion, the Higgsino. Mass of the Higgsino is protected by chiral symmetry. So since $\delta\mu \sim \mu$ for higgsino, same holds from Higgs.

SUSY broken by soft terms, but corrections due to breaking proportional to these terms.

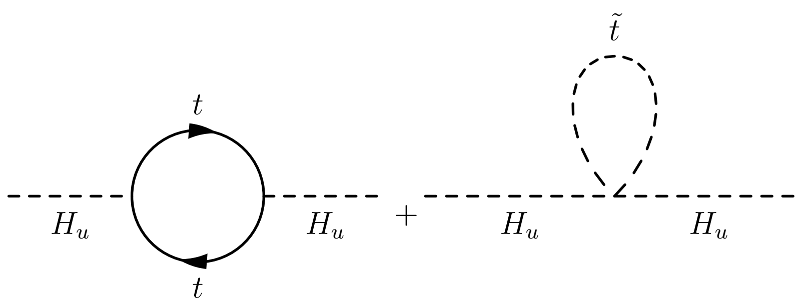
Still have a just-so problem. Symmetries allow masses

$$\mu\psi_u\psi_d + {\rm h.c.} + \mu^2|H_u^2| + \mu^2|H_d|^2$$

Need to explain value of μ, but once set, protected.

Now Higgs mass is calculable. Lots of ways to see it. Simple way: know particle content of MSSM, allow also unknown new physics at scale Λ. Assume new physics respects the symmetry. Then can compute loops up to cutoff as way of parameterizing our ignorance.

Relative to SM, now cancellations between loops of opposite statistics, e.g. top-stop loop



And find

$$\delta m_{H_u}^2 = -\frac{6y_t^2}{16\pi^2}\Lambda^2 + \frac{6y_t^2}{16\pi^2}\Lambda^2 - \frac{3y_t^2}{4\pi^2}m_{\tilde{t}}^2 \ln(\Lambda/m_{\tilde{t}}) + \ldots \tag{36}$$

The quadratic pieces cancel. No UV sensitivity! Key assumption is that Λ is same for both loops, true for UV physics respecting supersymmetry. Left only with physical threshold corrections (can compute in any scheme) from hew heavy states. At most logarithmic sensitivity to Λ. Can fix even this by writing down explicit theory to break SUSY.

Now that mass is finite, can use naturalness argument to determine where the new particles should enter. Now we see the hierarchy problem very explicitly. Rendered the Higgs mass calculable; now depends on masses of new partner particles. Masses can't be too large!

Two direct sources of concern, tree-level contributions and loop-level contributions. Both play a role primarily through the relation between the weak scale and soft parameters, viz.

$$m_h^2 = -2(m_{H_u}^2 + |\mu|^2) + \ldots \tag{37}$$

Then corrections to Higgs mass come from three places:

- The first is the tree-level potential, which involves certain combinations of soft masses that set the weak scale vev. At tree-level the naturalness of the weak scale implies something about the soft parameters $m_{H_u}^2$ and μ, which itself controls the higgsino masses. Higgsinos should be light! Naturalness suggests $\mu \lesssim 200$ GeV and correspondingly light Higgsinos.
- The second is immediate loop-level corrections. The soft mass parameter $m_{H_u}^2$ accumulates one-loop corrections from other soft parameters. By far the largest is due to the stops, since the top chiral superfields couple most strongly to H_u, with correction of order

  $$\delta m_{H_u}^2 = -\frac{3y_t^2}{4\pi^2}m_{\tilde{t}}^2 \ln(\Lambda/m_{\tilde{t}}) \tag{38}$$

  Naturalness requires stops ~ 400 GeV with a cutoff $\Lambda \sim 10$ TeV. Other particles are also tied to naturalness, though less directly. After the SM top loop, the gauge and Higgs loops drive the mass corrections, so unsurprisingly the wino and higgsino corrections play a role, with

  $$\delta m_{H_u}^2 = -\frac{3g^2}{8\pi^2}(m_{\tilde{W}}^2 + m_{\tilde{h}}^2) \ln(\Lambda/m_{\tilde{W}}) \tag{39}$$

Having already bounded Higgsinos, for winos this translates to $m_{\tilde{W}} \lesssim$ TeV. Note that sbottoms need not be directly connected to naturalness, but since the left-handed sbottom gauge eigenstate transforms in the same $SU(2)$ multiplet as the left-handed stop gauge eigenstate, at least one sbottom is typically found in the same mass range as the stops.

- The third is two-loop corrections, due to the naturalness of other sparticles. The stop mass is corrected by the gluino mass due to the size of g_3, so it is hard to separate the gluino substantially from the stops, with

$$\delta m_{\tilde{t}}^2 = \frac{2g_s^2}{3\pi^2} m_{\tilde{g}}^2 \ln\left(\Lambda/m_{\tilde{g}}\right) \tag{40}$$

  which ties $m_{\tilde{g}} \lesssim 2m_{\tilde{t}}$. Indeed, these corrections typically tie the masses of the gluino and all squark flavors quite tightly given even a modest amount of running.

Thus: SUSY provides a custodial symmetry to Higgs by relating mass to fermion. Makes clear the hierarchy problem in terms of threshold corrections from new scales. Implies new physics very close to weak scale, with known Standard Model quantum numbers and interactions fixed by symmetry.

3.2 Global symmetry

Now let's consider the second possible custodial symmetry, a global symmetry. The idea here is to make the Higgs boson a PNGB of a spontaneously broken global symmetry. Then the PNGB shift symmetry protects against a Higgs mass.

Let's go through a toy model of how the global symmetry story might work. Consider a triplet $\vec{\Phi}$ of real scalar fields, described by the Lagrangian

$$\mathcal{L} = \frac{1}{2}\partial_\mu \vec{\Phi}^T \partial^\mu \vec{\Phi} - \frac{g_*^2}{8}\left(\vec{\Phi}^T\vec{\Phi} - f^2\right)^2 . \tag{41}$$

For now, let's imagine this is a weakly coupled theory. The theory is invariant under SO(3) transformations acting on $\vec{\Phi}$ as

$$\vec{\Phi} \;\rightarrow\; g \cdot \vec{\Phi}\,, \qquad g = e^{i\alpha_A T^A} \in \mathrm{SO}(3)\,, \tag{42}$$

where the SO(3) generators, normalized to $\mathrm{Tr}[T^A T^B] = \delta^{AB}$, can be conveniently chosen as $T^A = \{T, \widehat{T}^i\}$

$$T = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \widehat{T}^i = \left\{ \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{bmatrix}, \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{bmatrix} \right\}, \tag{43}$$

with $i = 1, 2$. Geometrically, the three generators correspond to rotations in the 1-2, 1-3 and 2-3 planes.

When the field $\vec{\Phi}$ acquires a VEV, this breaks SO(3) to SO(2). There are 3 and 1 generators in these groups, leaving two broken generators. The tree-level minimization condition reads $\langle \vec{\Phi}^T \rangle \langle \vec{\Phi} \rangle = f^2$, so that the manifold of equivalent vacua is the two-sphere. We can choose the vacuum to be

$$\vec{F} = \begin{bmatrix} 0 \\ 0 \\ f \end{bmatrix}. \tag{44}$$

In order to study the fluctuations around the vacuum it is convenient to perform a field redefinition and to trade the three $\vec{\Phi}$ components for one radial coordinate σ plus two "angular" variables $\pi_{1,2}$ (the Goldstone fields) describing the fluctuations around the broken generators. We write

$$\vec{\Phi} = e^{i \frac{\sqrt{2}}{f} \pi^i(x) \widehat{T}_i} \begin{bmatrix} 0 \\ 0 \\ f + \sigma(x) \end{bmatrix}, \tag{45}$$

normalized to give canonical kinetic terms for the π_i. The exponential matrix is the "Goldstone matrix" $U[\pi]$, which can be defined for any $\mathcal{G} \to \mathcal{H}$ breaking and appears in composite Higgs models.

Explicitly,

$$U[\pi] = e^{i \frac{\sqrt{2}}{f} \pi_i(x) \widehat{T}^i} = \begin{bmatrix} \mathbf{1} - \left(1 - \cos\frac{\pi}{f}\right) \frac{\vec{\pi}\vec{\pi}^T}{\pi^2} & \sin\frac{\pi}{f} \frac{\vec{\pi}}{\pi} \\ -\sin\frac{\pi}{f} \frac{\vec{\pi}^T}{\pi} & \cos\frac{\pi}{f} \end{bmatrix}, \tag{46}$$

where $\pi = \sqrt{\vec{\pi}^T\vec{\pi}}$. In this parameterization we have

$$\vec{\Phi} = (f+\sigma)\left[\begin{array}{c} \sin\frac{\pi}{f}\frac{\vec{\pi}}{\pi} \\ \cos\frac{\pi}{f} \end{array}\right] . \tag{47}$$

The new variables furnish a full one-to-one parametrization of the field space, provided the Goldstones are restricted to the region $\pi \in [0, \pi f)$. Expanding out the Lagrangian we obtain

$$\begin{aligned} \mathcal{L} &= \frac{1}{2}\partial_\mu\sigma\partial^\mu\sigma - \frac{(g_* f)^2}{2}\sigma^2 - \frac{g_*^2 f}{2}\sigma^3 - \frac{g_*^2}{8}\sigma^4 \\ &+\frac{1}{2}\left(1+\frac{\sigma}{f}\right)^2\left[\frac{f^2}{\pi^2}\sin^2\frac{\pi}{f}\partial_\mu\vec{\pi}^T\partial^\mu\vec{\pi} + \frac{f^2}{4\pi^4}\left(\frac{\pi^2}{f^2}-\sin^2\frac{\pi}{f}\right)\partial_\mu\pi^2\partial^\mu\pi^2\right] . \end{aligned} \tag{48}$$

Unsurprisingly, the Lagrangian contains an infinite set of two-derivative local interactions. In analogy with the theory of QCD pions, where the role of f is played by the pion decay constant f_π, we will sometimes refer to f as the "Higgs decay constant". In agreement with the Goldstone theorem the π's describe two massless bosons associated with the two broken generators $\widehat{T}^{1,2}$.

The σ field has a mass

$$m_* = g_* f\,. \tag{49}$$

In analogy with a strongly coupled sector, which we would like to mimic by our example, the σ particle is called a "resonance".

Now the goldstone Lagrangian has an SO(2) symmetry under which $\vec{\pi}$ forms a doublet and transforms as

$$\vec{\pi} \to e^{i\alpha\sigma_2}\vec{\pi}\,. \tag{50}$$

This is a "linearly realized" symmetry as it acts in a linear and homogeneous way on the field variables. We can switch to more suggestive notation by writing

$$H = \frac{\pi_1 - i\,\pi_2}{\sqrt{2}}\,, \tag{51}$$

which we identify with the (abelian) Higgs field, with unit charge under U(1) = SO(2). Obviously, the linearly realized SO(2) invariance comes from an SO(3) rotation along the unbroken generator T

$$\vec{\pi} \to e^{i\alpha\sigma_2}\vec{\pi} \;\;\Leftrightarrow\;\; \vec{\Phi} \to e^{i\sqrt{2}\alpha T}\vec{\Phi}\,. \tag{52}$$

The goldstones also nonlinearly realize the symmetry of the broken generators,

$$\vec{\pi} \to \vec{\pi} + \pi \cot \frac{\pi}{f}\, \vec{\alpha} + \left(\frac{f}{\pi} - \cot \frac{\pi}{f} \right) \left(\vec{\alpha}^T \vec{\pi} \right) \frac{\vec{\pi}}{\pi}\,, \tag{53}$$
$$\Updownarrow$$
$$\vec{\Phi} \to \vec{\Phi} + i\alpha_i \widehat{T}^i \vec{\Phi}\,.$$

These symmetries of the goldstone Lagrangian forbid the generation of a mass term.

We can see that the goldstones are protected against $SO(3)$-symmetric UV corrections directly. In terms of the field Φ, cutoff-dependent corrections could give it a mass of the form

$$\Lambda^2 \vec{\Phi}^T \vec{\Phi} \to \Lambda^2 (f+\sigma)^2$$

i.e., independent of the π fields. As goldstones, the potential of the π is independent of any $SO(3)$ symmetric potential.

Now that a NGB Higgs scalar has been obtained the last ingredient to construct the Abelian Higgs model is a U(1) gauge field. We do this by gauging the unbroken U(1) subgroup, namely by replacing in the original Lagrangian

$$\partial_\mu \vec{\Phi} \;\to\; D_\mu \vec{\Phi} = \left(\partial_\mu - i\,\sqrt{2} e\, A_\mu T \right) \vec{\Phi}\,, \tag{54}$$

where A_μ is a U(1) gauge field with canonical kinetic term. The gauging, since it selects one generator among three, breaks SO(3) explicitly to SO(2). The composite Higgs has now became a pNGB.

We can finally write down our Abelian composite Higgs theory. The only effect of the gauging is to turn ordinary derivatives into covariant ones in Eq. (48), with

$$D_\mu \vec{\pi} = \left(\partial_\mu - i\, e\, A_\mu \sigma_2 \right) \vec{\pi}\,. \tag{55}$$

In terms of H the Lagrangian is

$$\frac{1}{2}\left(1+\frac{\sigma}{f}\right)^2 \quad \left[\frac{f^2}{|H|^2} \sin^2 \frac{\sqrt{2}|H|}{f} D_\mu H^\dagger D^\mu H \right. \tag{56}$$
$$\left. + \frac{f^2}{4|H|^4} \left(2\frac{|H|^2}{f^2} - \sin^2 \frac{\sqrt{2}|H|}{f} \right) \left(\partial_\mu |H|^2 \right)^2 \right]\,.$$

while the σ field Lagrangian remains unchanged.

Now that the Goldstone symmetry has been broken by the gauging two new important features emerge. The explicit breaking of the global symmetry by gauging the $U(1)$ means that the PNGB can now obtain a radiative potential. In particular, now it can obtain mass from gauge loops up to the cutoff of the form

$$\delta m_H^2 \sim \frac{e^2}{16\pi^2}\Lambda^2$$

This is because the gauge coupling violates the $SO(3)$ symmetry explicitly.

Second, the Higgs can acquire a vev and break the $U(1)$. By setting the Higgs to its VEV

$$H = \langle H \rangle \equiv \frac{V}{\sqrt{2}}\,, \tag{57}$$

the first term in the square bracket of Eq. (57) gives to the gauge field a mass

$$m_A = ef \sin\frac{V}{f} \equiv e\,v\,. \tag{58}$$

In the second equality of the above equation we have defined the scale v of U(1) symmetry breaking in analogy with the ordinary elementary Abelian Higgs mass formula. In the latter case the scale v is directly provided by the Higgs field VEV while in the composite case

$$v = f\sin\frac{V}{f} \quad \Rightarrow \quad \xi = \frac{v^2}{f^2} = \sin^2\frac{V}{f}\,. \tag{59}$$

Of course, this is just a toy model. If we want the SM Higgs doublet to be a PNGB, we need four broken generators. We also need the unbroken group to contain $SU(2) \times U(1)$. A nice possibility is $SO(5) \to SO(4) \simeq SU(2)_L \times SU(2)_R$, which has four broken and six unbroken generators. Our whole argument goes through in a natural generalization.

Moreover, we have just written down a linear sigma model. You should worry that the Φ fields are elementary scalars and they have a hierarchy problem of their own. There are many possible solutions. One is for supersymmetry to enter at some scale to protect the mass of the Φ.

The other, most common, is to actually assume that the global symmetry is broken by strong dynamics that become important at the scale Λ. Typically then we require $\Lambda \sim 4\pi f$. There is no elementary scalar Φ. There is still a set of goldstones parameterizing the vacuum manifold, with interactions given by the NLSM.

There is a final problem, however. We have written down a model in which we identify the Higgs boson with a goldstone of a spontaneously broken global symmetry. As such, its mass is protected against corrections respecting the global symmetry, but not against corrections breaking it. In the Standard Model we have yukawa and gauge couplings that manifestly violate the $SO(5)$ symmetry.

So even if the Higgs is a PNGB, if the theory has a cutoff Λ, then loops of SM states probing the cutoff still give

$$\delta m_H^2 = \frac{\Lambda^2}{16\pi^2}\left(-6y_t^2 + \frac{9}{4}g^2 + \frac{3}{4}g'^2 + 6\lambda\right)$$

and we have not made much progress relative to the Standard Model. We have made some progress – if the theory grows strong at a scale Λ, we have explained how to generate a light Higgs-like scalar at $m_H^2 \sim \frac{y_t^2}{16\pi^2}\Lambda^2$.

But this is not actually good enough. At the scale Λ, compositeness introduces lots of new resonances with Standard Model quantum numbers. These are strongly constrained by precision electroweak constraints, which force $\Lambda \gtrsim 5$ TeV. Now there is a problem – if the cutoff is this high, loop corrections to the Higgs are too large.

A solution is to extend the Standard Model sector so that some or all of its couplings respect the global symmetry. Consider the analog of the top quark sector in our toy model. A yukawa coupling of the form

$$\mathcal{L} \supset -y_t H t_L t_R^\dagger$$

explicitly breaks the $SO(3)$ symmetry and gives mass to H at one loop. Instead, we can imagine extending the theory so that there is an $SO(3)$ triplet of top quarks, $\chi_L = (\sigma^2 t_L, T_L)$, an extra right-handed top multiplet T_R, and $SO(3)$ invariant Yukawas of the form

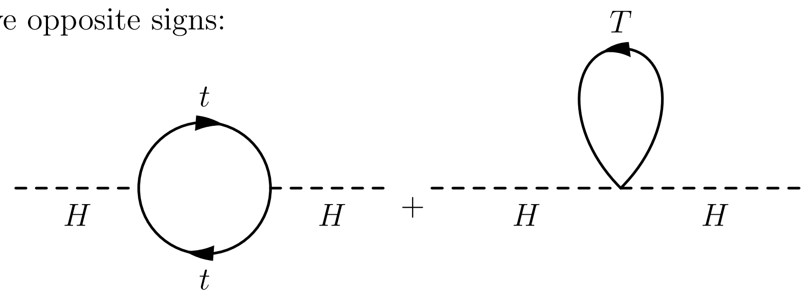
$$\mathcal{L} \supset -y_1(\vec{\Phi} \cdot \chi_L)t_R^\dagger - y_2 f T_L T_R^\dagger$$

Expanding out in terms of the goldstones, diagonalizing fermion masses, and integrating out the heavy mode σ, we end up with leading terms of the form

$$\mathcal{L} = -y_t t_R H t_L + \frac{y_t^2}{m_T} H^\dagger H T_R^\dagger T_L + \ldots$$

Now the T_L and T_R play the role of heavy fermionic top partners, much like the scalar top partners in SUSY. The specific form of their couplings is dictated by the pattern of spontaneous symmetry breaking.

Now if we compute loops up to the cutoff there is again a cancellation, since these terms have opposite signs:



which gives us

$$\delta m_H^2 = -\frac{6y_t^2}{16\pi^2}\Lambda^2 + \frac{6y_t^2}{16\pi^2}\Lambda^2 - \frac{3y_t^2}{4\pi^2} m_T^2 \ln\left(\Lambda/m_T\right) + \ldots \tag{60}$$

in exact analogy with the SUSY case, except now with fermionic top partners. This occurs because now the yukawas respect the global symmetry.

The story is now very parallel to SUSY. We can compute the mass of the Higgs, and in general we find that we require new partner particles that should be close to the weak scale.

3.3 Scale invariance?

Finally, let's turn to the last possibility, scale invariance (I will not distinguish between conformal symmetry and scale invariance.)

It has often been proposed that scale invariance might also serve to protect the Higgs. The general idea is that now it is invariance under scale transformations

$$x^\mu \to e^\sigma x^\mu \qquad \phi \to e^{-\sigma\Delta}\phi$$

that protects the Higgs, where Δ is the scaling dimension. Classically, a QFT is scale invariant if only marginal operators appear in the Lagrangian. Quantum mechanically, scale invariance is typically broken by quantum effects unless the theory is at a conformal fixed point.

Classical scale invariance can't do anything useful. It's almost a completely vacuous proposition. If the SM is a classically scale invariant sector, other scales can still feed in to the Higgs mass. If gravity doesn't give a scale, we hit the hypercharge Landau pole.

The sensible possibility is quantum scale invariance. Of course, the SM is not conformally invariant, so it must undergo a transition to a conformal fixed point at some scale.

There are two possibilities. The first is that the Standard Model and gravitational physics all merge into a conformal fixed point in the ultraviolet. In this case there are no physical thresholds associated with the gravitational sector. The second is that just the mundane non-gravitational physics of the Standard Model (plus potential other field theory states) merge into a CFT before the Planck scale, and assume that the unspecified physics of quantum gravity respects this conformal symmetry. This is an assumption.

This conformal fixed point should be interacting, rather than free, because both hypercharge and gravitational couplings grow large in the ultraviolet.

Either way, we do not have a concrete model. But we know general features. If this transition involves additional heavy states, then these new states will give threshold corrections to the Higgs and must enter near the weak scale to avoid reintroducing a hierarchy problem.

Alternately, perhaps the physics involed in the transition to an interacting fixed point in the UV is nonperturbative, and there are no apparent threshold corrections. This raises the question of how such a transition might impact the Higgs mass, if at all. This was studied beautifully by Schmaltz, Skiba, and Marquez-Tavares.

To study this, they computed corrections to the Higgs mass coming from loops in position space.

$$\delta m_h^2 = -iy_t^2\mu^{4-d}\int d^dx\langle 0|T\mathcal{O}^\dagger(x)\mathcal{O}(0)|0\rangle \tag{61}$$

For example, this reproduces the usual momentum-space result for top quark loops upon plugging in the normal-ordered operator $\mathcal{O} = \bar{t}P_L t$, contracting fields with propagators, and doing the integral over x:

$$\begin{aligned}
& \int d^dx\langle 0|T(\bar{t}P_R t)(x)(\bar{t}P_L t)(0)|0\rangle \\
= \quad & -N_c\int d^dx\,\mathrm{tr}\left[\int d^d\tilde{p}e^{ipx}P_L\frac{i(\not{p}+m_t)}{p^2-m_t^2+i\epsilon}P_R\int d^d\tilde{q}e^{-iqx}\frac{i(\not{q}+m_t)}{q^2-m_t^2+i\epsilon}\right] \\
= \quad & 2N_c\int d^d\tilde{p}\frac{p^2}{(p^2-m_t^2+i\epsilon)^2}
\end{aligned} \tag{62}$$

where $d^d\tilde{p} = \frac{d^dp}{(2\pi)^d}$.

Now we are in a position to consider theories that transition between two different scaling behaviors.

The Standard Model is a theory that is IR free. For conformal invariance to offer any help, we want to flow to an interacting UV fixed point. In this case the two-point function is of the form

$$\langle 0|T\mathcal{O}^\dagger(x)\mathcal{O}(0)|0\rangle = \left(\frac{1}{-x^2}\right)^{d-1} f(-x^2M^2) \tag{63}$$

where we are able to fix the factor of $(-x^2)^{1-d}$ by dimensional analysis, coinciding with the free two-point function. Then the dimensionless function f contains all the information about the interacting dynamics. In the conformal regime, f reproduces the power law of the anomalous dimension of $\mathcal{O}$, and it should interpolate between the free and interacting limits:

$$f(-x^2M^2) \to \begin{cases} 1 & \text{as} \quad -x^2M^2 \to \infty \text{ (IR)} \\ (-x^2M^2)^{-\gamma_{UV}} & \text{as} \quad -x^2M^2 \to 0 \text{ (UV)} \end{cases} \tag{64}$$

The important part is that the two-point function depends in some way on the scale M. Pushing forward the calculation, we have in general

$$\delta m_h^2 = -iy_t^2\mu^{4-d}\int d^dx\left(\frac{1}{-x^2}\right)^{d-1} f(-x^2M^2) = -M^2\frac{y_t^2\pi^{d/2}}{\Gamma(d/2)}\left(\frac{\mu^2}{M^2}\right)^{2-d/2}\int\frac{dy}{y^{d/2}}f(y) \tag{65}$$

where we have carried out a change of variables to $y = -x^2M^2$ to make the M dependence more transparent.

In general, the precise form of f is unknown, but we can try various test functions with the desired asymptotic behavior. For example,

$$f(y) = \begin{cases} 1 & \text{for} \quad y > 1 \\ y^{-\gamma_{UV}} & \text{for} \quad 1 > y > 0 \end{cases} \tag{66}$$

This leads to a UV divergent integral that we can regularize to preserve scale invariance (e.g. with dimensional regularization), giving the result

$$\delta m_H^2 = -M^2\pi^2y_t^2\frac{\gamma_{UV}}{1+\gamma_{UV}} \tag{67}$$

The Higgs mass is dominated by the scale M at which the transition occurs. Of course, this example is quite abrupt and unphysical. A more realistic test example might be one of the form

$$f(y) = \left(\frac{1}{y^{n\gamma_{UV}} + y^{n\gamma_{IR}}}\right)^{1/n} \tag{68}$$

where $\gamma_{UV} < \gamma_{IR} \leq 0$ and n controls the smoothness of the transition. The larger n, the more abrupt the transition.

In the case of $\gamma_{IR} = 0$ corresponding to an IR free theory, we have

$$\delta m_H^2 = -M^2y_t^2\pi^2\frac{\Gamma\left(\frac{1}{n}+\frac{1}{n\gamma_{UV}}\right)\Gamma\left(1-\frac{1}{n\gamma_{UV}}\right)}{\Gamma(1/n)} \tag{69}$$

Here the functional dependence is a bit more complicated, but the result is the same. The correction to the Higgs mass is dominated by the scale M.

The lesson here is that even if conformal symmetry is argued to protect the Higgs, the Standard Model must transition to a conformal fixed point. The transition to a conformal fixed point gives rise to new scales, even if they do not correspond to perturbative states, and so must be close to the weak scale to avoid generating a hierarchy problem.

Thus we conclude that the Higgs mass can be rendered calculable and sensitivity to UV physics eliminated by introducing custodial symmetries. In each case, this involves new states that give physical threshold corrections to the Higgs mass and should lie near the weak scale.